DENSITY ESTIMATES FOR JUMP DIFFUSION PROCESSES

ARTURO KOHATSU-HIGA, EULALIA NUALART AND NGOC KHUE TRAN

Abstract. We consider a real-valued diffusion process with a linear jump term driven by a Poisson point process and we assume that the jump amplitudes have a centered density with finite moments. We show upper and lower estimates for the density of the solution in the case that the jump amplitudes follow a Gaussian or Laplacian law. The proof of the lower bound uses a general expression for the density of the solution in terms of the convolution of the density of the continuous part and the jump amplitude density. The upper bound uses an upper tail estimate in terms of the jump amplitude distribution and techniques of the Malliavin calculus in order to bound the density by the tails of the solution. We also extend the lower bounds to the multidimensional case.

1. Introduction and main results

Consider the following integral equation with jumps

$$X_t^x = x + \int_0^t \sigma(X_s^x) dB_s + \int_0^t b(X_s^x) ds + \sum_{i=1}^{\infty} Y_i \, \mathbf{1}_{T_i \le t}, \quad t \ge 0, \tag{1.1}$$

where $x \in \mathbb{R}$ and $(B_t)_{t\ge0}$ is a standard Brownian motion. The jump amplitude sequence $Y = (Y_i)_{i\ge1}$ is formed with i.i.d. random variables which have mean zero, finite moments of all orders and probability density function φ. The jump times $(T_i)_{i\ge1}$ are the arrival times of a Poisson process $(N_t)_{t\ge0}$ with rate $\lambda > 0$. All sources of randomness are assumed to be mutually independent.

The coefficients $\sigma, b : \mathbb{R} \to \mathbb{R}$ are assumed to be twice differentiable with bounded derivatives of all orders. Set $c_1 := \|b\|_\infty$ and $c_2 := \|\sigma\|_\infty$. Moreover, we assume that $\inf_{y\in\mathbb{R}} |\sigma(y)| \ge \rho > 0$ for some constant $\rho > 0$.

Under these conditions it is well-known that there exists a unique càdlàg adapted Markov process $X^x = (X_t^x)_{t\ge0}$ solution to the integral equation (1.1), which satisfies that for all $T > 0$ and $p > 1$,

$$\mathrm{E}\left[\sup_{t\in[0,T]} |X_t^x|^p\right] < \infty,$$

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see for e.g. [4, Theorem III.2.32]. Moreover, it is also well-known that for all $t > 0$, the law of X_t^x has a density with respect to the Lebesgue measure on $\mathbb{R}$, that we denote by $f_t(x, \cdot)$, see [2, Theorem 2.5] or [9, Theorem 11.4.4].

In this paper, we are interested in obtaining upper and lower bound estimates for the density f. When equation (1.1) has no jumps, Gaussian estimates for the density are well-known. Indeed, if we denote by $Z^x = (Z_t^x)_{t\geq 0}$ the unique solution to the equation

$$Z_t^x = x + \int_0^t \sigma(Z_s^x) dB_s + \int_0^t b(Z_s^x) ds$$

and by $p_t(x, \cdot)$ its density function, then it is well-known that for all $T > 0$, there exist constants $A_T, a_T > 1$ such that for all $t \in (0, T]$ and $y \in \mathbb{R}$,

$$\frac{1}{A_T\sqrt{2\pi t}} e^{-\frac{a_T|y-x|^2}{2t}} \leq p_t(x,y) \leq \frac{A_T}{\sqrt{2\pi t}} e^{-\frac{|y-x|^2}{2a_T t}}, \tag{1.2}$$

see for e.g. [5, 6, 10]. However, in the presence of jumps, less is known about estimates for the density of the solution to (1.1). Similar estimates for the density function f as the ones we obtain in this paper are proved in [7] (see also the references therein) for a class of infinite activity Lévy processes. The main motivation to write this paper is the fact that the model (1.1) appears in some insurance problems and their statistical estimation requires in principle properties of their transition densities. The Laplace transform and Karamata-Tuberian theorems are traditionally used in order to obtain asymptotic results for the density, see for e.g. [1] and the references therein. Here we propose a more direct analysis of the density that replaces the machinery of Laplace transforms.

The aim of this paper is to obtain upper and lower bounds for the density f when the jump amplitudes follow the Gaussian or Laplace laws.

