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Metaheuristics for Combinatorial Optimization

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Combinatorial Optimization

- Combinatorial Optimization problem
  - Given a set of elements $E = \{1, 2, ..., n\}$
  - Set of feasible solutions F
    - * Each element of F is a subset of E.
  - Objective function $f(x): F \to R$.
  - In the minimization version the problem consists in
    - * Finding $x^* \in F$, such that $f(x^*) \leq f(x) \ \forall x \in F$.
      - – Discrete Optimization
      - – Graph models

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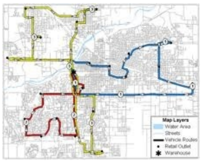
Combinatorial Optimization Problems

- Traveling Salesman Problem (TSP)
- Routing problems
  - Vehicle Routing Problem (VRP)
  - Heterogeneous Vehicle Routing Problem (HVRP)
- Location problems
- Scheduling problems
  - Job-shop scheduling problem
  - Parallel machines
- Other …
  - Clique problems



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Traveling Salesman Problem

- Traveling Salesman Problem
  - Given a number of cities and the costs (distances) of traveling from any city to any other city…
  - What is the least-cost round-trip route that visits each city exactly once and then returns to the starting city?
  - http://www.math.uwaterloo.ca/tsp/

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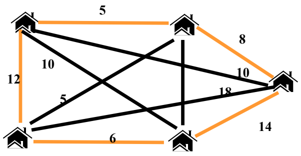
Traveling Salesman Problem

- Traveling Salesman Problem
  - Traveling Salesman Problem
    - * E: set of edges, each has a cost c(e);
    - * A: any subset of edges forming a Hamilton cycle;
    - * c(x): total cost of the edges in x.



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Traveling Salesman Problem

- Problem difficult to solve…
  - the size of the solution space is O(n)! for $n > 2$
    - * where n is the number of cities.

#cities	#asymetric tours	#symetric tours
5	24	12
6	120	60
7	720	360
8	5040	2520
9	40320	20160
10	362880	181440
20	1.2165E+17	6.08226E+16
25	6.2045E+23	3.10224E+23

Traveling Salesman Problem

- Mathematical programming model (asymmetric)

$$x_{ij} = \begin{cases} 1, & \text{if } (i,j) \text{ is in the tour;} \\ 0, & \text{otherwise.} \end{cases}$$

m = number of edges;
n = number of vertices;
c_{ij} = cost/distance of edge (i,j).

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

s.t.

$$\sum_{i=1}^{n} x_{ij} = 1 \quad j = 1,\ldots,n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad i = 1,\ldots,n$$

$$\sum_{i,j \in S} x_{ij} \le |S| - 1, \quad \text{for all } S \subset \{1,\ldots,n\};$$

$$x_{ij} \in \{0,1\}, \quad i = 1,\ldots,n;\ j = 1,\ldots,n;$$

Example TSP

- Real applications
  - Time windows
  - Distance vs. Cost vs. Time
  - Customer constraints
  - Several vehicles
  - Different capacities
  - Etc.…

Facility Location Models

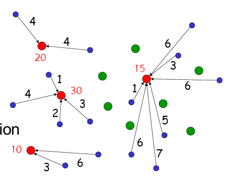
- Important problem in Logistics, Health Care, Public Sector, Telecommunications…
  - Where to locate new facilities.
    - Retailers, warehouses, factories.
  - Very complex problems
- Warehouse location problem
  - to locate a set of warehouses in a distribution network
- Cost of locating a warehouse at a particular site:
  - fixed cost vs variable cost
    - cost of open facility vs. transportation cost

Facility Location Models

- Where to locate the facilities?
  - Warehouses
  - Schools
  - Hospitals
  - Etc…
- How to meet customer demands from the facilities?
  - Which facility (facilities) serves each customer?
  - How much demand is met from each facility?
- Costs
  - Transportation, warehousing, customer service, …

Facility Location Models

- Facility Location Models
  - p-Median
    - ILP Model
  - Covering problems
    - Maximal covering location problem
  - Capacitated Facility Location
    - Single-source capacitated facility location
  - And many more …



Facility Location: p-median

- Location of p facilities to serve n customers.
- Which is the best location?
- Which facility should serve each customer?
  - Minimizing costs, distances, etc.

