An introduction to the stochastic heat equation: local existence and blowup.

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Abstract

In this course we will first define the concept of mild solution to the stochastic heat equation driven by a space-time white noise by recalling the theory of stochastic integrals with respect to Gaussian random fields. Then, under the assumption that the coefficients are locally Lipschitz functions, we will define and show local existence and uniqueness of the solution. Then, we will prove recent results that give necessary and sufficient conditions on the coefficients for the solution to blowup in finite time, which are related to the well-known Osgood condition for ordinary differential equations.

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Chapter 1

Introduction: from the heat equation to the stochastic heat equation

In order to introduce the stochastic heat equation, we first need to recall some basics about the heat equation. For more detailed exposition about the heat equation, one can consult, for instance, the monograph by David Borthwick [1] and the references therein.

Let us start with the following observation: it is easy to check that the one-dimensional centered Gaussian density with variance $2\kappa t$, where $\kappa > 0$ and $t > 0$, given by

$$p_t(x) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right),$$

satisfies the partial differential equation (PDE)

$$\partial_t(p_t(x)) = \kappa \partial^2_{xx}(p_t(x)).$$

This PDE is well-known to be the basic mathematical model for heat conduction and is called the heat equation, developed by Joseph Fourier in the early 19th century.

Consider a metal bar of length $L > 0$, sufficiently thin that can be parametrized as a one-dimensional system on the x-axis, with one end at the origin and the other at $x = L$. Let $u = u(t\,, x)$ denote the temperature of the bar at time t and position x. We assume that the bar is perfectly insulated except possibly at its endpoints, and that the temperature is constant on each cross section and therefore depends only on t and x. We also assume that the thermal properties of the bar are independent of x and t. In this case, it can be shown (see for example Section 6.1 of [1]) that u satisfies the heat equation on $[0, L]$ given by

$$\partial_t(u(t\,, x)) = \kappa \partial^2_{xx}(u(t\,, x)), \quad 0 < x < L,\ t > 0, \tag{1.0.1}$$

where $\kappa > 0$ is fixed and determined by the thermal properties.

To determine u we need to specify the temperature at any point in the bar when $t = 0$, say

$$u(0\,,x) = u_0(x), \quad 0 \leqslant x \leqslant L.$$

We call this the initial condition and in this course we assume that u_0 a continuous function on $[0, L]$. We also need to specify the boundary conditions that u must satisfy at the end of the bar for all $t > 0$. We consider homogeneous Dirichlet boundary conditions, that is,

$$u(t\,,0) = u(t\,,L) = 0.$$

By the method of separation of variables and the theory of Fourier series, one can show (see for example Chapter 8 of [1]) that the unique solution to (1.0.1) is given by

$$u(t\,,x) = (G_t u_0)(x),$$

where

$$(G_t f)(x) = \int_0^L g_t(x\,,y) f(y)\,\mathrm{d}y, \tag{1.0.2}$$

and the function $g_t(x\,,y)$ is known as the Dirichlet heat kernel on $[0, L]$ which is given by

$$g_t(x\,,y) = \sum_{n=1}^{\infty} \Phi_n(x)\Phi_n(y)\exp(-\lambda_n \kappa t), \tag{1.0.3}$$

where $\lambda_n = \frac{n^2\pi^2}{L^2}$ and $\Phi_n(x) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$, $n = 1, 2, \ldots$ are respectively the eigenvalues and eigenfunctions of the eigenvalue problem

$$X''(x) = -\lambda X(x), \quad 0 < x < L$$

with initial condition $X(0) = X(L) = 0$. Observe that $\{\Phi_n\}_{n\geqslant 1}$ forms an orthonormal basis of $L^2([0\,,L])$. Moreover, the solution belongs to $\mathcal{C}^\infty((0,\infty)\times[0,L])$ and for all $x \in [0\,,L]$,

$$\lim_{t\to 0} u(t\,,x) = u_0(x). \tag{1.0.4}$$

In the case that the length of the bar is assumed to be infinite we obtain the heat equation on the real line, that is,

$$\partial_t(u(t\,,x)) = \kappa\partial^2_{xx}(u(t\,,x)), \quad x \in \mathbf{R},\ t > 0, \tag{1.0.5}$$

with initial condition $u(0\,,x) = u_0(x)$. In this case we assume that u_0 is a continuous and bounded function on the real line. Then, one can show (see for example Theorem 6.2 in [1]) that a solution to (1.0.5) is given by

$$u(t\,,x) = (P_t u_0)(x),$$

where

$$(P_t f)(x) = \int_{\mathbf{R}} p_t(x-y) f(y)\, \mathrm{d}y, \tag{1.0.6}$$

and recall that $p_t(x)$ is the density of a Brownian motion $B_{2\kappa t}$ which is known as the heat kernel. Moreover, under the assumption that u is bounded on $[0,T] \times \mathbf{R}$ for each $T > 0$, this is the unique solution which belongs to $\mathcal{C}^\infty((0,\infty) \times \mathbf{R})$ and (1.0.4) holds for all $x \in \mathbf{R}$ (see for example Theorems 6.3 and 6.4 in [1]).

The relationship between both heat kernels is the following. On one hand, the Dirichlet heat kernel $g_t(x\,,y)$ is the probability density function at a point y of a Brownian motion $B_{2\kappa t}$ starting at x and killed if it leaves the interval $[0,L]$. This implies that

$$g_t(x\,,y) \leqslant p_t(x-y). \tag{1.0.7}$$

On the other hand, the heat kernel is the fundamental solution to the heat equation in the sense that it solves equation (1.0.5) when the initial condition is given by the Dirac function $u(0\,,x) = \delta_0(x)$. In this case, since $\delta_0(x)$ is not a function, equation (1.0.5) needs to be understood in the sense of distributions. Then, by the method of images (see for example Section 12.5 in [1]), the Dirichlet heat kernel can be written as

$$g_t(x\,,y) = \sum_{n=-\infty}^{\infty} \left\{ p_t(y-x-2nL) - p_t(y+x-2nL) \right\}.$$

One can also verify directly that $g_t(x\,,y)$ solves equation (1.0.1).

The operator P_t is known as the heat operator which satisfies the semigroup property

$$P_t f = P_{t-s} P_s f, \qquad 0 < s < t.$$

Observe that $u(s\,,x) = (P_s u_0)(x)$ is the solution to the heat equation up to time s. Then, we can assume that we start again at time s up to time t and the semigroup property says that the solution at time t is given by $u(t\,,x) = (P_{t-s} u(s\,,\cdot))(x)$.

We next assume that the heat equation (1.0.5) is perturbed by a reaction term $f(t\,,x)$. Then we obtain the inhomogeneous heat equation on the real line given by

$$\partial_t(u(t\,,x)) = \kappa \partial^2_{xx}(u(t\,,x)) + f(t\,,x), \quad x \in \mathbf{R},\ t > 0, \tag{1.0.8}$$

with initial condition $u_0(x)$ as above. By Duhamel's principle (see for example p.54 of [1]) a solution to equation (1.0.8) is given by

$$u(t\,,x) = (P_t u_0)(x) + \int_0^t \int_{\mathbf{R}} p_{t-s}(x-y) f(s\,,y)\, \mathrm{d}y \mathrm{d}s. \tag{1.0.9}$$

The idea of Duhamel's principle is to write the solution as $u(t\,,x) = u_h(t\,,x) + u_p(t\,,x)$, where $u_h(t\,,x)$ is the solution to the homogeneous heat equation (1.0.5) with initial condition

u_0, and $u_p(t\,,x)$ is a solution to the inhomogeneous heat equation (1.0.8) with zero initial condition. Duhamel's principle consists of showing that

$$u_p(t\,,x) = \int_0^t (P_{t-s}f(s\,,\cdot))(x)\,\mathrm{d}s. \tag{1.0.10}$$

In fact, as $(P_{t-s}f(s\,,\cdot))(x)$ solves the heat equation with initial condition $f(s,x)$ at time s,

$$\begin{aligned}(\partial_t - \kappa\partial^2_{xx})(u_p(t\,,x)) &= \int_0^t (\partial_t - \kappa\partial^2_{xx})((P_{t-s}f(s\,,\cdot))(x))\,\mathrm{d}s + (P_0 f(t\,,\cdot))(x)\\ &= 0 + f(t\,,x),\end{aligned}$$

which proves that (1.0.10) indeed solves the inhomogeneous heat equation with zero initial condition, and thus (1.0.9) is a solution to equation (1.0.8).

Similarly, we can consider the inhomogeneous heat equation on $[0\,,L]$ given by

$$\partial_t(u(t\,,x)) = \kappa\partial^2_{xx}(u(t\,,x)) + f(t\,,x), \quad 0 < x < L,\ t > 0,$$

with initial condition $u_0(x)$ and homogeneous Dirichlet boundary conditions as above. By Duhamel's principle the solution is given by

$$u(t\,,x) = (G_t u_0)(x) + \int_0^t \int_0^L g_{t-s}(x\,,y) f(s\,,y)\,\mathrm{d}y\mathrm{d}s.$$

The inhomgeneous heat equation has many applications in physics, chemistry and biology. For example, it appears in the study of neurons, which can be seen as thin bars that act as electrical cables. The function $u(t\,,x)$ represents the electric potential at time t and point x, and is known to be governed by the cable equation given by

$$\partial_t(u(t\,,x)) = \kappa\partial^2_{xx}(u(t\,,x)) - \gamma u(t\,,x), \tag{1.0.11}$$

where $\gamma > 0$. Note that the solution to (1.0.11) can be written as $u(t\,,x) = e^{-\gamma t}v(t\,,x)$, where $v(t\,,x)$ denotes the solution to the homogeneous heat equation (1.0.1).

We next assume that the heat equation is perturbed by a random term, that is,

$$\partial_t(u(t\,,x)) = \kappa\partial^2_{xx}(u(t\,,x)) + b(u(t\,,x)) + \sigma(u(t\,,x))\dot{W}(t\,,x), \quad x \in \mathbf{R},\ t > 0, \tag{1.0.12}$$

with initial condition u_0, a bounded and continuous function on $\mathbf{R}$. The term $\dot{W}$ denotes a space-time white noise, which is a Gaussian noise in both time and space. The coefficients b and σ are nonrandom measurable functions on the real line known as the drift and noise coefficient of the stochastic heat equation (SHE), respectively. Equation (1.0.12) is formal since

$$\dot{W}(t\,,x) = \frac{\partial^2 W(t\,,x)}{\partial x \partial t},$$

where $W(t\,,x)$ is a two-parameter Wiener process, and the latter process is nowhere differentiable. Hence, $\dot{W}(t\,,x)$ is not a function but a distribution. Therefore, one needs to interpret the solution to (1.0.12) in the weak sense, that is, integrating with respect to a test function. Then one can show that such solution exists if and only if the following integral equation has a solution

$$\begin{aligned} u(t,\,x) = (P_t u_0)(x) &+ \int_0^t \int_{\mathbf{R}} p_{t-s}(x-y) b(u(s\,,y))\,\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_{\mathbf{R}} p_{t-s}(x-y)\sigma(u(s\,,y)) W(\mathrm{d}s\,\mathrm{d}y), \end{aligned} \tag{1.0.13}$$

where the last integral is a stochastic integral with respect to the white-noise measure W and is known as Walsh's stochastic integral. In Chapter 2 we provide the rigorous definition of the white-noise measure and the construction of Walsh's stochastic integral. Equation (1.0.13) is called the mild formulation or random field solution to the formal equation (1.0.12) and was introduced by John B. Walsh in 1984 in his famous Saint Flour lecture notes [24]. We refer to Walsh's lecture notes for the proof of the equivalence between the mild formulation (1.0.13) and the weak solution to (1.0.12). In this course, we will use the mild equation (1.0.13) as the definition of the solution to the SHE.