Theorem 1.1. *Assume that φ is the centered Gaussian density with variance $\beta > 0$. Then for all $T > 0$ there exist constants $C_T, c_T > 1$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}$,*

$$C_T^{-1}\left(e^{-c_T|y-x|\sqrt{\ln_+(\frac{|y-x|}{t})}} + \frac{\mathbf{1}_{x=y}}{\sqrt{t}}\right) \leq f_t(x,y) \leq \frac{C_T}{\sqrt{t}} e^{-c_T^{-1}|y-x|\sqrt{\ln_+(\frac{|y-x|}{t})}},$$

where $\ln_+(x) = \max(\ln x, 0)$.

Theorem 1.2. *Assume that φ is the centered Laplace density with scale parameter $1/\mu$ where $\mu > 0$, that is,*

$$\varphi(z) = \frac{1}{2}\mu e^{-\mu|z|}. \tag{1.3}$$

Then for all $T > 0$ there exist constants $C_T, c_T > 1$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}$,

$$C_T^{-1}\left(e^{-c_T|y-x|} + \frac{\mathbf{1}_{x=y}}{\sqrt{t}}\right) \leq f_t(x,y) \leq \frac{C_T}{\sqrt{t}} e^{-c_T^{-1}|y-x|}.$$

The proof of these two theorems may be applied to other probability density functions φ. Indeed, first, we obtain an expression for the density f in terms of a convolution of the density of the continuous part p and the jump amplitude density φ. This expression is given in Proposition 2.1 below and turns out to be very suitable in order to obtain lower bounds for the density. Second, we show an upper tail estimate for the solution to equation (1.1) in terms of the jump distribution of Y that will be crucial for the upper bounds, see Proposition 2.3 below. Finally, in Proposition 2.5 below we appeal to the techniques of the Malliavin calculus in order to bound the density f_t in terms of the tail probabilities of X_t^x. These three results are proved for a general jump amplitude density φ. Then, we will show how to apply these general results for the two particular cases of φ defined in Theorems 1.1 and 1.2.

The rest of the paper is organized as follows. Section 2 presents the key preliminary results explained above for a general density φ. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2. Finally, in Section 4 we explore how the main results extend in the multidimensional case.

2. Preliminary results

In this section we present some preliminary results that will be crucial for the proof of Theorems 1.1 and 1.2.

We start proving an expression for the density that will be very suitable in order to establish lower bounds. For any $t>0$ we consider the random variable Z_t^x+Y and we denote by $q_t(x,y)$ its probability density function, where recall that Z_t^x is the continuous part of X_t^x and has density p_t, and Y is the jump amplitude which has density φ. As Y and Z_t^x are independent then we have that $q_t(x,y)=(p_t*\varphi)(x,y)$ where $*$ denotes the space convolution of the functions $p_t(x,\cdot)$ and $\varphi(\cdot)$. That is,

$$q_t(x,y)=\int_{\mathbb{R}}\varphi(y-v)p_t(x,v)dv. \tag{2.1}$$

Given two measurable functions $g(x,y)$ and $k(x,y)$ in $\mathbb{R}^2$ we define the product

$$(g\star k)(x,y)=\int_{\mathbb{R}}g(x,v)k(v,y)dv.$$

The following result gives an expression for the density of X_t^x in terms of q.

Proposition 2.1. *For any $t>0$ and $x,y\in\mathbb{R}$, the density $f_t(x,y)$ of X_t^x solution to equation (1.1) can be written as (here $t_0\equiv 0$):*

$$\begin{aligned} f_t(x,y)&=p_t(x,y)e^{-\lambda t}\\ &\quad+\sum_{n=1}^{\infty}\int_{t_1<\cdots<t_n<t<t_{n+1}}(q_{t_1-t_0}\star(\cdots\star(q_{t_n-t_{n-1}}\star p_{t-t_n})\cdots))(x,y)\lambda^{n+1}e^{-\lambda t_{n+1}}dt_1\cdots dt_{n+1}. \end{aligned} \tag{2.2}$$

Proof. First, we write

$$f_t(x,y)=\mathrm{E}[\delta_y(X_t^x)]=\mathrm{E}[\delta_y(Z_t^x)\,\mathbf{1}_{t<T_1}]+\sum_{n=1}^{\infty}\mathrm{E}[\delta_y(X_t^x)\,\mathbf{1}_{T_1<\cdots<T_n<t<T_{n+1}}], \tag{2.3}$$

where we recall that $(T_i)_{i\geq 0}$ are the arrival times of a Poisson process $(N_t)_{t\geq 0}$ with parameter $\lambda>0$. Remark that an abuse of notation is used here when we write the delta distribution function $\delta_y(x)$. The formal argument can be obtained by proper approximation arguments which are left to the reader.