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Facility Location: p-median

- Locate 3 schools to serve 9 areas.
- Each area should be assigned to one and only one school.
- The distances between the potential facility location (school) and the areas is indicated as follows...

Facility Location: p-median

- Which is the best location for the 3 schools that minimize the total distance?

Tiempo entre distritos

Distrito	1	2	3	4	5	6	7	8	9
1	0	5	4	2	1	7	2	9	1
2	5	0	12	10	8	4	2	1	9
3	4	12	0	11	2	2	7	10	12
4	2	10	11	0	2	7	1	1	14
5	1	8	2	2	0	6	4	4	5
6	7	4	2	7	6	0	4	3	2
7	2	2	7	1	4	4	0	8	5
8	9	1	10	1	4	3	8	0	9
9	1	9	12	14	5	2	5	9	0

Facility Location: p-median

- Variables
  - x_{ij} = 1 if area i is assigned to school at j ; 0 otherwise, for i, j=1, … , 9.
  - y_j = 1 if a school is located at area j, otherwise, for j=1, … , 9.
    - * Binary Variables.

Facility Location: p-median

$$\min \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij}$$

$$\begin{aligned}
\text{st} \quad & \sum_{j=1}^{n} x_{ij} = 1 && \forall i && (1)\\
& x_{ij} \le y_j && \forall i,j && (2)\\
& \sum_{j=1}^{m} y_j = p && && (3)\\
& x_{ij}, y_i \text{ binary } \forall i,j
\end{aligned}$$

Maximal Covering Location Model

- Maximal Covering Location model
  - This problem identifies the minimal number and the location of facilities, which ensures that no demand point will be farther than the maximal service distance or time from a facility.

Maximal Covering Location Model

- The objective is to locate the minimum number of centers to satisfy a demand within 5 units of time
- Traveling times (units of time) from population i (row) to facility j (column):

Distrito	1	2	3	4	5	6	7	8	9	10	11	12
1	0	5	4	2	1	7	2	9	1	4	2	8
2	5	0	12	10	8	4	2	1	9	7	7	9
3	4	12	0	1	2	2	7	10	12	11	6	8
4	2	10	1	0	2	7	1	1	14	18	1	2
5	1	8	2	2	0	6	4	4	5	3	6	8
6	7	4	2	7	6	0	4	3	2	2	10	4
7	2	2	7	1	4	4	0	8	5	4	8	6
8	9	1	10	1	4	3	8	0	9	7	8	3
9	1	9	12	14	5	2	5	9	0	8	4	5
10	4	7	11	18	3	2	4	7	8	0	5	10
11	2	7	6	1	6	10	8	8	4	5	0	12
12	8	9	8	2	8	4	6	3	5	10	12	0

Maximal Covering Location Model

- Data
  - m= number of potential location for the center
  - n= number of populations to be covered
  - d_{ij} = distance between population i and center (facility) j
  - D_{max} = maximum service time.

Maximal Covering Location Model

- Variables
  - x_{ij} = 1 if population i is covered by a center at j ; 0 otherwise, for i, j=1, ... , 12.
  - y_j = 1 if a center is open at location j, otherwise, for j=1, ... , 12.