As in the deterministic case, the SHE has many applications. For example, going back to the cable equation (1.0.11) that studies the electric potential of a neuron, one observes that the surface of the neuron is covered by synapses, through which it receives impulses of currents that are random. This impulses are generally small and independent and can be modeled as a space-time white noise. Hence, $\dot{W}(t\,,x)$ represents the intensity of the impulse at time t and point x. Since the response of the neuron to a current impulse may depend on local potential we have the term $\sigma(u(t\,,x))$ in front of the noise. Then one obtains the following stochastic heat equation, which is called the stochastic cable equation:

$$\partial_t(u(t\,,x)) = \kappa\partial^2_{xx}(u(t\,,x)) - \gamma u(t\,,x)) + \sigma(u(t\,,x))\dot{W}(t\,,x).$$

The case $\sigma(u) = u$ is of particular interest in practise. In fact, equation (1.0.12) with $b = 0$ and $\sigma(u) = u$ is known as the Parabolic Anderson model (PAM) with space-time white noise and has been extensively studied in the physics and probability literature due to its intermittency properties. See for example the course by Davar Khoshnevisan [14].

The aim of this course is to give some answers to the question of existence and uniqueness of the mild formulation (1.0.13) of the SHE. In order to simplify the exposition we will assume that $\kappa = 1$, and we will consider both the SHE on the real line and on the interval $[0, 1]$. This latter is written as

$$\partial_t(u(t\,,x)) = \partial^2_{xx}(u(t\,,x)) + b(u(t\,,x)) + \sigma(u(t\,,x))\dot{W}(t\,,x), \quad 0 < x < 1,\ t > 0,$$

with initial condition a continuous function u_0 on $[0,1]$ and homogeneous Dirichlet boundary conditions. In this case, the mild formulation is written as

$$u(t\,,x) = (G_t u_0)(x) + \int_0^t \int_0^1 g_{t-s}(x\,,y) b(u(s\,,y))\,\mathrm{d}y\mathrm{d}s + \int_0^t \int_0^1 g_{t-s}(x\,,y)\sigma(u(s\,,y)) W(\mathrm{d}s\,\mathrm{d}y).$$

In order to obtain the existence and uniqueness of the mild solution needs sufficient conditions on the drift and noise coefficients of the SHE. When both coefficients are assumed to be globally Lipschitz continuous (for instance, PAM), it is well-known that the mild solution exists globally, that is, for all $(t\,,x)$ a.s. and is a.s. unique and continuous. We will give the proof of this fact in Chapter 2. When the coefficients are only assumed to be locally Lipschitz continuous, then the problem becomes more delicate. In this case, for the SHE on the interval $[0,1]$, we will show in Chapter 2 that the mild solution exists locally, that is, a.s. up to a stopping time $t < \tau$, and is unique and continuous a.s. Then, in Chapter 3, we will study the a.s. blowup in finite time of the mild solution, that is, when $\tau < \infty$ a.s. and

$$\lim_{t\uparrow\tau} \sup_{x\in[0,1]} |u(t\,,x)| = \infty, \qquad \text{a.s.}$$

We will show that when the drift is locally Lipschitz and the diffusion coefficient is a positive constant, the classical Osgood condition for ordinary differential equations is a necessary and sufficient condition for the solution to blowup in finite time a.s. We observe that the blowup of solutions to the SHE is very different if (say) $\sigma(u) = u$ (PAM) or if σ is (say) bounded away from zero and infinity. Thus, we will restrict the theory to the case that σ is a constant since this is the simplest and first case studied in the literature, and we will mention recent extensions to more general cases.

The stopping time argument used to show the existence of a local solution does not apply for the SHE on the real line since the solution is unbounded a.s., that is, for all $t > 0$,

$$\sup_{x\in\mathbf{R}} |u(t\,,x)| = \infty, \qquad \text{a.s.}$$

If the coefficients are both locally Lipschitz continuous with at most linear growth, one expects to obtain global existence and uniqueness of the mild solution, but this is still an open problem in the literature. For the SHE on the real line, we will show in Chapter 3 that when σ is a positive constant, the Osgood condition is sufficient for the non-existence of the global mild solution. We will also mention some recent extensions of this result.

An example of locally Lipschitz function that is not globally Lipschitz is $f(u) = u^\alpha$ with $\alpha > 1$. Another interesting case which is not covered in this course is the existence and uniqueness of solutions for the case $\sigma(u) = u^\alpha$ with $\alpha < 1$. This latter case appears in the theory of branching processes and is studied for example in the paper by Carl Mueller, Leonard Mytnik, and Ed Perkins [19], and the references therein. For the case $\sigma(u) = u^{\alpha}$ with $\alpha > 1$, we refer to the paper by Carl Mueller [18] and the references therein.

Chapter 2

Global and local existence of the stochastic heat equation

The aim of this chapter is twofold. We first introduce the notion of space-time white noise and its properties, and we define Walsh's stochastic integral. Then we give sufficient conditions on the drift and noise coefficients of the stochastic heat equation in $\mathbf{R}$ and $[0,1]$ for the mild solution to exist globally and locally, and be a.s. unique and continuous. The material of this chapter is very classical and follows essentially the lecture notes mentioned in the introduction [24] and [14]. We also refer to the lectures by Davar Khoshnevisan [2], the monograph by David Nualart [20], and the recent book by Robert C. Dalang and Marta Sanz-Solé [5].

2.1 Space-time white noise and Walsh's stochastic integral

A space-time white noise on $\mathbf{R}_+ \times \mathbf{R}$ is a zero-mean Gaussian process $W = \{W(h), h \in L^2(\mathbf{R}_+ \times \mathbf{R})\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with covariance

$$\mathrm{E}(W(h)W(g)) = \langle h, g\rangle_{L^2(\mathbf{R}_+\times\mathbf{R})}, \qquad h, g \in L^2(\mathbf{R}_+ \times \mathbf{R}), \tag{2.1.1}$$

where $\langle h, g\rangle_{L^2(\mathbf{R}_+\times\mathbf{R})}$ denotes the scalar product on $L^2(\mathbf{R}_+ \times \mathbf{R})$.

Notice that for every $\{h_1, \ldots, h_n\} \subset L^2(\mathbf{R}_+ \times \mathbf{R})$, the matrix $(\langle h_i, h_j\rangle_{L^2(\mathbf{R}_+\times\mathbf{R})})_{i,j=1,\ldots,n}$ is a $n \times n$ positive semidefinite matrix.

We can derive the following property of a space-time white noise.

Proposition 2.1.1. *For all $a_1, \ldots, a_n \in \mathbf{R}$ and $h_1, \ldots, h_n \in L^2(\mathbf{R}_+ \times \mathbf{R})$,*

$$W\left(\sum_{i=1}^{n} a_i h_i\right) = \sum_{i=1}^{n} a_i W(h_i), \qquad a.s.$$

Proof. By induction it suffices to prove that for all $a \in \mathbf{R}$ and $h, g \in L^2(\mathbf{R}_+ \times \mathbf{R})$: (i) $W(ah) = aW(h)$ a.s.; and (ii) $W(h+g) = W(h) + W(g)$ a.s. We verify (i) using (2.1.1) in order to see that $\mathrm{E}\left(|W(ah) - aW(h)|^2\right) = 0$. Similarly, we can prove (ii) by checking, using (2.1.1), that $\mathrm{E}\left(|W(h+g) - W(h) - W(g)|^2\right) = 0$. □

Proposition 2.1.1 tells us that $W : L^2(\mathbf{R}_+ \times \mathbf{R}) \to L^2(\Omega)$ is a linear isometry and the relation (2.1.1) is called the Wiener isometry. If $h \in L^2(\mathbf{R}_+ \times \mathbf{R})$ then the square-integrable random variable $W(h)$ is called the Wiener integral of h . We will write

$$W(h) = \int_{\mathbf{R}_+} \int_{\mathbf{R}} h(s\,, y) W(\mathrm{d}s\,\mathrm{d}y).$$

We denote by $\mathcal{B}(\mathbf{R}_+ \times \mathbf{R})$ the Borel σ-algebra of subsets of $\mathbf{R}_+ \times \mathbf{R}$, and by $\mathcal{A}(\mathbf{R}_+ \times \mathbf{R})$ the set of Borel sets in $\mathcal{B}(\mathbf{R}_+ \times \mathbf{R})$ with finite Lebesgue measure λ. If $A \in \mathcal{A}(\mathbf{R}_+ \times \mathbf{R})$ we abuse notation and write $W(A)$ for $W(\mathbf{1}_A)$. We observe from the Wiener isometry (2.1.1) that if $A_1, A_2 \in \mathcal{A}(\mathbf{R}_+ \times \mathbf{R})$, then

$$\mathrm{E}(W(A_1)W(A_2)) = \lambda(A_1 \cap A_2). \tag{2.1.2}$$

From this notation we have the following result.

Proposition 2.1.2. *The following properties hold:*

1. $W(A_1 \cup A_2) = W(A_1) + W(A_2) - W(A_1 \cap A_2)$ *for all* $A_1, A_2 \in \mathcal{A}(\mathbf{R}_+ \times \mathbf{R})$.
2. *If* $\{A_i\}_{i \geqslant 1}$ *is a decreasing sequence in* $\mathcal{A}(\mathbf{R}_+ \times \mathbf{R})$ *such that* $\cap_{i=1}^\infty A_i = \emptyset$, *then*

$$L^2(\Omega) - \lim_{n \to \infty} W(A_n) = 0.$$

Proof. We start proving 1. Observe that if A_1 and A_2 are disjoint, from (2.1.2), we get that

$$\mathrm{E}\left(|W(A_1 \cup A_2) - W(A_1) - W(A_2)|^2\right) = 0.$$

The general case follows by induction from this one. The proof of 2. follows easily from the fact that $\mathrm{E}\left(|W(A_n)|^2\right) = \lambda(A_n)$ for every $n \geqslant 1$. □

The preceding proposition implies that the mapping $A \to W(A)$ from $\mathcal{A}(\mathbf{R}_+ \times \mathbf{R})$ to $L^2(\Omega)$ defines a σ-additive $L^2(\Omega)$-valued measure, since we can deduce from it that if $\{A_i\}_{i \geqslant 1}$ is a sequence in $\mathcal{A}(\mathbf{R}_+ \times \mathbf{R})$ where the A_i's are pairwise disjoint and such that $\cup_{i=1}^\infty A_i$ also belongs to $\mathcal{A}(\mathbf{R}_+ \times \mathbf{R})$, then

$$W\left(\cup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty W(A_i) \qquad \text{a.s.,}$$

where the sum converges in $L^2(\Omega)$.

For every time $t \geqslant 0$ and $A \in \mathcal{A}(\mathbf{R})$, we define the white noise process $\{W_t\}_{t\geqslant 0}$ by $W_t(A) = W([0,t] \times A)$. Observe that for fixed A, this is a proper stochastic process. In particular, the process $\{\lambda(A)^{-1/2}W_t(A)\}_{t\geqslant 0}$ is a Brownian motion.

Let $\mathcal{F} = (\mathcal{F}_t)_{t\geqslant 0}$ be the filtration of the process $\{W_t\}_{t\geqslant 0}$. That is, for all $t \geqslant 0$, $\mathcal{F}_t$ is the σ-algebra generated by $\{W_s(A); 0 \leqslant s \leqslant t, A \in \mathcal{A}(\mathbf{R})\}$. We say that a process $X : \mathbf{R}_+ \times \mathbf{R} \times \Omega \to \mathbf{R}$ is adapted if for any $t > 0$ and $x \in \mathbf{R}$, the random variable $X(t\,,x)$ is measurable with respect to $\mathcal{F}_t$.