As Z_t^x and T_1 are independent, we have

$$\mathrm{E}[\delta_y(Z_t^x)\,\mathbf{1}_{t<T_1}]=p_t(x,y)\mathrm{P}(t<T_1)=p_t(x,y)e^{-\lambda t},$$

which gives the first term in (2.2).

We next show how to obtain the second term. First recall that for each $n \geq 1$, the density function of the random vector $(T_1, T_2, \dots, T_n)$ is given by

$$g_{T_1,T_2,\dots,T_n}(t_1,\dots,t_n) = \lambda^n e^{-\lambda t_n} \mathbf{1}_{0\leq t_1<\cdots<t_{n-1}<t_n}(t_1,\dots,t_n).$$

Let $X_t^{x,n} \equiv X_t^{x,n}(t_1,...,t_n)$ denote the solution to the following integral equation with deterministic jump times

$$X_t^{x,n} = x + \int_0^t \sigma(X_s^{x,n})dB_s + \int_0^t b(X_s^{x,n})ds + \sum_{i=1}^n Y_i \mathbf{1}_{t_i\leq t}.$$

Then, using again the independence between the jump times and the other random components in $X^{x,n}$, we obtain

$$\mathrm{E}[\delta_y(X_t^x)\,\mathbf{1}_{T_1<\cdots<T_n<t<T_{n+1}}] = \int_{t_1<\cdots<t_n<t<t_{n+1}} \mathrm{E}[\delta_y(X_t^{x,n})]\lambda^{n+1}e^{-\lambda t_{n+1}}dt_1\cdots dt_{n+1}.$$

Finally, appealing to Chapman-Kolmogorov's equation yields to

$$\begin{aligned}\mathrm{E}[\delta_y(X_t^{x,n})] &= \int_{\mathbb{R}^n} q_{t_1-t_0}(x,y_1)q_{t_2-t_1}(y_1,y_2)\cdots q_{t_n-t_{n-1}}(y_{n-1},y_n)p_{t-t_n}(y_n,y)dy_1\cdots dy_n \\ &= (q_{t_1-t_0}\star(\cdots\star(q_{t_n-t_{n-1}}\star p_{t-t_n})\cdots))(x,y),\end{aligned}$$

which gives the second term in (2.2) and completes the proof. □

Remark 2.2. *Observe that in the linear case, that is, $b=0$ and $\sigma=1$, (2.2) reads as*

$$f_t(x,y) = e^{-\lambda t}\sum_{n=0}^\infty (\Phi_t * \varphi^{*n})(y-x)\frac{(\lambda t)^n}{n!}$$

where Φ_t denotes the $N(0,t)$ density.

The second result of this section is an upper bound for the tail probability of X_t^x in terms of the distribution of the jump amplitude Y. This result is an extension of Lemma 26.4 in [11], where a similar tail probability estimate is obtained for Lévy processes. Let us first introduce some notation which is similar to that in [11]. Define

$$C := \{u\in\mathbb{R} : \mathrm{E}[e^{uY}]<\infty\} \quad \text{and} \quad s := \sup(C).$$

Note that C is an interval and assume that $s>0$. Set, for $u\in C$,

$$\Psi(u) := \frac{1}{2}u^2c_2^2 + \lambda\mathrm{E}[e^{uY}-1].$$

Then

$$\Psi'(u) = uc_2^2 + \lambda\mathrm{E}[Ye^{uY}]$$

and

$$\Psi''(u) = c_2^2 + \lambda\mathrm{E}[Y^2e^{uY}].$$

Notice that $\Psi\in C^\infty$, and $\Psi''>0$ in the interior of C.

Let $u=\theta(\xi)$ be the inverse function of $\xi=\Psi'(u)$, that is,

$$\xi = \theta(\xi)c_2^2 + \lambda\mathrm{E}[Ye^{\theta(\xi)Y}], \qquad \xi\in(0,\Psi'(s-)).$$

Proposition 2.3. *For all $t > 0$ and $x, y \in \mathbb{R}$ such that $\frac{|y-x|}{t} - c_1 \in (0, \Psi'(s-))$, we have*

$$\mathrm{P}(|X_t^x - x| > |y - x|) \le 2e^{-t \int_0^{\frac{|y-x|}{t} - c_1} \theta(\xi) d\xi}.$$