Maximal Covering Location Model

$$
\begin{array}{llll}
 & \min \sum_{j=1}^{m} y_j & & \\
\text{st} & \sum_{j=1}^{n} x_{ij} = 1 & \forall i & (1) \\
 & x_{ij} \le y_j & \forall i,j & (2) \\
 & d_{ij} x_{ij} \le D_{\max} & \forall i,j & (3) \\
 & x_{ij}, y_j \text{ binarias } \forall i,j & &
\end{array}
$$

Maximal Covering Location Model

- Simple Formulation
- y_j = 1 if a center is open at location j, otherwise, for j=1, ... , m.

$$
\begin{array}{l}
\min \sum_{j=1}^{m} y_j \\
\text{st} \\
\sum_{j \in N_i} y_j \ge 1 \quad \text{for} \quad i = 1,\ldots,n \text{ and } N_i = \{ j : d_{ij} \le D \} \\
y_j \text{ binarias } \forall j
\end{array}
$$

Capacitated Facility Location

- Location of facilities to serve *n* customers.
  - Number of location is not fixed in advanced.
  - Fixed cost to open a facility.
  - Each customer only served by only one facility.
  - Each customer has a annual demand to be served.
  - Capacity constraints in the facilities.
- Which is the best location?
- Which facility should serve each customer?
  - Minimizing costs.

Capacitated Facility Location

- **Data**
  - n retailers
  - m potential sites for warehouses
  - d_i demand of retailer I
  - f_j fixed cost of open warehouse at location j
  - q_j demand capacity at warehouse at location j
  - c_{ij} unit transportation cost between warehouse j and retailer i
- **Binary Variables**
  - y_j = open/close warehouse at location j
  - x_{ij} if retailer i is served or not by warehouse j

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Capacitated Facility Location

$$(1)\ min\ \ f(x)=\sum_{j=1}^{m}\sum_{i=1}^{n}c_{ij}x_{ij}+\sum_{j=1}^{m}f_jy_j$$

$$s.a$$

$$(2)\ \sum_{i=1}^{n}d_ix_{ij}\le q_jy_j,\quad j=1,...,m$$

$$(3)\ \sum_{j=1}^{m}x_{ij}=1,\quad i=1,...,n$$

$$(4)\ x_{ij},y_j\in\{0,1\},\quad i=1,...,n;\ j=1,...,m$$

Capacitated Facility Location

- If the open locations are decided…
  - If demand of all customers is 1…
    - The subproblem is the Assignment Model
    - Easy to solve
  - If demand of all customers is an integer …
    - The subproblem is the Generalized Assignment Model
    - Difficult to solve

Assignment problem

- Assign each task to each individual
  - $x_{ij} = \begin{cases} 1, \text{ if individual } i \text{ is assigned to task } j \\ 0, \text{ otherwise} \end{cases}$
- One to one assignment

$$\min z=\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}x_{ij}$$

$$s.a$$

$$\sum_{j=1}^{n}x_{ij}=1,\qquad i=1,\dots,n$$

$$\sum_{i=1}^{n}x_{ij}=1,\qquad j=1,\dots,n$$

$$x_{ij}\ge 0\quad \text{for all}\quad i,j$$

Generalized Assignment

- I : set of tasks (i=1, ..., n)
- J : set of agents (j=1, ..., m)
- a_j = resource capacity of agent j
- b_{ij} = resource needed if task i is assigned to agent j
- c_{ij} = cost of task i if assigned to agent j
- The variables:
  - x_{ij} = 1, if task i is assigned to agent j; 0, otherwise.

Generalized Assignment

$$(1)\quad min\ \ f(x)=\sum_{j=1}^{m}\sum_{i=1}^{n}c_{ij}x_{ij}$$

$$s.a$$

$$(2)\quad \sum_{i=1}^{n}b_{ij}x_{ij}\le a_j\ j=1,\dots,m$$

$$(3)\quad \sum_{j=1}^{m}x_{ij}=1\ i=1,\dots,n$$

$$(4)\quad x_{ij}\in\{0,1\}\ i=1,\dots,n;\ j=1,\dots,m$$

Research links on Location models

- EWGLA
  - European working Group on Locational Analysis
    - http://www.vub.ac.be/EWGLA/
- Location Analysis –INFORMS
    - http://location.section.informs.org/