We next extend the definition of the Wiener integral to include random integrands, and the result is called Walsh's stochastic integral. The construction follows similarly as the Itô's stochastic integral. We first consider simple random fields $X : \mathbf{R}_+ \times \mathbf{R} \times \Omega \to \mathbf{R}$ of the form

$$X(t\,,x)(\omega) = X(\omega)\varphi(x)\mathbf{1}_{(a,b]}(t),$$

where $0 \leqslant a < b$, $X \in L^2(\Omega, \mathcal{F}_a)$ and $\varphi \in \mathcal{C}_0(\mathbf{R})$ (continuous function in $\mathbf{R}$ with compact support). Then, if $X(t\,,x)$ is a simple random field with the preceding representation, we define the stochastic integral of X as

$$W(X) = X \int_{(a,b]} \int_{\mathbf{R}} \varphi(y) W(\mathrm{d}s\,\mathrm{d}y),$$

where the integral $\int_{(a,b]} \int_{\mathbf{R}} \varphi(y)W(\mathrm{d}s\,\mathrm{d}y)$ is the Wiener integral defined above, which is independent of the σ-algebra $\mathcal{F}_a$. Thus, X and $\int_{(a,b]} \int_{\mathbf{R}} \varphi(y)W(\mathrm{d}s\,\mathrm{d}y)$ are independent. In particular,

$$\mathrm{E}(W(X)) = 0 \tag{2.1.3}$$

and

$$\mathrm{E}([W(X)]^2) = \|X\|^2_{L^2(\mathbf{R}_+ \times \mathbf{R} \times \Omega)}. \tag{2.1.4}$$

We next consider the set of elementary random fields $\mathcal{E}$ given by the processes $X : \mathbf{R}_+ \times \mathbf{R} \times \Omega \to \mathbf{R}$ that can be written as

$$X(t\,,x)(\omega) = \sum_{i=1}^{n} X_i(\omega)\varphi_i(x)\mathbf{1}_{(a_i,b_i]}(t),$$

where $0 \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_n < b_n$, the X_i's are random variables in $L^2(\Omega, \mathcal{F}_{a_i})$ and $\varphi_i \in \mathcal{C}_0(\mathbf{R})$.

We define the stochastic integral of an elementary random field $X \in \mathcal{E}$ as

$$W(X) = \sum_{i=1}^{n} X_i \int_{(a_i,b_i]} \int_{\mathbf{R}} \varphi_i(y) W(\mathrm{d}s\,\mathrm{d}y).$$

By the properties of Wiener integrals it is easy to see that the definition of the integral does not depend on the representation of $X(t\,,x)$ in the sense that if we have two different representations as a sum of simple random fields then the stochastic integral coincides a.s. Moreover, it is also easy to check that (2.1.3) and (2.1.4) also hold for all $X \in \mathcal{E}$. The identity (2.1.4) is called Walsh's isometry and shows that the integral operator $X \in \mathcal{E} \to W(X) \in L^2(\Omega)$ is a linear isometry.

The next results extends the definition of the integral to general integrable random fields.

Theorem 2.1.3. *Let $\mathcal{W}$ denote the completion of $\mathcal{E}$ with respect to the $L^2(\mathbf{R}_+ \times \mathbf{R} \times \Omega)$-norm. Then $W : \mathcal{W} \to L^2(\Omega)$ is a linear isometry that satisfies* (2.1.3) *and* (2.1.4).

Proof. Let $\{X^{(n)}\}_{n\geqslant 1}$ be a sequence of element of $\mathcal{E}$ that is Cauchy in $L^2(\mathbf{R}_+ \times \mathbf{R} \times \Omega)$. By completeness, $X := \lim_{n\to\infty} X^{(n)}$ exists in $L^2(\mathbf{R}_+ \times \mathbf{R} \times \Omega)$. According to Walsh's isometry (2.1.4), the sequence $\{W(X^{(n)})\}_{n\geqslant 1}$ is Cauchy in $L^2(\Omega)$. Therefore $W(X) := \lim_{n\to\infty} W(X^{(n)})$ exists in $L^2(\Omega)$ and satisfies (2.1.3) and (2.1.4). □

For any $X \in \mathcal{W}$, we call $W(X)$ Walsh's stochastic integral of X and we write

$$W(X) = \int_{\mathbf{R}_+} \int_{\mathbf{R}} X(s\,,y) W(\mathrm{d}s\,\mathrm{d}y).$$

Several observations are in order. First we observe that the class of non-random elements of $\mathcal{W}$ coincides with $L^2(\mathbf{R}_+ \times \mathbf{R})$. Moreover, Walsh's integral of $X \in L^2(\mathbf{R}_+ \times \mathbf{R})$ is the same as the Wiener integral of X. Second, it can shown that $\mathcal{W}$ contains the set of jointly measurable and adapted processes X that are square integrable, that is,

$$\|X\|^2_{L^2(\mathbf{R}_+\times\mathbf{R}\times\Omega)} = \mathrm{E}\left(\int_0^\infty \int_{\mathbf{R}} (X(s\,,y))^2 \mathrm{d}y\,\mathrm{d}s\right) < \infty.$$

We next give Burkholder-Davis-Gundy inequality for Walsh's stochastic integral.

Proposition 2.1.4. *If X is element of $\mathcal{W}$ then the process*

$$t \to M_t(X) := \int_0^t \int_{\mathbf{R}} X(s\,,y) W(\mathrm{d}s\,\mathrm{d}y)$$

defines a continuous $L^2(\Omega)$-martingale with quadratic variation

$$t \to \|X\|^2_{L^2([0,t]\times\mathbf{R}\times\Omega)}.$$

Moreover, for all $p \geqslant 2$, there exists a constant $c_p > 0$ such that

$$\mathrm{E}(|W(X)|^p) \leqslant c_p \|X\|^p_{L^2(\mathbf{R}_+\times\mathbf{R}\times\Omega)}. \tag{2.1.5}$$

Proof. The first statement follows directly if X is and element of $\mathcal{E}$. If $X \in \mathcal{W}$ then we can find a sequence $\{X^{(n)}\}_{n\geqslant 1}$ in $\mathcal{E}$ that converges to X in $L^2(\mathbf{R}_+ \times \mathbf{R} \times \Omega)$ as $n \to \infty$. Thanks to Walsh's isometry, $W(X^{(n)}) \to W(X)$ in $L^2(\Omega)$ as $n \to \infty$. Moreover, by Doob's maximal inequality for continuous $L^2(\Omega)$-martingales (see for e.g. Theorem 1.7 in [13]),

$$\begin{aligned} \mathrm{E}\left(\sup_{t\geqslant 0} |M_t(X^{(n)}) - M_t(X^{(m)})|^2\right) &\leqslant 4 \sup_{t\geqslant 0} \mathrm{E}\left(|M_t(X^{(n)} - X^{(m)})|^2\right) \\ &= 4\|X^{(n)} - X^{(m)}\|^p_{L^2(\mathbf{R}_+\times\mathbf{R}\times\Omega)}, \end{aligned}$$

which goes to zero as $n, m \to \infty$. Thus, the first statement is true. In order to prove (2.1.5), we see that

$$W(X) = \lim_{t\to\infty} M_t(X),$$

where the limit holds in probability. Hence, thanks to Fatou's lemma, for all $p \geqslant 0$, we get that

$$\mathrm{E}(|W(X)|^p) \leqslant \sup_{t\geqslant 0} \mathrm{E}(|M_t(X)|^p)$$

Thus, from the classical Burkholder-Davis-Gundy inequality for continuous $L^2(\Omega)$ martingales (see for e.g. Theorem 4.1 in [22]), we conclude the proof of (2.1.5).

□

2.2 Global and local existence of the stochastic heat equation

Consider the stochastic heat equation

$$\partial_t(u(t\,,x)) = \partial^2_{xx}(u(t\,,x)) + b(u(t\,,x)) + \sigma(u(t\,,x))\dot{W}(t\,,x), \quad x \in [0,1] \text{ or } \mathbf{R},\ t > 0, \tag{2.2.1}$$

with homogeneous Dirichlet boundary conditions in the $[0,1]$ case and W a space-time white noise. The initial condition is assumed to be a continuous function on $[0,1]$ and in the real line case is assumed to be a continuous and bounded function in $\mathbf{R}$.

As explained in the introduction, equation (2.2.1) is formal and we define its solution through the mild formulation as follows.

Definition 2.2.1. *A local mild solution to equation* (2.2.1) *on* $[0,1]$ *is a jointly measurable and adapted process* $u = \{u(t\,,x)\}_{(t,x)\in\mathbf{R}_+\times[0,1]}$ *that satisfies the following integral equation*

$$\begin{aligned} u(t\,,x) = (G_t u_0)(x) &+ \int_0^t \int_0^1 g_{t-s}(x\,,y) b(u(s\,,y))\,\mathrm{d}y\,\mathrm{d}s \\ &+ \int_0^t \int_0^1 g_{t-s}(x\,,y)\sigma(u(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y), \end{aligned} \tag{2.2.2}$$

for all $t \in (0, \tau)$, where τ is a stopping time. We recall that G_t is the semigroup defined in (1.0.2) *and $g_t(x\,,y)$ is the Dirichlet heat kernel on* $[0,1]$ *defined in* (1.0.3)*. If we can take $\tau = \infty$, then the local solution is also a global one. In* $\mathbf{R}$*, the mild formulation can be written as*

$$\begin{aligned} u(t\,,x) = (P_t u_0)(x) &+ \int_0^t \int_{\mathbf{R}} p_{t-s}(x-y) b(u(s\,,y))\,\mathrm{d}y\,\mathrm{d}s \\ &+ \int_0^t \int_{\mathbf{R}} p_{t-s}(x-y)\sigma(u(s\,,y))\, W(\mathrm{d}s\,\mathrm{d}y), \end{aligned} \tag{2.2.3}$$

where P_t is the semigroup defined in (1.0.6) *and $p_t(x)$ is the heat kernel which is the $N(0,2t)$ Gaussian density function.*

We recall that $f : \mathbf{R} \to \mathbf{R}$ is a locally Lipschitz continuous if for any compact set $K \subset \mathbf{R}$ there exist a positive constant L_K such that for all $x, y \in \mathbf{R}$,

$$|f(x) - f(y)| \leqslant L_K |x-y|,$$

If the constant L_K is uniform over all compact sets in $\mathbf{R}$ we say that f is a globally Lipschitz continuous function.

The next result proves existence and uniqueness to the SHE (2.2.1) under globally Lipschitz coefficients.

Theorem 2.2.2. *Assume that σ and b are globally Lipschitz continuous functions. Then there exists a global mild solution to equation* (2.2.1) *on* $[0,1]$ *that satisfies for all $T > 0$ and $p > 0$,*

$$\sup_{t\in[0,T]} \sup_{x\in[0,1]} \mathrm{E}\left(|u(t\,,x)|^p\right) < \infty. \tag{2.2.4}$$

The solution is unique (in the sense of versions) among all mild solutions that satisfy (2.2.4) *with $p = 2$. The same statement holds for equation* (2.2.1) *in* $\mathbf{R}$ *by replacing* $[0,1]$ *by* $\mathbf{R}$.