Proof. Let $u \in (0, s)$ and let $(M_t)_{t\ge 0}$ denote the Itô-Lévy process given by

$$M_t = u \int_0^t \sigma(X_s^x) dB_s + u \sum_{i=1}^{\infty} Y_i \, \mathbf{1}_{T_i \le t} - \frac{1}{2} u^2 \int_0^t \sigma^2(X_s^x) ds - \lambda t \int_{\mathbb{R}} (e^{uz} - 1) \varphi(z) dz.$$

Observe that

$$u(X_t^x - x) = M_t + u \int_0^t b(X_s^x) ds + \frac{1}{2} u^2 \int_0^t \sigma^2(X_s^x) ds + \lambda t \int_{\mathbb{R}} (e^{uz} - 1) \varphi(z) dz.$$

By Itô's formula,

$$e^{M_t} = 1 + u \int_0^t e^{M_s} \sigma(X_s^x) dB_s + \sum_{i=1}^{\infty} e^{M_{T_i -}} \left(e^{uY_i} - 1\right) \mathbf{1}_{T_i \le t} - \lambda \int_0^t \int_{\mathbb{R}} e^{M_{s-}} (e^{uz} - 1) \varphi(z) dz ds.$$

In particular, $(e^{M_t})_{t\ge 0}$ is a martingale and $\mathrm{E}[e^{M_t}] = 1$.

Using Markov's inequality and the fact that σ and b are bounded, we have that

$$\begin{aligned}
\mathrm{P}(X_t^x - x > |y - x|) &= \mathrm{P}(e^{u(X_t^x - x)} > e^{u|y-x|}) \\
&= \mathrm{P}(e^{M_t + u \int_0^t b(X_s^x) ds + \frac{1}{2} u^2 \int_0^t \sigma^2(X_s^x) ds + \lambda \int_0^t \int_{\mathbb{R}} (e^{uz} - 1) \varphi(z) dz ds} > e^{u|y-x|}) \\
&\le \mathrm{P}(e^{M_t + u c_1 t + \frac{1}{2} u^2 c_2^2 t + \lambda t \int_{\mathbb{R}} (e^{uz} - 1) \varphi(z) dz} > e^{u|y-x|}) \\
&= \mathrm{P}\left(e^{M_t} > e^{(|y-x| - c_1 t) u - \frac{1}{2} u^2 c_2^2 t - \lambda t \mathrm{E}[e^{uY} - 1]}\right) \\
&\le \min_{0<u<s} e^{-(|y-x| - c_1 t) u + t\Psi(u)} = \min_{0<u<s} e^{t(\Psi(u) - uz)},
\end{aligned}$$

where $z := \frac{|y-x|}{t} - c_1$. The rest of the proof follows as in Lemma 26.4 in [11]. Indeed, we have $z \in (0, \Psi'(s_-))$. As $\Psi'(u) - z$ changes value from negative to positive at $u = \theta(z)$ and $\Psi(0) = 0$, we have

$$\begin{aligned}
\min_{0<u<s} (\Psi(u) - uz) &= \Psi(\theta(z)) - \theta(z) z = \int_0^{\theta(z)} \Psi'(u) du - \theta(z) z = \int_0^z \xi d\theta(\xi) - \theta(z) z \\
&= z\theta(z) - \lim_{\xi \downarrow 0} \xi \theta(\xi) - \int_0^z \theta(\xi) d\xi - \theta(z) z = - \int_0^z \theta(\xi) d\xi,
\end{aligned}$$

since

$$\lim_{\xi \downarrow 0} \xi \theta(\xi) = \lim_{u \downarrow 0} \Psi'(u) u = 0.$$

This shows the upper bound for $\mathrm{P}(X_t^x - x > |y - x|)$.

Proceeding exactly along the same lines, we can consider the martingale $-M_t$ and show the same upper bound for $\mathrm{P}(-(X_t^x - x) > |y - x|)$. Thus, the desired result follows. □

The last result of this section is an upper bound for the density of X_t^x in terms of its upper tails. For this, we appeal to techniques of the Malliavin calculus with respect to the Brownian motion B. We denote by D the Malliavin derivative operator with respect to B and by $\mathbb{D}^{2,\infty}$ the Sobolev space of twice differentiable random variables with finite moments of all orders. See the monographs [8] or [9] for the precise definitions. The next result is classical and shows that the solution to (1.1) is twice differentiable in the Malliavin sense and gives an expression for the Malliavin derivatives, see [2, 3] and [9, Theorem 11.4.3] for its proof.