Design of logistics network

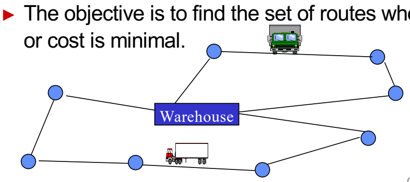
- Commercial Software
  - Llamasoft
    - https://www.llamasoft.com/solutions/network-optimization/
    - LogicTools LogicNet Plus XE
      - https://www.youtube.com/watch?v=KksQbzRMwvQ
      - https://www.youtube.com/watch?v=yfhHgEVLuVU
  - JDA
    - Network Design and Optimization
      - https://jda.com/solutions/adaptable-manufacturing-distribution-solutions/manufacturing-planning/network-design-and-optimization
      - https://www.youtube.com/watch?v=Ecc-IOhrQz8
  - Amazon Supply Chain Optimization Technologies
    - https://www.youtube.com/watch?v=ncwsr1Of6Cw

Vehicle Routing

- A large part of many logistics systems involves the management of a fleet of vehicles used to serve
  - warehouses
  - retailers
  - customers
- General class of vehicle routing problems.
- Can be applied to other areas too…
  - Humanitarium Logistics
  - Retailing and Logistics
  - E-commerce
  - Flow routing in telecommunications

Vehicle Routing

- A set of customers at known geographical locations has to be supplied by a fleet of vehicles from a single depot.
- Each customer has a specific demand.
- Each route starts and finish at the depot.
- The objective is to find the set of routes whose total length or cost is minimal.



Vehicle Routing

- Set of clients (vertex set) V={0,...,n}
  - Location 0 correspond to the depot
  - d_j demand of client j
- Set of arcs A={1,...,m}
  - c_{ij} nonnegative cost associate with each arc (travel cost)
- k vehicles with identical capacity Q.
- Customer are visited by one and only one vehicle.
- Find de set of routes minimizing the total cost.

Vehicle Routing model

- A set of customers at known geographical locations has to be supplied by a fleet of vehicles from a single depot.
- Set of clients (vertex set) *V={0,...,n}*
  - Location 0 correspond to the depot
- *m* vehicles with identical capacity *Q*
  - Each route starts and finishes at the depot.
- Each customer has a specific demand.
  - q_i demand of client *i*; *i=1,…n*
- The objective is to find the set of routes whose total length or cost is minimal.
  - c_{ij} nonnegative cost associate with each arc (travel cost between *i* and *j*).

Vehicle Routing model

- Variables

$$x_{ijk} = \begin{cases} 1 & \text{if vehicle } k \text{ visits customer } j \text{ immediately after customer } i; \\ 0 & \text{otherwise} \end{cases}$$

$$i = 0,\dots,n;\ j = 0,\dots,n;\ k = 1,\dots,m$$

$$y_{ik} = \begin{cases} 1 & \text{if } i \text{ is visited by vehicle } k; \\ 0 & \text{otherwise} \end{cases} \qquad i = 0,\dots,n;\ k = 1,\dots,m$$

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Vehicle Routing model

$$\min \sum_{j=0}^{m}\sum_{i=0}^{n} c_{ij} \sum_{k=1}^{m} x_{ijk}$$

$s.t.$

$$\sum_{k=1}^{m} y_{ik} = 1, \quad i = 1,\ldots,n \qquad (1)$$

$$\sum_{k=1}^{m} y_{ik} = m, \quad i = 0 \qquad (2)$$

$$\sum_{i=1}^{n} q_i y_{ik} \le Q, \quad k = 1,\ldots,m \qquad (3)$$

$$\sum_{j=0}^{n} x_{ijk} = \sum_{j=0}^{n} x_{jik} = y_{ik}, \quad i = 1,\ldots,n;\ k = 1,\ldots,m \qquad (4)$$

$$\sum_{i,j\in S} x_{ijk} \le |S| - 1, \quad \text{for all } S \subset \{1,\ldots,n\};\ k = 1,\ldots,m \qquad (5)$$

$$x_{ijk} \in \{0,1\}, \quad i = 0,\ldots,n;\ j = 0,\ldots,n;\ k = 1,\ldots,m$$

$$y_{ik} \in \{0,1\}, \quad i = 0,\ldots,n;\ k = 1,\ldots,m$$

Vehicle Routing

- Solve the TSP model
  - Heuristics
    - Savings Heuristics - Clarke & Wright (1964)
  - EXCEL
    - Same difficulties as for the TSP…
  - Commercial Software
    - ArcRoute Logistics (ESRI)
    - …… many more
  - Metaheuristics
    - Local search heuristic
    - … many more