Proof. We start with equation (2.2.1) on $[0,1]$. We first show the existence. Observe that since u_0 is bounded on $[0,1]$, the first term on the right hand side of (2.2.2) is bounded. We next consider the Picard iteration scheme. Let $u_0(t\,,x) = u_0(x)$, and then iteratively define

$$\begin{aligned} u_{n+1}(t\,,x) = (G_t u_0)(x) &+ \int_0^t \int_0^1 g_{t-s}(x\,,y) b(u_n(s\,,y))\,\mathrm{d}y\,\mathrm{d}s \\ &+ \int_0^t \int_0^1 g_{t-s}(x\,,y)\sigma(u_n(s\,,y))\, W(\mathrm{d}s\,\mathrm{d}y). \end{aligned} \tag{2.2.5}$$

Define $d_n(t\,,x) = u_{n+1}(t\,,x) - u_n(t\,,x)$ and let $H_n(t) = \sup_{(s,y)\in[0,t]\times[0,1]} \mathrm{E}\left(|d_n(s\,,y)|^p\right)$, where $p \geqslant 2$ is fixed. Note that since $L^p(\Omega)$-norms increase with p it suffices to take $p \geqslant 2$.

Observe that

$$\begin{aligned}d_n(t\,,x) = &\int_0^t\int_0^1 g_{t-s}(x\,,y)[b(u_n(s\,,y)) - b(u_{n-1}(s\,,y))]\,\mathrm{d}y\,\mathrm{d}s\\&+\int_0^t\int_0^1 g_{t-s}(x\,,y)[\sigma(u_n(s\,,y)) - \sigma(u_{n-1}(s\,,y))]\,W(\mathrm{d}s\,\mathrm{d}y).\end{aligned}$$

Using the Lipschitz continuity property of b and Hölder's inequality, we obtain

$$\begin{aligned}&\mathrm{E}\left(\left|\int_0^t\int_0^1 g_{t-s}(x\,,y)[b(u_n(s\,,y)) - b(u_{n-1}(s\,,y))]\,\mathrm{d}y\,\mathrm{d}s\right|^p\right)\\&\leqslant c_p\left(\int_0^t\int_0^1 g_{t-s}(x\,,y)\,\mathrm{d}y\,\mathrm{d}s\right)^{p-1}\int_0^t\int_0^1 g_{t-s}(x\,,y)\mathrm{E}(|d_{n-1}(s\,,y)|^p)\,\mathrm{d}y\,\mathrm{d}s\\&\leqslant c_{p,T}\int_0^t\int_0^1 g_{t-s}(x\,,y)\mathrm{E}(|d_{n-1}(s\,,y)|^p)\,\mathrm{d}y\,\mathrm{d}s,\end{aligned}$$

for some positive constants c_p and $c_{p,T}$ that may change from line to line. In the last inequality we have used the fact that by (1.0.7),

$$\int_0^t\int_0^1 g_{t-s}(x\,,y)\,\mathrm{d}y\,\mathrm{d}s \leqslant \int_0^t\int_0^1 p_{t-s}(x-y)\,\mathrm{d}y\,\mathrm{d}s < t.$$

Similarly, from the Lipschitz continuity property of σ, Burkholder-Davis-Gundy inequality (2.1.5) and Hölder's inequality, we get that

$$\begin{aligned}&\mathrm{E}\left(\left|\int_0^t\int_0^1 g_{t-s}(x\,,y)[\sigma(u_n(s\,,y)) - \sigma(u_{n-1}(s\,,y))]\,W(\mathrm{d}s\,\mathrm{d}y)\right|^p\right)\\&\leqslant c_p\left(\int_0^t\int_0^1 g^2_{t-s}(x\,,y)\,\mathrm{d}y\,\mathrm{d}s\right)^{\frac{p}{2}-1}\int_0^t\int_0^1 g^2_{t-s}(x\,,y)\mathrm{E}(|d_{n-1}(s\,,y)|^p)\,\mathrm{d}y\,\mathrm{d}s\\&\leqslant c_{p,T}\int_0^t\int_0^1 g^2_{t-s}(x\,,y)\mathrm{E}(|d_{n-1}(s\,,y)|^p)\,\mathrm{d}y\,\mathrm{d}s,\end{aligned}$$

where again in the last inequality we have used that by (1.0.7),

$$\int_0^t\int_0^1 g^2_{t-s}(x\,,y)\,\mathrm{d}y\,\mathrm{d}s \leqslant \int_0^t\int_0^1 p^2_{t-s}(x-y)\,\mathrm{d}y\,\mathrm{d}s < c\sqrt{t}.$$

Therefore, using again inequality (1.0.7), we obtain

$$\mathrm{E}(|d_n(t\,,x)|^p) \leqslant c_{p,T}\int_0^t\left(\frac{1}{|t-s|^{1/2}}+1\right)\sup_{y\in[0,1]}\mathrm{E}(|d_{n-1}(s\,,y)|^p)\,\mathrm{d}s.$$

We next apply Hölder's inequality with $\alpha \in (1,2)$ and β such that $\alpha^{-1}+\beta^{-1}=1$ to find that for all $t \in [0,T]$,

$$H_n(t) \leqslant c_{p,T}\left(\int_0^t H_{n-1}^{\beta}(s)\,\mathrm{d}s\right)^{1/\beta}.$$

Apply Gronwall's Lemma A.0.1 with $\phi_n = H_n^{\beta}$ to find that $\sum_{n=1}^{\infty} H_n(t) < \infty$. This implies that the sequence $u_n(t\,,x)$ converges in $L^p(\Omega)$ uniformly for all t and x to a process $u(t\,,x)$. Moreover, the process $u(t\,,x)$ is jointly measurable, adapted, and satisfies (2.2.2) and (2.2.4).

We finally prove uniqueness. Set $d(t\,,x) = u(t\,,x) - v(t\,,x)$, where u and v both solve (2.2.2) and satisfy the integrability condition (2.2.4) with $p = 2$. Then, proceeding as above, it follows that

$$H(t) \leqslant c_T\left(\int_0^t H^{\beta}(s)\,\mathrm{d}s\right)^{1/\beta},$$

where $H(t) = \sup_{(s,y)\in[0,t]\times[0,1]} \mathrm{E}\left(|d(s\,,y)|^2\right)$ Apply Gronwall's Lemma A.0.1 with $\phi_n = H^{\beta}$ for all $n \geqslant 1$ to find that $H(t) \equiv 0$. Thus, u and v are modifications of one another, that is, $u(t\,,x) = v(t\,,x)$ a.s. for all $t > 0$ and $x \in [0,1]$.

The proof for the equation in $\mathbf{R}$ follows exactly along the same lines by replacing $[0,1]$ by $\mathbf{R}$. □

We have the following estimate of the increments of the solution to (2.2.1).

Theorem 2.2.3. *Consider the solution to equation* (2.2.1) *on* $[0,1]$ *with* $u_0 = 0$*. Then, for all* $p \geqslant 2$ *and* $T > 0$ *there exists a constant* $c_{p,T} > 0$ *such that for all* $x, y \in [0,1]$ *and* $s, t \in [0,T]$,

$$\mathrm{E}\left(|u(t\,,x) - u(s\,,y)|^p\right) \leqslant c_{p,T}(|t-s|^{1/4} + |x-y|^{1/2})^p. \tag{2.2.6}$$

The same holds for the solution to equation (2.2.1) *in* $\mathbf{R}$ *by replacing* $[0,1]$ *by* $\mathbf{R}$.

Proof. Fix $0 \leqslant s \leqslant t \leqslant T$ and $x, y \in [0,1]$. Using the mild formulation (2.2.2), we have that

$$\mathrm{E}\left(|u(t\,,x) - u(s\,,y)|^p\right) \leqslant c_p(I_1 + I_2),$$

where

$$\mathrm{E}\left(\left|\int_0^t\int_0^1 \left(g_{t-r}(x\,,z) - g_{s-r}(y\,,z)\mathbf{1}_{[0,s)}(r)\right) b(u(r\,,z))\,\mathrm{d}z\,\mathrm{d}r\right|^p\right)$$

and

$$\mathrm{E}\left(\left|\int_0^t\int_0^1 \left(g_{t-r}(x\,,z) - g_{s-r}(y\,,z)\mathbf{1}_{[0,s)}(r)\right) \sigma(u(r\,,z))\,W(\mathrm{d}r\,\mathrm{d}z)\right|^p\right).$$

Since b is globally Lipschitz continuous it has linear growth. Hence, using Hölder's inequality and (2.2.4), we obtain that

$$I_1 \leqslant c_{p,T} \left(\int_0^t \int_0^1 |g_{t-r}(x\,,z) - g_{s-r}(y\,,z)\mathbf{1}_{[0,s)}(r)| \,\mathrm{d}z \,\mathrm{d}r \right)^p .$$

Similarly, from the linear growth property of σ, Burkholder-Davis-Gundy inequality (2.1.5) and Hölder's inequality, we get that

$$I_2 \leqslant c_{p,T} \left(\int_0^t \int_0^1 |g_{t-r}(x\,,z) - g_{s-r}(y\,,z)\mathbf{1}_{[0,s)}(r)|^2 \,\mathrm{d}z \,\mathrm{d}r \right)^{p/2} .$$

We now use the following inequality of the Dirichlet heat kernel (see (B.2.5) in [5]): there exists $c > 0$ such that for any $0 \leqslant s \leqslant t$ and $x, y \in [0,1]$,

$$\int_0^t \int_0^1 |g_{t-r}(x\,,z) - g_{s-r}(y\,,z)\mathbf{1}_{[0,s)}(r)|^2 \,\mathrm{d}z \,\mathrm{d}r \leqslant c(|t-s|^{1/2} + |x-y|).$$

Thus, using Cauchy-Schwarz inequality, we conclude the proof of (2.2.6).

The proof of (2.2.6) for equation (2.2.1) in $\mathbf{R}$ follows exactly along the same lines using the following estimate of the heat kernel (see (B.1.7) in [5]): there exists $c > 0$ such that for any $s, t \geqslant 0$ and $x, y \in \mathbf{R}$,

$$\int_0^\infty \int_{\mathbf{R}} |p_{t-r}(x-z) - p_{s-r}(y-z)|^2 \,\mathrm{d}z \,\mathrm{d}r \leqslant c(|t-s|^{1/2} + |x-y|). \tag{2.2.7}$$

This completes the desired proof. □

As a consequence of Theorem 2.2.3 and Kolmogorov's continuity criterion (Theorem A.0.2) we can derive the a.s. continuity of the solution to equation (2.2.1).

Corollary 2.2.4. *Consider the solution to equation* (2.2.1) *on* $[0,1]$ *or on* $\mathbf{R}$. *Then, there exists a modification of the solution which is continuous in* (t, x).

The next result gives the local existence and uniqueness for the SHE on $[0,1]$. We refer to Theorem 4.5.8 in [5] for a more detailed proof.

Theorem 2.2.5. *If* σ *and* b *are locally Lipschitz functions then there exists an a.s. unique local mild solution to equation* (2.2.1) *on* $[0,1]$ *which is almost surely continuous in* $(t\,, x)$.

Proof. For each $N \geqslant 1$, one can define the truncated functions

$$\sigma_N(u) := \mathbf{1}_{\{|u| \leqslant N\}}\sigma(u) + \mathbf{1}_{\{u > N\}}\sigma(N) + \mathbf{1}_{\{u < -N\}}\sigma(-N)$$

and

$$b_N(u) := \mathbf{1}_{\{|u|\leqslant N\}} b(u) + \mathbf{1}_{\{u>N\}} b(N) + \mathbf{1}_{\{u<-N\}} b(-N).$$

Since σ_N and b_N are globally Lipschitz continuous function, using to Theorem 2.2.2, for each $N \geqslant 1$, we obtain the existence of a unique global mild solution $u_N(t,x)$ to equation (2.2.1) where σ and b are replaced by σ_N and b_N, respectively. Moreover, $u_N(t,x)$ is continuous in (t,x) a.s. by Corollary 2.2.4.