Lemma 2.4. *For all $t > 0$ and $x \in \mathbb{R}$, X_t^x belongs to $\mathbb{D}^{2,\infty}$ and the Malliavin derivative $(D_r X_t^x, r \leq t)$ satisfies the following linear equation*

$$D_r X_t^x = \sigma(X_r^x) + \int_r^t \sigma'(X_s^x) D_r X_s^x dB_s + \int_r^t b'(X_s^x) D_r X_s^x ds,$$

for $r \leq t$, a.e., and $D_r X_t^x = 0$ for $r > t$, a.e. Moreover, for all $p > 1$,

$$\sup_{r \in [0,T]} \mathrm{E}\left[\sup_{t \in [r,T]} |D_r X_t^x|^p\right] < \infty.$$

Furthermore, the iterated Malliavin derivative $(D^2_{r_1,r_2} X_t^x, r_1 \vee r_2 \leq t)$ satisfies the equation

$$D^2_{r_1,r_2} X_t^x = D_{r_1}\sigma(X_{r_2}^x) + D_{r_2}\sigma(X_{r_1}^x) + \int_{r_1 \vee r_2}^t D^2_{r_1,r_2}(\sigma(X_s^x))\, dB_s + \int_{r_1 \vee r_2}^t D^2_{r_1,r_2}(b(X_s^x))\, ds,$$

for $r_1 \vee r_2 \leq t$, a.e., and $D^2_{r_1,r_2} X_t^x = 0$ otherwise. Moreover, for all $p > 1$,

$$\sup_{r_1, r_2 \in [0,T]} \mathrm{E}\left[\sup_{r_1 \vee r_2 \leq t \leq T} \left|D^2_{r_1,r_2} X_t^x\right|^p\right] < \infty.$$

We are now ready to state the last result of this section which bounds the density $f_t(x,y)$ of X_t^x in terms of its upper tail probability. Since the jump term is linear, the proof follows exactly along the same lines as for a continuous diffusion (see [8]). For the sake of completeness, we provide the proof.

Proposition 2.5. *For any $T > 0$ and $q > 1$, there exists a constant $C_{q,T} > 0$ such that for all $t \in (0,T]$ and $x, y \in \mathbb{R}$,*

$$f_t(x,y) \leq \frac{C_{q,T}}{\sqrt{t}} (\mathrm{P}(|X_t^x - x| > |y - x|))^{1/q}.$$

Proof. Appealing to Proposition 2.1.2 in [8], we have

$$f_t(x,y) \leq c_{q,\alpha,\beta} \left(\mathrm{P}(|X_t^x - x| > |y-x|)\right)^{1/q} \Bigg(\mathrm{E}\left[\|DX_t^x\|_H^{-1}\right] + \left(\mathrm{E}\left[\|D^2 X_t^x\|^\alpha_{H \otimes H}\right]\right)^{1/\alpha} \times \left(\mathrm{E}\left[\|DX_t^x\|_H^{-2\beta}\right]\right)^{1/\beta} \Bigg), \tag{2.4}$$

for some constant $c_{q,\alpha,\beta} > 0$, where $\frac{1}{q} + \frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $H = L^2([0,t];\mathbb{R})$.

Using Lemma 2.4 and Hölder's inequality, we get for any $\alpha > 1$,

$$\begin{aligned}\mathrm{E}\left[\|D^2X_t^x\|_{H\otimes H}^{\alpha}\right] &= \mathrm{E}\left[\left(\|D^2X_t^x\|_{H\otimes H}^{2}\right)^{\frac{\alpha}{2}}\right] \\ &= \mathrm{E}\left[\left(\int_0^t\int_0^t \left(D_{r,v}^2X_t^x\right)^2 drdv\right)^{\frac{\alpha}{2}}\right] \leq C_{\alpha,T}t^{\alpha}.\end{aligned}$$

Thus,

$$\left(\mathrm{E}\left[\|D^2X_t^x\|_{H\otimes H}^{\alpha}\right]\right)^{1/\alpha} \leq C_{\alpha,T}t \leq C'_{\alpha,T}\sqrt{t}. \tag{2.5}$$

Now, we denote by $J_t^x := \partial_x X_t^x$ the derivative of X_t^x with respect to the initial condition x, which satisfies the linear equation

$$J_t^x = 1 + \int_0^t \sigma'(X_s^x)J_s^x dB_s + \int_0^t b'(X_s^x)J_s^x ds.$$

By Itô's formula, the inverse $H_t^x := (J_t^x)^{-1}$ satisfies the linear equation

$$H_t^x = 1 - \int_0^t \sigma'(X_s^x)H_s^x dB_s - \int_0^t (b'(X_s^x) - (\sigma'(X_s^x))^2)H_s^x ds.$$