Extensions Vehicle Routing

- Vehicle routing problem with vehicles with different capacities.
- Vehicle routing problem with time windows.
- Vehicle routing with simultaneous pickups and deliveries.
- Reverse Logistics
- Forward (products) and reverse (packages) channel for the same customer.
- Vehicle routing with multi depots.
  - Location, assignment and routing decisions

Scheduling

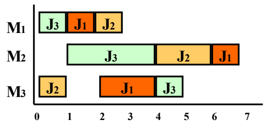
- Allocation of limited resources to the processing of tasks.
  - Resources
    - Machines, crews, vehicles, planes, buses, personal…
  - Tasks
    - Jobs, flights, distribution operations, projects...
- Important role in most manufacturing, logistics and service industries.
- Strategic and operations decisions.

Scheduling

- Allocation of limited resources to the processing of tasks.
  - Resources
    - Machines, crews, vehicles, planes, buses, personal…
  - Tasks
    - Jobs, flights, distribution operations, projects...
- with the objective of...
  - Minimize completion time; cost; etc.
- Important role in most manufacturing, logistics and service industries.
- Strategic and operations decisions.

The Goals of Scheduling

- By scheduling effectively, companies use assets more effectively and create greater capacity per dollar invested, which, in turn, lowers cost.
- Faster delivery and better use of resources.
- Better customer service.
- Good scheduling is a competitive advantage.



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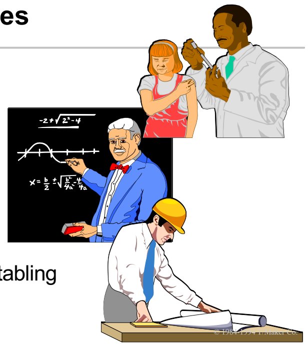
Scheduling

- Applications Areas
  - Logistics
  - Transportation
    - Many applications (train, bus, truck, airlines, etc.)
    - Crew Scheduling
    - Vehicle Scheduling
  - Distribution
  - Production and operations
  - Information processing and communications

Scheduling Examples

- Transportation
  - Airlines
  - Train
  - Public transportation
- Hospital
  - Outpatient treatments
  - Operating rooms
- Course Scheduling-Timetabling



The Goals of Scheduling

- Allocation of limited resources to the processing of tasks, with the objective of...
  - Minimize completion time;
  - Minimize WIP inventory;
    - Keep inventory levels low
  - Maximize utilization;
    - Make effective use of personnel and equipment
  - Minimize customer waiting time.
- … so the goods or services are at the right place at the right time.

The Goals of Scheduling

- By scheduling effectively, companies use assets more effectively and create greater capacity per dollar invested, which, in turn, lowers cost.
- Faster delivery and better use of resources.
- Better customer service.
- Good scheduling is a competitive advantage.

Scheduling Models

- Classify the problems
  - Machine configurations
  - Processing characteristics and constraints
  - Objectives and performance measures

Scheduling Models

- Notation
  - n jobs $J=\{J_1, J_2, ..., J_n\}$
  - m machines $M=\{M_1, M_2, ..., M_n\}$
  - r_j : release date
  - d_j : due date
  - q_j : delivery time
  - w_j : priority factor or weight
  - C_{ij} : completion time

Scheduling Models

- Machine Configurations
  - Single-machine models
  - Parallel-machine models
    - * Any machine can process the jobs.
  - Flow-shop models
    - * multiple operations on different machines
    - * all jobs have identical routes
  - Job-shop models
    - * multiple operations on different machines

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Scheduling Models

- Processing characteristics and constraints:
  - precedence constraints
  - routing constraints
  - sequence-dependent setup times
  - preemptions
  - tooling constraints
  - personal scheduling constraints