We next consider the stopping time

$$\tau_N := \inf\left\{ t > 0 : \sup_{x\in[0\,,1]} |u_N(t\,,x)| > N \right\},$$

where $\inf \emptyset := \infty$. It is easy to see that for all $x \in [0,1]$ and $t \in [0\,,\tau_N)$, $u_N(t,x) = u_{N+1}(t,x)$ a.s. This implies that $\tau_N \leqslant \tau_{N+1}$ a.s. We set $\tau = \lim_{N\to\infty} \tau_N$. Then, we can define $u(t\,,x)$ for all $x \in [0,1]$ and $t < \tau$ by setting $u(t\,,x) = u_N(t\,,x)$ whenever $t < \tau_N$ for some N. This defines a local mild solution to equation (2.2.1) on $[0,1]$ which is continuous in $(t\,,x)$ a.s.

Finally, using a Gronwall-type argument similar as in the proof of Theorem 2.2.2 on the event $\{t < \tau\}$, we deduce that if we have two local solutions u and v to equation (2.2.1) on $[0,1]$ which are continuous in $(t\,,x)$ a.s., then $u(t\,,x) = v(t\,,x)$ a.s. on $\{t < \tau\}$ and $x \in [0,1]$, which gives uniqueness. The proof is completed. □

We observe that the latter stopping time argument does not work to show existence of a local mild solution to equation (2.2.1) on the real line since the solution is unbounded almost surely, that is, for any $t > 0$,

$$\sup_{x\in\mathbf{R}} |u(t\,,x)| = \infty \qquad \text{a.s.}$$

In the recent paper by Davar Khoshnevisan, Mohammud Foondun and Eulalia Nualart [7] the authors construct a local solution to equation (2.2.1) on the real line, under the assumption that σ is a bounded and globally Lipschitz continuous function and b is locally Lipschitz continuous, nonnegative, and nondecreasing on $(0,+\infty)$. The construction also uses both a truncation and stopping time argument by exploiting the fact that b is nondecreasing.

Consider equation (2.2.1) on the real line where b is replaced by $b^{(n)} = b \wedge n$ for every $n \in \mathbb{N}$. Because $b^{(n)}$ is globally Lipschitz continuous, by Theorem 2.2.2 this equation has a unique solution that we denote by $u_{b^{(n)}}(t\,,x)$ for every $n \geqslant 1$. The monotonicity of b implies that $b^{(n)} \leqslant b^{(m)}$ when $n \leqslant m$. Therefore, by a comparison theorem (see [17] and [11]), we get that $u_{b^{(n)}} \leqslant u_{b^{(m)}}$ whenever $n \leqslant m$. It follows that the random field

$$u = \lim_{n\to\infty} u_{b^{(n)}}$$

exists and has lower-semicontinuous sample functions.

$$b^{(n)}\left(u_{b^{(n)}}(t\,,x)\right) \leqslant b^{(m)}\left(u_{b^{(m)}}(t\,,x)\right) \qquad \text{whenever } n \leqslant m,$$

off a single null set that does not depend on $(b\,,n\,,m)$. Since

$$b^{(n)}(x) = \frac{b(x)+n-|b(x)-n|}{2},$$

it follows that

$$\lim_{n\to\infty} b^{(n)}\left(u_{b^{(n)}}(t\,,x)\right) = b(u(t\,,x)) \qquad \text{for all } t>0 \text{ and } x\in\mathbf{R}, \tag{2.2.8}$$

again off a single null set. Therefore, the monotone convergence theorem yields

$$\lim_{n\to\infty}\int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)b^{(n)}(u^{(n)}(s\,,y))\,\mathrm{d}s\,\mathrm{d}y = \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)b(u(s\,,y))\,\mathrm{d}s\,\mathrm{d}y,$$

where $b(\infty)=\sup b$.

Next, let us consider the $[0\,,\infty]$-valued random variable

$$\tau = \inf\left\{t>0:\ u(t\,,x)=\infty \quad \text{for some } x\in\mathbf{R}\right\}. \tag{2.2.9}$$

Then one can show that τ is a stopping time and since σ is a bounded and continuous function, we get that

$$\begin{aligned}&\lim_{n\to\infty}\mathrm{E}\left(\left|\int_{(0,t\wedge\tau)\times\mathbf{R}} p_{t-s}(y-x)[\sigma(u^{(n)}(s\,,y))-\sigma(u(s\,,y))]\,W(\mathrm{d}s\,\mathrm{d}y)\right|^2\right)\\ &=\mathrm{E}\left(\int_{(0,t\wedge\tau)\times\mathbf{R}}\left[p_{(t\wedge\tau)-s}(y-x)\right]^2\lim_{n\to\infty}[\sigma(u^{(n)}(s\,,y))-\sigma(u(s\,,y))]^2\mathrm{d}s\,\mathrm{d}y\right)=0.\end{aligned}$$

Therefore, if $\tau>0$ then u is a local solution to equation (2.2.1).

Chapter 3

Blowup of the stochastic heat equation

In Chapter 2 we defined the concept of local mild solution to the stochastic heat equation, which means that the solution is only defined up to a stopping time τ, see Definition 2.2.1. Then in Theorem 2.2.5, we show existence and uniqueness of the local solution for the stochastic heat equation on $[0, 1]$, when the coefficients are locally Lipschitz functions. Moreover, this local solution is a.s. continuous for all $(t\,,x)$. Observe that by the construction of τ in the proof of Theorem 2.2.5, this stopping time is the blowup time of the solution, since

$$\tau = \inf\left\{ t > 0 : \sup_{x\in[0\,,1]} |u(t\,,x)| = \infty \right\}$$

and $\lim_{t\uparrow\tau} \sup_{x\in[0\,,1]} |u(t\,,x)| = \infty$ a.s. Then, we say that the solution blowsup in finite time almost surely or with positive probability if $\tau < \infty$ a.s or $\mathrm{P}(\tau < \infty) > 0$, respectively. For the stochastic heat equation in $\mathbf{R}$, the argument after the proof of Theorem 2.2.5 shows how to construct local solutions up to its blowup time τ defined as

$$\tau = \inf\left\{ t > 0 : u(t\,,x) = \infty \quad \text{for some } x \in \mathbf{R} \right\}.$$

In this chapter we address the question of blowup in finite time almost surely for the solution to the stochastic heat equation when the diffusion coefficient is constant and that the drift satisfies the following condition.

Assumption 3.0.1. *The function $b : \mathbf{R} \to \mathbf{R}_+$ is locally Lipschitz continuous, nonnegative and nondecreasing on $(0, \infty)$.*

An example of function satisfying this assumption is $b(u) = u^\alpha$ with $\alpha > 1$.

We first recall some results on the blowup in finite time for stochastic differential equations.

3.1 Blowup of stochastic differential equations

Suppose that b satisfies Assumption 3.0.1 and consider the following ordinary differential equation

$$\dot{x} = b(x), \qquad x(0) = a \geqslant 0.$$

Observe that the integral formulation of this equation is given by

$$x(t) = a + \int_0^t b(x(s))\,\mathrm{d}s, \quad t \geqslant 0.$$

This equation admits a unique solution up to its blowup time defined as

$$\tau := \inf\{t > 0 : |x(t)| = \infty\}.$$

Then we say that the solution blows up in finite time if $\tau < \infty$. One can show that this blowup time is equal to the following

$$\tau = \int_a^\infty \frac{1}{b(s)}\,\mathrm{d}s. \tag{3.1.1}$$

This can be seen rewritting the equation as $t = \int_0^t \frac{1}{b(x(s))}\,\mathrm{d}x(s)$. After a change of variable, we obtain

$$t = \int_{x(0)}^{x(t)} \frac{1}{b(s)}\,\mathrm{d}s,$$

which implies (3.1.1).

We next introduce the *Osgood condition* which states that the blowup time in (3.1.1) is finite.

Assumption 3.1.1.

$$\int_1^\infty \frac{1}{b(s)}\,\mathrm{d}s < \infty, \tag{3.1.2}$$

where $1/0 = \infty$.

We now consider the following stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma \mathrm{d}B_t, \quad X_0 = a \geqslant 0, \tag{3.1.3}$$

where $(B_t)_{t\geqslant 0}$ is a Brownian motion and $\sigma > 0$. This can be written as the following integral equation

$$X_t = a + \int_0^t b(X_s)\,\mathrm{d}s + \sigma B_t.$$

If b is locally Lipschitz continuous, the same argument of the proof of Theorem 2.2.5 shows that this equation admits a unique solution up to its blowup time

$$\tau := \inf\{t > 0 : |X_t| = \infty\}. \tag{3.1.4}$$

The next result known as Feller's test, gives necessary and sufficient conditions for the blowup of this process, see Proposition 5.32 in [13] for the proof.

Proposition 3.1.2. *Suppose that* $b : \mathbf{R} \to \mathbf{R}$ *is a continuous function. Let*

$$\rho(x) = \int_1^x \exp\left(-2\sigma^{-2}\int_1^s b(r)\,\mathrm{d}r\right)\mathrm{d}s \quad \textit{and} \quad v(x) = 2\sigma^{-2}\int_1^x \frac{\rho(x) - \rho(y)}{\rho'(y)}\,\mathrm{d}y.$$

The blowup time of the solution to (3.1.3) *defined in* (3.1.4) *is finite a.s. if and only if one of the following conditions holds:*

1. $v(\infty) < \infty$ *and* $v(-\infty) < \infty$,
2. $v(\infty) < \infty$ *and* $\rho(-\infty) = -\infty$,
3. $v(-\infty) < \infty$ *and* $\rho(\infty) = \infty$.

The next result (see Theorem 5.2 in [16]) shows how the Osgood condition and the conditions in Feller's test are related.

Proposition 3.1.3. *Assume that* b *is locally Lipschitz continuous and nonngative. Then*

(i) $\rho(-\infty) = -\infty$;

(ii) *if* b *is nondecreasing on* $(0, +\infty)$, *then* $\int_1^\infty \frac{1}{b(s)}\,\mathrm{d}s < \infty$ *if and only if* $v(\infty) < \infty$;

(iii) *if* b *is convex on* $(0, +\infty)$ *and* $\int_1^\infty \frac{1}{b(s)}\,\mathrm{d}s < \infty$, *then* $v(\infty) < \infty$.