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Scheduling Models

- Objectives and performance measures:
  - Makespan objectives

$$C_{\max} = \max\{C_1, C_2, \ldots, C_n\}$$
$$C_i = \max\{C_{i1}, C_{i2}, \ldots, C_{im}\}, i = 1, \ldots, n$$

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Scheduling Models

- Objectives and performance measures:
  - Due date related objectives
  - Lateness $L_j = C_j - d_j$
  - Minimize maximum lateness (L_{max})
  - Tardiness $T_j = \max\{C_j - d_j, 0\}$
  - Minimize the weighted tardiness / Total weighted completion time

$$\sum_{j=1}^{n} w_j T_j \qquad \sum_{j=1}^{n} w_j C_j$$

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Scheduling Methods

- Solution Methods:
  - General purpose scheduling procedures
    - * Priority dispatching rules
  - Exact methods
    - * Branch-and-bound
    - * Branch-and-cut
  - **Heuristics**
    - * Constructive heuristics
      - – Priority Rules
      - – One-machine scheduling relaxations
  - **METAHEURISTICS**

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Priority Rules for Dispatching Jobs

- First come, first served (FCFS)
  - The first job to arrive at a work center is processed first
- Earliest due date (EDD)
  - The job with the earliest due date is processed first
- Shortest processing time (SPP)
  - The job with the shortest processing time is processed first
- Longest processing time (LPT)
  - The job with the longest processing time is processed first

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First Come, First Served Rule

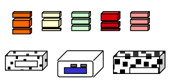
- Process first job to arrive at a work center first
- Average performance on most scheduling criteria
- Appears ‘fair’ & reasonable to customers
  - Important for service organizations
    - Example: Restaurants

Earliest Due Date Rule

- Process job with earliest due date first
- Widely used by many companies
  - If due dates important
  - If MRP used
    - Due dates updated by each MRP run
- Performs well on many scheduling criteria

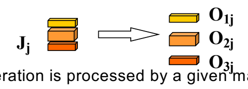
The Job-Shop Scheduling

- A set of n jobs, $J_1, \dots, J_n$
- A set of m machines, $M_1, \dots, M_m$
- Schedule the processing of each job by the machines.
- The objective is to minimize the maximum completion time (makespan)
  - Min C_{max}= max C_j.

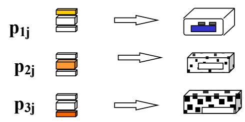


The Job-Shop Scheduling

- Each job consists of a sequence of operations:

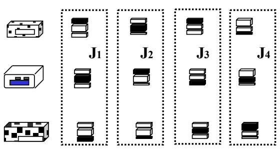


- Each operation is processed by a given machine during uninterrupted processing time p_{ij}.



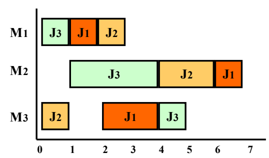
The Job-Shop Scheduling

- Constraints
  - Each machine can process at most one job at a time.
  - Each job can be processed by at most one machine at a time.
  - Jobs have different processing order by the machines.
  - No preemption



The Job-Shop Scheduling

- A schedule is an allocation of the operations to time intervals on the machines.
  - Example:
    - J_1: M_1 / 1; M_3 / 2; M_2 / 1;
    - J_2: M_3 / 1; M_1 / 1; M_2 / 2;
    - J_3: M_1 / 1; M_2 / 3; M_3 / 1;



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The Job-Shop Scheduling

p_{ij} = processing time of job j on machine i

y_{ij} = starting time of job j on machine i

Disjunctive Model

Minimize C_{max}

Subject to

$y_{kj} - y_{ij} \geq p_{ij}$	For all edges (i, j) → (k, j)
$C_{max} - y_{ij} \geq p_{ij}$	For all operations (i, j)
$y_{ij} - y_{i\ell} \geq p_{i\ell}$ or $y_{i\ell} - y_{ij} \geq p_{ij}$	For all edges (i, ℓ) — (i, j)
$y_{ij} \geq 0$	For all operations (i,j)

Transform in a ILP model

The Job-Shop Scheduling

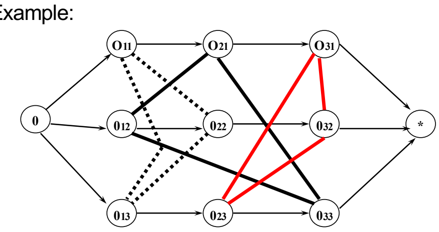
- Disjunctive graph G=(O,D,N)
  - * Roy & Sussman (1964)
  - Operation Oij - Node with weight pij.
    - * O = Set of nodes
  - Arc for each consecutive operations of a job.
    - * D = set of (conjunctive) arcs
  - Disjunctive arc for each pair of operations that are to be processed in the same machine.
    - * N = set of disjunctive arcs (edges)

The Job-Shop Scheduling

- Disjunctive graph G=(O,D,N)
  - Initial node 0
    - * dummy arcs from 0 to the first operation of each job.
  - Final node *
    - * dummy arcs from the last operation of each job to *.

The Job-Shop Scheduling

- Example:

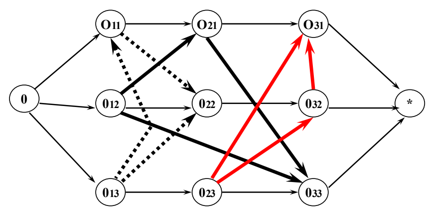


The Job-Shop Scheduling

- Solution:
  - Orient each disjunctive arc in one of two the possible way.
  - Sequence the operations in each machine.
    - * complete orientation
    - * consistent (acyclic)
- Makespan
  - Value of the longest path in the oriented graph.

The Job-Shop Scheduling

- Feasible solution:



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The Job-Shop Scheduling

- NP-hard in the strong sense
  - Garey,Johnson and Sethi (1976).
  - Earned a reputation for intractability.
- MT10 remained unsolved for over 20 years.
  - 10 jobs and 10 machines
- Very few special cases can be solved in polynomial time:
  - 2 machines, 2 operations/job;
  - 2 machines, unit processing times.

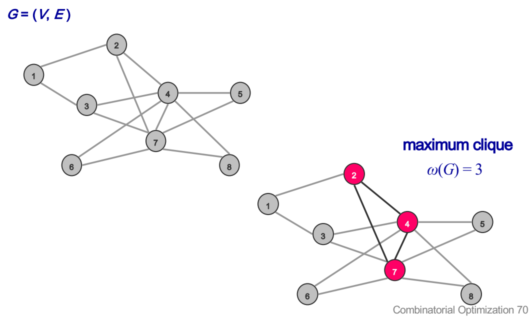
Clique problems

- Find set of high related elements
- Applications:
  - Consider a social network, the elements represent people, find a largest subset of people who all know each other… or buy the same product… or…
  - Marketing basket applications
  - Other application in Bioinformatics, Chemistry etc.
- http://en.wikipedia.org/wiki/Clique_problem

Maximum Clique Problem

- Given a simple and undirected graph $G = (V, E)$, with $V = \{1, ..., n\}$ the set of nodes and $E \subseteq V \times V$ the set of edges
- $C \subseteq V$ is a clique if $(i,j) \in E$, for all $i,j \in C$
- A clique C is maximum if it is the largest clique in G.
  - Clique cardinality
  - $\omega(G) = \max\{ |C| : C \text{ is a clique of } G \}$