Proof. We first prove (i). Observe that since b is nonnegative,

$$\begin{aligned}\rho(-\infty) &= -\int_{-\infty}^1 \exp\left(2\sigma^{-2}\int_s^1 b(r)\,\mathrm{d}r\right)\mathrm{d}s \\ &\leqslant -\int_{-\infty}^1 \exp(0)\,\mathrm{d}s = -\infty.\end{aligned}$$

We next prove (ii). Suppose that $\int_1^\infty \frac{1}{b(s)}\,\mathrm{d}s < \infty$. Then, since b is nonnegative and nondecreasing on $(0,+\infty)$,

$$\begin{aligned}
v(\infty) &= 2\sigma^{-2}\int_1^\infty\int_y^\infty \exp\left(-2\sigma^{-2}\int_1^s b(r)\,\mathrm{d}r\right)\exp\left(2\sigma^{-2}\int_1^y b(t)\,\mathrm{d}t\right)\mathrm{d}s\,\mathrm{d}y \\
&\leqslant 2\sigma^{-2}\int_1^\infty\int_y^\infty \frac{b(s)}{b(y)}\exp\left(-2\sigma^{-2}\int_1^s b(r)\,\mathrm{d}r\right)\exp\left(2\sigma^{-2}\int_1^y b(t)\,\mathrm{d}t\right)\mathrm{d}s\,\mathrm{d}y \\
&= 2\sigma^{-2}\int_1^\infty \frac{1}{b(y)}\left(\int_y^\infty b(s)\exp\left(-2\sigma^{-2}\int_y^s b(r)\,\mathrm{d}r\right)\mathrm{d}s\right)\mathrm{d}y \\
&= \int_1^\infty \frac{1}{b(y)}\left(1-\exp\left(-2\sigma^{-2}\int_y^\infty b(r)\,\mathrm{d}r\right)\right)\mathrm{d}y \\
&\leqslant \int_1^\infty \frac{1}{b(y)}\,\mathrm{d}y.
\end{aligned}$$

Conversely, assume that $v(\infty) < \infty$. First note that

$$\begin{aligned}
\int_1^\infty \frac{1}{b(s)}\exp\left(-2\sigma^{-2}\int_1^s b(r)\,\mathrm{d}r\right)\mathrm{d}s &\leqslant \int_1^\infty \frac{1}{b(s)}\exp\left(-2\sigma^{-2}b(1)(s-1)\right)\mathrm{d}s \\
&\leqslant \frac{1}{b(1)}\int_1^\infty \exp\left(-2\sigma^{-2}b(1)(s-1)\right)\mathrm{d}s < \infty.
\end{aligned}$$

On the other hand, Fubini's theorem yields

$$\begin{aligned}
v(\infty) &\geqslant 2\sigma^{-2}\int_1^\infty\int_1^s \frac{b(y)}{b(s)}\exp\left(-2\sigma^{-2}\int_1^s b(r)\,\mathrm{d}r\right)\exp\left(2\sigma^{-2}\int_1^y b(t)\,\mathrm{d}t\right)\mathrm{d}y\,\mathrm{d}s \\
&= \int_1^\infty \frac{1}{b(s)}\exp\left(-2\sigma^{-2}\int_1^s b(r)\,\mathrm{d}r\right)\left(\exp\left(2\sigma^{-2}\int_1^s b(r)\,\mathrm{d}r - 1\right)\right)\mathrm{d}s \\
&= \int_1^\infty \frac{1}{b(s)}\left(1-\exp\left(-2\sigma^{-2}\int_1^s b(r)\,\mathrm{d}r\right)\right)\mathrm{d}s.
\end{aligned}$$

Hence, $\int_1^\infty \frac{1}{b(s)}\,\mathrm{d}s < \infty$, and the proof of (ii) is completed.

We finally show (iii). The fact that b is convex on $(0,+\infty)$ and $\int_1^\infty \frac{1}{b(s)}\,\mathrm{d}s < \infty$ imply that b is nondecreasing on (x_0,∞) for some $1 < x_0 < \infty$. We then write

$$\begin{aligned}
v(\infty) &= 2\sigma^{-2}\int_1^\infty\int_y^\infty \exp\left(-2\sigma^{-2}\int_y^s b(r)\,\mathrm{d}r\right)\mathrm{d}s\,\mathrm{d}y \\
&= \int_1^{x_0}\int_y^{x_0}(\cdots)\,\mathrm{d}s\,\mathrm{d}y + \int_1^{x_0}\int_{x_0}^\infty(\cdots)\,\mathrm{d}s\,\mathrm{d}y + \int_{x_0}^\infty\int_y^\infty(\cdots)\,\mathrm{d}s\,\mathrm{d}y.
\end{aligned}$$

Since b is continuous, the first integral in the above display is clearly finite. For the second integral, we use Cauchy-Schwarz inequality to see that

$$
\begin{aligned}
&2\sigma^{-2}\int_1^{x_0}\int_{x_0}^{\infty}\exp\left(-2\sigma^{-2}\int_y^s b(r)\,\mathrm{d}r\right)\mathrm{d}s\,\mathrm{d}y\\
&\leqslant 2\sigma^{-2}(x_0-1)\int_{x_0}^{\infty}\exp\left(-2\sigma^{-2}\int_{x_0}^s b(r)\,\mathrm{d}r\right)\mathrm{d}s\,\mathrm{d}y\\
&\leqslant 2\sigma^{-2}(x_0-1)\left(\int_{x_0}^{\infty}\frac{1}{b(s)}\,\mathrm{d}s\right)^{1/2}\left(\int_{x_0}^{\infty}b(s)\exp\left(-4\sigma^{-2}\int_{x_0}^s b(r)\,\mathrm{d}r\right)\,\mathrm{d}s\right)^{1/2}\\
&=\sigma^{-1}(x_0-1)\left(\int_{x_0}^{\infty}\frac{1}{b(s)}\,\mathrm{d}s\right)^{1/2}\left(1-\exp\left(-4\sigma^{-2}\int_{x_0}^{\infty}b(r)\,\mathrm{d}r\right)\right)^{1/2},
\end{aligned}
$$

which is finite as $\int_1^{\infty}\frac{1}{b(s)}\,\mathrm{d}s<\infty$. Finally, since b is nondecreasing on (x_0,∞), proceeding exactly as in part (ii), we see that the third integral is also finite. This completes the proof. □

Feller's test is proved using Itô's formula, thus it does not extend if we replace Brownian motion by a process which is not a semi-martingale. In order to overcome this fact, Jorge A. León and José Villa [16] extend the ordinary differential equation explosion time to functions that satisfy the following assumption.

Assumption 3.1.4. *$g:[0\,,\infty)\to\mathbf{R}$ is a continuous function such that*

$$\limsup_{t\to\infty}\inf_{0\leqslant h\leqslant 1} g(t+h)=\infty.$$

Then we have the following result obtained in Theorem 3.1 of [16].

Proposition 3.1.5. *Let $a\geqslant 0$ and suppose that Assumptions 3.0.1 and 3.1.4 hold. Then the solution to the integral equation*

$$X_t=a+\int_0^t b(X_s)\,\mathrm{d}s+g(t) \tag{3.1.5}$$

blows up in finite time if and only if the function b satisfies the Osgood condition.

Proof. Suppose that the solution blows up at finite time τ. Since g is continuous, we can set

$$M:=\sup_{s\leqslant\tau}|g(s)|.$$

Let $t\leqslant\tau$. Upon noting that b is nonnegative, (3.1.5) gives

$$X_t\leqslant a+M+\int_0^t b(X_s)\,\mathrm{d}s.$$

The nonnegativity of b together with the continuity of g imply that X_t can only blowup to positive infinity. Let $Y_t = a + M + 1 + \int_0^t b(Y_s)\,\mathrm{d}s$. Then by Lemma A.0.3, we have $X_t \leqslant Y_t$ on $[0\,,\tau]$. But since X_t blows up at time τ, Y_t should also blowup by time τ. This means that b satisfies the Osgood condition.

We now suppose that X_t does not blowup in finite time and that the Osgood condition holds. Let $\{t_n\}_{n=1}^{\infty}$ be some sequence which tends to infinity. Since b is nonnegative and nondecreasing, we get that

$$\begin{aligned} X_{t+t_n} &\geqslant a + \int_{t_n}^{t+t_n} b(X_s)\,\mathrm{d}s + g(t+t_n) \\ &\geqslant a + \int_0^t b(X_{s+t_n})\,\mathrm{d}s + g(t+t_n) \\ &\geqslant a + \inf_{0\leqslant h\leqslant 1} g(h+t_n) + \int_0^t b(X_{s+t_n})\,\mathrm{d}s, \end{aligned}$$

where the last inequality holds whenever $0 \leqslant t \leqslant 1$. This means that $X_{t+t_n} \geqslant Z_t$ where

$$Z_t = \frac{1}{2}\left(a + \inf_{0\leqslant h\leqslant 1} g(h+t_n)\right) + \int_0^t b(Z_s)\,\mathrm{d}s.$$

Since we are assuming that X_t does not blowup in finite time, the blowup time of Z_t has to be greater than 1, which implies that

$$\int_{\frac{1}{2}(a+\inf_{0\leqslant h\leqslant 1} g(h+t_n))}^{\infty} \frac{1}{b(s)}\,\mathrm{d}s > 1.$$

But from Assumption 3.1.4, we can find a sequence $t_n \to \infty$ such that

$$\frac{1}{2}(a + \inf_{0\leqslant h\leqslant 1} g(h+t_n)) \to \infty.$$

This contradicts the Osgood condition and the proof is complete. □

One can show that almost surely Brownian motion B_t satisfies Assumption 3.1.4. We will prove this fact for a more general class of processes in Section 3.3. Hence, the Osgood condition is a necessary and sufficient condition for blowup of the solution to equation (3.1.3). As showed in [16], one can replace the Brownian motion by a more general class of processes including the bifractional Brownian motion for which Feller's test for explosions is not applicable.

The bifractional Brownian motion was introduced by Christian Houdré and José Villa in [12] and is a generalization of the fractional Brownian motion. It is defined as the centered Gaussian process $B^{H,K} = (B_t^{H,K}, t \geqslant 0)$ with covariance

$$R^{H,K}(t\,,s) = 2^{-K}\left((t^{2H} + s^{2H})^K - |t-s|^{2HK}\right),$$

where $H \in (0,1)$ and $K \in (0,1]$. Note that if $K = 1$, then $B^{H,1}$ is a fractional Brownian motion with Hurst parameter H and $B^{\frac{1}{2},1}$ is a Brownian motion.

Set

$$\psi_{H,K}(t) := t^{HK}\sqrt{2\log\log t}, \quad t > e.$$

The bifractional Brownian motion satisfies the following law of iterated logarithm; see for instance Lemma 4.1 of [16] for the proof.

Lemma 3.1.6. *Almost surely,*

$$\limsup_{t\to\infty} \frac{B_t^{H,K}}{\psi_{H,K}(t)} = 1.$$

As a consequence of Proposition 3.1.5 and Lemma 3.1.6, we have the following result.

Proposition 3.1.7. *Let $a \geqslant 0$ and $\sigma > 0$, and suppose that Assumption 3.0.1 holds. Then, if the Osgood condition holds, the solution to the stochastic integral equation*

$$X_t = a + \int_0^t b(X_s)\,\mathrm{d}s + \sigma B_t^{H,K}, \qquad t \geqslant 0, \tag{3.1.6}$$

blows up in finite time almost surely. Moreover, if the solution blows up in finite time with positive probability then the Osgood condition holds.

Proposition 3.1.7 will be a consequence of the results in Sections 3.2 and 3.3 since similar results will be proved for the stochastic heat equation, and we will see that the same arguments apply to the solution to equation (3.1.6).

3.2 Blowup of the stochastic heat equation on $[0, 1]$

Consider the stochastic heat equation on $[0, 1]$,

$$\partial_t(u(t\,,x)) = \partial^2_{xx}(u(t\,,x)) + b(u(t\,,x)) + \sigma\dot{W}(t\,,x), \quad x \in [0,1],\ t > 0, \tag{3.2.1}$$

with homogeneous Dirichlet boundary conditions, $\sigma > 0$, and initial condition $u_0(x)$ a nonnegative and continuous function on $[0, 1]$.

The following theorem, due to Pablo Groisman and Julian Fernández Bonder [6], shows the sufficiency of the Osgood condition for the solution to (3.2.1) to blowup in finite time a.s. with the additional condition that b is convex instead of nondecreasing.