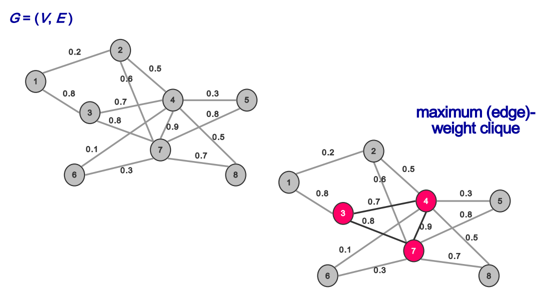
Maximum Clique Problem



Maximum Weight Clique Problem

- Maximum (edge) Weight Clique Problem
- Given a simple and undirected graph $G = (V, E)$, with $V = \{1, ..., n\}$ the set of nodes and $E \subseteq V \times V$ the set of edges
- $C \subseteq V$ is a clique if $(i,j) \in E$, for all $i,j \in C$
- If we assign weights a_{ij} to each edge $(i,j) \in E$
- Let C be a clique and $A(C) = \sum_{i,j \text{ belong } C} a_{ij}$
- **we want to find a clique C with maximum $A(C)$**

Maximum Weight Clique Problem



Maximum Clique Problem

- Known formulations

$$x_i = \begin{cases} 1 & \text{if nodes } i \text{ is in the clique} \\ 0 & \text{otherwise} \end{cases} \quad , \quad i \in V$$

Edge Formulation (EF)

$$\max \sum_{i \in V} x_i$$

$$\text{s. t} \quad x_i + x_j \le 1 \quad , \quad \forall (i,j) \in \overline{E}$$

$$x_i \in \{0,1\} \quad , \quad \forall i \in V$$

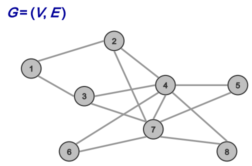
Maximum Clique Problem

$$Max \quad Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8$$

s.t.

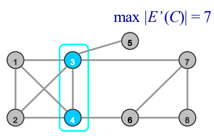
$x_1 + x_4 \le 1$	$x_3 + x_5 \le 1$
$x_1 + x_5 \le 1$	$x_3 + x_6 \le 1$
$x_1 + x_6 \le 1$	$x_3 + x_8 \le 1$
$x_1 + x_7 \le 1$	$x_5 + x_6 \le 1$
$x_1 + x_8 \le 1$	$x_5 + x_8 \le 1$
$x_2 + x_3 \le 1$	$x_6 + x_8 \le 1$
$x_2 + x_5 \le 1$	
$x_2 + x_6 \le 1$	
$x_2 + x_8 \le 1$	

$x_i \ge 0, \quad \text{for } i = 1,\ldots,8$



Maximum Cut-Clique Problem

- Given a clique C, its edge neighborhood (cut-clique) is defined by the set of edges $E'(C)=\{(i,j) \in E : i \in C \text{ and } j \in V \setminus C\}$, and $|E'(C)|$ is its size. Denote $N(i)=\{j \in V:(i,j) \in E\}$.
- Maximum Cut-Clique
  - Maximum edge neighborhood clique



Maximum Cut-Clique Problem

- Formulation (1/2)

$G=(V,E)$ and $Q = \{ q_{\min},\ldots, q_{\max} \}$ a set of all possible clique's sizes

- Variables

$$x_i = \begin{cases} 1 & \text{if nodes } i \text{ is in the clique} \\ 0 & \text{otherwise} \end{cases} \quad , \quad i \in V$$

$$w^q = \begin{cases} 1 & \text{if the clique size is equal to } q \\ 0 & \text{otherwise} \end{cases} \quad , \quad q \in Q$$

Maximum Cut-Clique Problem

- Formulation (2/2)

$$\max \sum_{i \in V} |N(i)| x_i - \sum_{q \in Q} q \cdot (q-1) w^q$$

All the edges incident on the clique

Minus the ones inside the clique

subject to

$$x_i + x_j \le 1, \quad \forall (i,j) \in \overline{E}$$

$$\sum_{i \in V} x_i = \sum_{q \in Q} q w^q$$

$$\sum_{q \in Q} w^q = 1$$

$$x_i \in \{0,1\}, \ \forall i \in V \quad ; \quad w^q \in \{0,1\}, \ \forall q \in Q$$

Combinatorial Optimization Models

- Many real problems are on combinatorics nature…
- How to solve these problems?
- In particular when they are of large scale…
- And need a rapid response …
- Sometimes a online solution…

Metaheuristics is the answer!