Theorem 3.2.1. *Suppose that $b : \mathbf{R} \to \mathbf{R}_+$ is locally Lipschitz continuous, nonnegative, convex, and satisfies the Osgood condition. Then, the solution to* (3.2.1) *blows up in finite time a.s.*

Proof. We consider the process

$$F_t = \int_0^1 u(t\,, y)\phi(y)\,\mathrm{d}y,$$

where $\phi(y) = \frac{\pi}{\sqrt{8}}\Phi_1(y)$ and Φ_1 is the principle eigenfunction of the Dirichlet heat kernel on $[0, 1]$. See (1.0.3). We multiply equation (3.2.1) by ϕ and we integrate to obtain that

$$F_t - F_0 = -\lambda_1 \int_0^t F_s\,\mathrm{d}s + \int_0^t \int_0^1 \phi(y)b(u(s\,, y))\,\mathrm{d}y\ \mathrm{d}s + \sigma \int_0^t \int_0^1 \phi(y)\,W(\mathrm{d}s\,\mathrm{d}y).$$

Since b is convex, Jensen's inequality implies that

$$\int_0^1 \phi(y)b(u(s\,, y))\,\mathrm{d}y \geqslant b\left(\int_0^1 \phi(y)u(s\,, y)\,\mathrm{d}y\right) = b(F_s).$$

Moreover, since $\int_{-\infty}^{\infty} \phi^2 = \pi^2/8$, it is easy to see that

$$B_t = \frac{\sqrt{8}}{\pi} \int_0^t \int_0^1 \phi(y)W(\mathrm{d}s\,\mathrm{d}y),$$

is a standard Brownian motion. Therefore, we obtain that

$$\mathrm{d}F_t \geqslant (-\lambda_1 F_t + b(F_t))\mathrm{d}t + \sigma\frac{\pi}{\sqrt{8}}\mathrm{d}B_t.$$

Define z_t to be the one-dimensional process that satisfies the SDE given by

$$\mathrm{d}z_t = (-\lambda_1 z_t + b(z_t))\mathrm{d}t + \sigma\frac{\pi}{\sqrt{8}}\mathrm{d}B_t,$$

with initial condition $z_0 = F_0$. Set $e_t = F_t - z_t$. Then e_t verifies the deterministic differential inequality

$$\mathrm{d}e_t \geqslant \left(-\lambda_1 e_t + \frac{b(F_t) - b(z_t)}{F_t - z_t}e_t\right)\mathrm{d}t,$$

with $e_0 = 0$. It is easy to check that $e_t \geqslant 0$ upto its blowup time. Therefore, $F_t \geqslant z_t$ up to the blowup time of z. We now apply Feller's test (Proposition 3.1.2) to the SDE satisfied by z_t. On one hand, since b is nonnegative, similarly as in the proof of Proposition 3.1.3(i), we get that

$$\begin{aligned}\rho(-\infty) &= -\int_{-\infty}^1 \exp\left(c\int_s^1 (-\lambda_1 r + b(r))\,\mathrm{d}r\right)\mathrm{d}s\\ &\leqslant -\int_{-\infty}^1 \exp\left(-c\lambda_1/2\right)\mathrm{d}s = -\infty,\end{aligned}$$

where $c = 16\sigma^{-2}\pi^{-2}$. On the other hand, the fact that b is convex and $\int_1^\infty \frac{1}{b(s)}\,\mathrm{d}s < \infty$ implies that there exists $x_0 \in (1, \infty)$ such that $b(r) \geqslant 2\lambda_1 r$ for all $r \geqslant x_0$. In particular, $b(r) - \lambda_1 r \geqslant \frac{b(r)}{2}$ for all $r \geqslant x_0$. Then, using this fact, we can proceed exactly along the same lines as in the proof of Proposition 3.1.3(iii) and we get that

$$v(\infty) = c\int_1^\infty \int_y^\infty \exp\left(-c\int_y^s (b(r) - \lambda_1 r)\,\mathrm{d}r\right)\mathrm{d}s\,\mathrm{d}y < \infty.$$

Hence, Proposition 3.1.2 implies that the blowup time of z is finite a.s., which concludes the proof. □

The next result proves the converse of Theorem 3.2.1 and was obtained by Mohammud Foondun and Eulalia Nualart [8]. The proof follows that of Proposition 3.1.5 but it relies on the fact that the stochastic term in the random field formulation is continuous and that the equation itself is defined on an interval.

Theorem 3.2.2. *Suppose that Assumption 3.0.1 holds. Then, if the solution to* (3.2.1) *blows up in finite time with positive probability then b satisfies the Osgood condition.*

Proof. Set

$$\tau := \inf\{t > 0 : \sup_{x\in[0\,,1]} |u(t\,,x)| = \infty\}.$$

Since the solution blows up in finite time with positive probability, we can find a set Ω that satisfies $\mathrm{P}(\Omega) > 0$ such that for any $\omega \in \Omega$, we have $\tau(\omega) < \infty$. We now fix such an ω but for the sake of notational convenience, we will not indicate the dependence on ω in what follows. We recall the mild formulation

$$\begin{aligned} u(t\,,x) = (G_t u_0)(x) + \int_0^t \int_0^1 g_{t-s}(x\,,y) b(u(s\,,y))\,\mathrm{d}y\,\mathrm{d}s \\ + \sigma\int_0^t \int_0^1 g_{t-s}(x\,,y)\,W(\mathrm{d}s\,\mathrm{d}y). \end{aligned}$$

The third term in the above display is almost surely continuous. Therefore the following quantity is finite almost surely:

$$M := \sup_{x\in[0\,,1]\,t\in(0,\tau)} \left|\int_0^t \int_0^1 g_{t-s}(x\,,y)\,W(\mathrm{d}s\,\mathrm{d}y)\right|.$$

Moreover, since b and u_0 are nonnegative,

$$u(t\,,x) \geqslant \sigma\int_0^t \int_0^1 g_{t-s}(x\,,y)\,W(\mathrm{d}s\,\mathrm{d}y).$$

This means that

$$\inf_{t\in[0\,,\tau],x\in[0\,,1]} u(t\,,x) \geqslant -\sigma M.$$

Since u_0 is bounded, we have

$$|(G_t u_0)(x)| \leqslant a,$$

for some positive constant a. Denote $\mathcal{A} := \{s \in (0\,,t), y \in (0\,,1); -\sigma M \leqslant u(s,y) \leqslant 0\}$ and $\mathcal{B} := \{s \in (0\,,t), y \in (0\,,1); u(s,y) > 0\}$ and write

$$\begin{aligned}\int_0^t \int_0^1 g_{t-s}(x\,,y) b(u(s\,,y))\,\mathrm{d}y\,\mathrm{d}s &= \iint_{\mathcal{A}} g_{t-s}(x\,,y) b(u(s\,,y))\,\mathrm{d}y\,\mathrm{d}s \\ &+ \iint_{\mathcal{B}} g_{t-s}(x\,,y) b(u(s\,,y))\,\mathrm{d}y\,\mathrm{d}s \\ &:= I_1 + I_2.\end{aligned}$$

Since we are assuming that b is nonnegative and nondecreasing on $(0, \infty)$, this gives us

$$I_2 \leqslant \int_0^t b(Y_s)\,\mathrm{d}s,$$

where $Y_t := \sup_{x\in[0\,,1]} u(t\,,x)$. Since b is assumed to be continuous, we have $I_1 \leqslant K$, where K is an almost sure finite quantity. Putting all these estimates together, we obtain

$$Y_t \leqslant a + \sigma M + K + \int_0^t b(Y_s)\,\mathrm{d}s.$$

We can now proceed as in the proof of Proposition 3.1.5 to conclude the proof. □

3.3 Blowup of the stochastic heat equation on R

Consider the stochastic heat equation

$$\partial_t(u(t\,,x)) = \partial^2_{xx}(u(t\,,x)) + b(u(t\,,x)) + \sigma \dot{W}(t\,,x), \quad x \in \mathbf{R},\ t > 0, \tag{3.3.1}$$

where $\sigma > 0$ and the initial condition $u_0(x)$ a nonnegative and bounded function.

We recall that, when it exists, the mild solution $u = \{u(t\,,x)\}_{(t,x)\in\mathbf{R}_+\times\mathbf{R}}$ satisfies the following integral equation

$$u(t\,,x) = \int_{\mathbf{R}} p_t(x-y) u_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbf{R}} p_{t-s}(x-y) b(u(s\,,y))\,\mathrm{d}y\,\mathrm{d}s + \sigma g(t\,,x), \tag{3.3.2}$$

where

$$g(t\,,x) := \int_0^t \int_{\mathbf{R}} p_{t-s}(x-y) W(\mathrm{d}s\,\mathrm{d}y).$$

Observe that $g(t\,,x)$ is the solution to the stochastic heat equation (3.3.1) with zero drift, zero initial condition, and $\sigma = 1$. It is shown in [15] by Pedro Lei and David Nualart that for a fixed $x \in \mathbf{R}$, the process $(g(t\,,x), t \geqslant 0)$ is a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$ multiplied by a constant. In fact, one can easily check that the covariance of $g(t\,,x)$ is given by

$$\mathrm{E}(g(t\,,x)g(s\,,x)) = \frac{1}{\sqrt{2\pi}}(\sqrt{t+s} - \sqrt{|t-s|}).$$

The following estimates on the increments of $g(t\,,x)$ are a direct consequence of Burkholder-Davis-Gundy inequality (2.1.5) and the estimate (2.2.7).

Lemma 3.3.1. *For any $p \geqslant 2$ there exists $c_p > 0$ such that for any $s, t \geqslant 0$ and $x, y \in \mathbf{R}$,*

$$\mathrm{E}\left(|g(t\,,x) - g(s\,,y)|^p\right) \leqslant c_p(|t-s|^{1/4} + |x-y|^{1/2})^p.$$

As a consequence of Lemma 3.3.1 and Lemma 4.5 in Robert Dalang, Davar Khoshnevisan and Eulalia Nualart [3] we have the following estimate. Lemma 4.5 in [3] is based on an improvement of the classical Garsia's lemma obtained in Proposition A.1 of [3] for general metric spaces, and in Lemma 4.5 this lemma is applied to the parabolic metric $|t-s|^{1/4} + |x-y|^{1/2}$.

Proposition 3.3.2. *For all $p \geqslant 2$ there exists a constant $A_p > 0$ such that for any integer $n \geqslant 1$,*

$$\mathrm{E}\left(\sup_{s,t\in[n,n+2],x,y\in[0,1]} |g(t\,,x) - g(s\,,y)|^p\right) \leqslant A_p 2^{p/4}.$$

Proof. Using Lemma 4.5 in [3] and Lemma 3.3.1, we get that for all $p \geqslant 2$, there exists a constant $A_p > 0$ such that for any $\epsilon > 0$,

$$\mathrm{E}\left(\sup_{|t-s|^{1/4}+|x-y|^{1/2}\leqslant\epsilon} |g(t\,,x) - g(s\,,y)|^p\right) \leqslant A_p \epsilon^p.$$

Then, using this inequality with $\epsilon = 22^{1/4}$ implies the desired result. □

We can now use Proposition 3.3.2 to get the following almost sure result.

Proposition 3.3.3. *Almost surely,*

$$\sup_{s,t\in[n,n+2],x,y\in[0,1]} \frac{|g(t\,,x) - g(s\,,y)|}{\psi_{\frac{1}{2},\frac{1}{2}}(n)} \longrightarrow 0, \qquad \textit{as } n \to \infty.$$

Proof. Proposition 3.3.2 implies that for $p > 4$,

$$\mathrm{E}\left(\sum_{n=1}^{\infty} \sup_{s,t\in[n,n+2],x,y\in[0,1]} \frac{|g(t\,,x) - g(s\,,y)|^p}{\psi_{\frac{1}{2},\frac{1}{2}}(n)^p}\right) \leqslant \sum_{n=1}^{\infty} \frac{A_p 2^{p/4}}{\psi_{\frac{1}{2},\frac{1}{2}}(n)^p} < \infty,$$

which gives us the desired result. □

As a consequence of Proposition 3.3.3, we get the following estimate.

Proposition 3.3.4. *Almost surely, there exists a sequence $t_n \to \infty$ such that*

$$\inf_{h\in[0,1],x\in[0,1]} g(t_n + h\,,x) \to \infty \quad \text{as} \quad n \to \infty.$$

Proof. Fix $x_0 \in [0,1]$. Choose ω such that both Proposition 3.3.3 and Lemma 3.1.6 hold. We now write

$$\begin{aligned}
\inf_{h\in[0,1],x\in[0,1]} g(t+h\,,x) &= g(t,x_0) + \inf_{h\in[0,1],x\in[0,1]} (g(t+h\,,x) - g(t\,,x_0)) \\
&\geqslant g(t,x_0) + \inf_{h\in[0,1],x\in[0,1]} (-|g(t+h\,,x) - g(t\,,x_0)|) \\
&\geqslant \frac{g(t\,,x_0)}{\psi_{\frac{1}{2},\frac{1}{2}}(t)}\psi_{\frac{1}{2},\frac{1}{2}}(t) - \sup_{h\in[0,1],x\in[0,1]} \frac{|g(t+h\,,x) - g(t\,,x_0)|}{\psi_{\frac{1}{2},\frac{1}{2}}([t])}\psi_{\frac{1}{2},\frac{1}{2}}([t]).
\end{aligned}$$

We use Proposition 3.3.3 and Lemma 3.1.6 to choose an appropriate sequence t_n and finish the proof. □

We are now ready to show the main result of this Section, which shows that if the Osgood condition holds, then there is no global solution to equation (3.3.1) and is due to Mohammud Foondun and Eulalia Nualart [8].

Theorem 3.3.5. *Suppose that Assumption 3.0.1 holds. Then, if b satisfies the Osgood condition, then almost surely, there is no global solution to equation* (3.3.1).

Proof. Let $\{t_n\}$ be a sequence of positive numbers which we are going to choose later. From the mild formulation of the solution and the nonnegativity of the function b, we obtain

$$\begin{aligned}
u(t+t_n\,,x) &= \int_{\mathbf{R}} p_{t+t_n}(x-y)u_0(y)\,\mathrm{d}y + \int_0^{t+t_n}\int_{\mathbf{R}} p_{t+t_n-s}(x-y)b(u(s\,,y))\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \sigma\int_0^{t+t_n}\int_{\mathbf{R}} p_{t+t_n-s}(x-y)W(\mathrm{d}s\,\mathrm{d}y) \\
&\geqslant \int_{\mathbf{R}} p_{t+t_n}(x-y)u_0(y)\,\mathrm{d}y + \int_0^{t}\int_{\mathbf{R}} p_{t-s}(x-y)b(u(s+t_n\,,y))\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \sigma\int_0^{t+t_n}\int_{\mathbf{R}} p_{t+t_n-s}(x-y)W(\mathrm{d}s\,\mathrm{d}y).
\end{aligned}$$

Let $0 \leqslant t \leqslant 1$ and $x \in (0,1)$. Recall that

$$g(t+t_n\,,x) := \int_0^{t+t_n} \int_{\mathbf{R}} p_{t+t_n-s}(x-y) W(\mathrm{d}s\,\mathrm{d}y).$$

By Proposition 3.3.4, we can find a sequence $t_n \to \infty$ so that the above quantity is positive for $0 \leqslant t \leqslant 1$ and $x \in (0,1)$. Therefore $u(t+t_n\,,x)$ is also positive for any $x \in (0\,,1)$ and any $0 \leqslant t \leqslant 1$. We now use the fact that b is nondecreasing on $(0\,,\infty)$ to bound the second term as follows. For fixed $x \in (0,\, 1)$,

$$\begin{aligned}\int_0^t \int_{\mathbf{R}} & p_{t-s}(x-y) b(u(s+t_n\,,y))\,\mathrm{d}y\,\mathrm{d}s \\ &\geqslant \int_0^t b\left(\inf_{y\in(0\,,1)} u(s+t_n\,,y)\right) \int_0^1 p_{t-s}(x-y)\,\mathrm{d}y\,\mathrm{d}s \\ &\geqslant \int_0^t b\left(\inf_{y\in(0\,,1)} u(s+t_n\,,y)\right)\,\mathrm{d}s,\end{aligned}$$

where we have used that fact that $p_t(x-y) \geqslant \frac{c}{t^{1/2}}$ whenever $|x-y| \leqslant t^{1/2}$. We now set $Y_t := \inf_{y\in(0\,,1)} u(t+t_n\,,y)$ and combine the above estimates to obtain

$$Y_t \geqslant \inf_{0\leqslant h\leqslant 1, x\in(0\,,1)} \left\{ \int_{\mathbf{R}} p_{h+t_n}(x-y) u_0(y)\,\mathrm{d}y + \sigma g(h+t_n\,,x)\right\} + \int_0^t b(Y_s)\,\mathrm{d}s.$$

We now choose ω as in Proposition 3.3.4, and we can therefore find a sequence $t_n \to \infty$ such that $\inf_{0\leqslant h\leqslant 1, x\in(0\,,1)} g(h+t_n\,,x)$ goes to infinity. By the proof of Proposition 3.1.5, we have the required result. □

3.4 Remarks and extensions

Concerning the stochastic heat equation on $[0,1]$, Theorem 3.2.2 can be easily extended to the case that σ is locally Lipschitz continuous and bounded, see Mohammud Foondun and Eulalia Nualart [8]. Moreover, Robert Dalang, Davar Khoshnevisan, and Tusheng Zhang [4] established the existence and uniqueness of the global solution with superlinear and locally Lipschitz coefficients satisfying

$$|b(z)| = O(|z|\log|z|) \quad \text{and} \quad |\sigma(z)| = o\left(|z|(\log|z|)^{1/4}\right),$$

which shows the optimality of the Osgood condition. Last but not least, Michael Salins [23] demonstrated that if b does not satisfy the Osgood condition, then to guarantee the existence of global solutions one can allow σ to grow superlinearly as long as it satisfies an appropriate Osgood-type condition.

On the other hand, as regards the stochastic heat equation on $\mathbf{R}$, Theorem 3.3.5 is extended by Mohammud Foondun, Davar Khoshnevisan, and Eulalia Nualart in [7] to the case that σ is globally Lipschitz continuous and bounded away from zero and infinity, and a stronger statement is proved. Namely, it is shown that the Osgood condition implies that the solution blowsup instantaneously and everywhere meaning that for every $t > 0$ and $x \in \mathbf{R}$, $u(t\,,x) = \infty$ a.s. This implies that τ defined in (2.2.9) satisfies that $\tau = 0$.

As we started these notes with the deterministic heat equation, we should end with the comparison of the results of Chapter 3 with the blowup of such equation. Namely, we consider the heat equation perturbed by a drift satisfying Assumption 3.0.1, that is,

$$\partial_t(u(t\,,x)) = \kappa\partial^2_{xx}(u(t\,,x)) + u(t\,,x)^\alpha, \quad x \in \mathbf{R},\ t > 0, \tag{3.4.1}$$

with $\alpha > 1$ and initial condition $u_0(x)$ a nonnegative, continuous and bounded function. Then, it is shown by Hiroshi Fujita [9] that when $\alpha \leqslant 3$ and $u_0(x_0) > 0$ for some x_0, then there is no global solution to equation (3.4.1). However, when $\alpha > 3$, one can construct nontrivial global solutions when u_0 is small enough. More precisely, for any $\kappa > 0$, there exists $\delta > 0$ such that equation (3.4.1) has a global solution whenever

$$0 \leqslant u_0(x) \leqslant \delta \exp\left(-\kappa|x|^2\right).$$

The exponent $\alpha_c = 3$ is called the Fujita exponent. Observe that the drift $b(u) = u^\alpha$ satisfies the Osgood condition if and only if $\alpha > 1$. Therefore, Theorem 3.3.5 shows that for the stochastic heat equation $\alpha_c = \infty$ meaning that there is no global solution no matter how small the initial condition is. This shows that the Fujita phenomenon does not occur in this stochastic setting.

We next consider equation (3.4.1) on $[0, 1]$, that is,

$$\partial_t(u(t\,,x)) = \kappa\partial^2_{xx}(u(t\,,x)) + u(t\,,x)^\alpha, \quad x \in [0,1],\ t > 0, \tag{3.4.2}$$

with homogeneous Dirichlet boundary conditions, $\alpha > 1$, and initial condition $u_0(x)$ a nonnegative and continuous function on $[0, 1]$. In this case, for any $\alpha > 1$, one can always construct nontrivial global solutions by taking u_0 small enough. When u_0 is large enough, there is no global solution for any $\alpha > 1$. More precisely, if

$$\int_0^1 u_0(x)\phi(x)\,\mathrm{d}x \geqslant \lambda_1^{1/(\alpha-1)},$$

then the solution to (3.4.2) blowsup in finite time, where ϕ and λ_1 are as in the proof of Theorem 3.2.1. See Hiroshi Fujita [10]. Again this phenomena does not occur for the stochastic heat equation on $[0, 1]$ as it is proved in Theorem 3.2.1. For the heat equation on both $\mathbf{R}$ and $[0, 1]$ this phenomenon holds for more general convex drifts satisfying the Osgood condition. See [10] and Section 17 in [21] for precise statements.

Appendix A

Some useful inequalities

The following Gronwall's lemma can be easily proved by induction. See for e.g. [1].

Lemma A.0.1. *Suppose $\phi_1, \phi_2, \dots : [0,T] \to \mathbf{R}_+$ are measurable and non-decreasing functions. Suppose that there exists a constant A such that for all integers $n \geqslant 1$ and all $t \in [0,T]$,*

$$\phi_{n+1}(t) \leqslant A \int_0^t \phi_n(s)\,\mathrm{d}s.$$

Then, for all $n \geqslant 1$ and $t \in [0,T]$,

$$\phi_n(t) \leqslant \phi_1(T)\frac{(At)^{n-1}}{(n-1)!}.$$

The next Theorem is a version of Kolmogorov's continuity criterion for random fields. See Theorem A.3.1 in [5].

Theorem A.0.2. *Let $Z = (Z(t\,,x))_{(t\,,x)\in\mathbf{R}_+\times D}$ a real-valued random field, where D is a closed domain in $\mathbf{R}$. Assume that for some $p > 6$ and $T > 0$ there exists a constant $c_{p,T} > 0$ such that for all $s, t \in [0,T]$ and $x, y \in D$,*

$$\mathrm{E}\left(|Z(t\,,x) - Z(s\,,y)|^p\right) \leqslant c_{p,T}(|t-s|^{1/4} + |x-y|^{1/2})^p.$$

Then there exists a modification of the process Z which is continuous in $(t,x) \in \mathbf{R}_+ \times D$.

The following is a comparison principle for ordinary differential equations. See for instance Lemma 2.1 in [16].

Lemma A.0.3. *Assume that b satisfies Hypohtesis 3.0.1. Let $a_1 > a_2$ and $T > 0$. Assume that u and v are two measurable functions on $[0,T]$ such that*

$$v(t) \geqslant a_1 + \int_0^t b(v(s))\mathrm{d}s, \qquad t \in [0,T]$$

and

$$u(t) = a_2 + \int_0^t b(u(s))\mathrm{d}s, \qquad t \in [0,T]$$

Then, $v \geqslant u$ *on* $[0,T]$*. Moreover, if* $a_1 < a_2$ *and*

$$v(t) \leqslant a_1 + \int_0^t b(v(s))\mathrm{d}s, \qquad t \in [0,T]$$

and

$$u(t) = a_2 + \int_0^t b(u(s))\mathrm{d}s, \qquad t \in [0,T]$$

Then, $v \leqslant u$ *on* $[0,T]$*.*

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