Using the assumptions on b and σ and Gronwall type arguments, we have that for all $p \geq 1$,

$$\mathrm{E}\Big[\sup_{t\in[0,T]} |J_t^x|^p\Big] \leq C_{p,T}, \qquad \mathrm{E}\Big[\sup_{t\in[0,T]} |H_t^x|^p\Big] \leq C_{p,T}.$$

Moreover, we have that

$$D_rX_t^x = J_t^x H_r^x \sigma(X_r^x)\mathbf{1}_{[0,t]}(r).$$

Consequently, for any $p \geq 1$,

$$\begin{aligned}\mathrm{E}\left[\|DX_t^x\|_H^{-p}\right] &= \mathrm{E}\left[\left(\int_0^t (J_t^x)^2(H_r^x)^2\sigma^2(X_r^x)dr\right)^{-p/2}\right] \\ &\leq \frac{1}{\rho^p t^{p/2}}\mathrm{E}\left[\sup_{0\leq r\leq T}|J_r^x|^p \sup_{0\leq r\leq T}|H_r^x|^p\right] \leq \frac{C_{p,T}}{\rho^p t^{p/2}}.\end{aligned} \tag{2.6}$$

Hence, using (2.5) and (2.6), we conclude that

$$\left(\mathrm{E}\left[\|D^2X_t^x\|_{H\otimes H}^{\alpha}\right]\right)^{1/\alpha}\left(\mathrm{E}\left[\|DX_t^x\|_H^{-2\beta}\right]\right)^{1/\beta} \leq C_{\alpha,T}\sqrt{t}\frac{C_{\beta,T}}{t} = \frac{C_{\alpha,\beta,T}}{\sqrt{t}},$$

which together with (2.4) finishes the proof of the upper bound.

Observe that Lemma 2.4, (2.6) and criterion in [8, Theorem 2.1.1] imply the existence of the density. $\square$

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Assume that φ is $N(0,\beta)$. We start proving the upper bound. By Proposition 2.5, it suffices to apply Proposition 2.3. For $u>0$, we have that

$$\Psi(u)=\frac{1}{2}u^2c_2^2+\lambda(e^{\frac{u^2\beta}{2}}-1).$$

Thus, $\Psi'(u)=u(c_2^2+\lambda\beta e^{\frac{u^2\beta}{2}})$ and its inverse function $u=\theta(\xi)$ satisfies

$$\xi=\theta(\xi)\big(c_2^2+\lambda\beta e^{\frac{\theta^2(\xi)\beta}{2}}\big),\quad \xi\in(0,\infty).$$

Let us now estimate $\theta(\xi)$. Observe that $\theta(\xi)\uparrow\infty$ as $\xi\uparrow\infty$ and for $\alpha<\frac{2}{\beta}$, $\xi e^{-\frac{\theta^2(\xi)}{\alpha}}\to 0$ as $\xi\uparrow\infty$. Hence, there is $\xi_1>1$ such that for all $\xi>\xi_1$, $\xi e^{-\frac{\theta^2(\xi)}{\alpha}}<\frac{1}{2}$ and $c_1e^{-\frac{\theta^2(\xi)}{\alpha}}<\frac12$. Thus, for $\xi>\xi_1$, $(\xi+c_1)e^{-\frac{\theta^2(\xi)}{\alpha}}<1$ and $\theta(\xi)>\sqrt{\alpha\ln(\xi+c_1)}$. Therefore, by Proposition 2.3, for $\frac{|y-x|}{t}-c_1>\xi_1$,

$$\mathrm{P}(|X_t^x-x|>|y-x|)\le 2e^{-t\int_{\xi_1}^{\frac{|y-x|}{t}-c_1}\sqrt{\alpha\ln(\xi+c_1)}d\xi}=2e^{-t\int_{\xi_1+c_1}^{\frac{|y-x|}{t}}\sqrt{\alpha\ln\xi}d\xi}.$$

As for z sufficiently large $\int_{\xi_1+c_1}^{z}\sqrt{\ln\xi}d\xi>\frac{z}{2}\sqrt{\ln z}$, the desired result follows.

In the case that $0<\frac{|y-x|}{t}-c_1\le\xi_1$, one clearly has that $(|y-x|,\frac{|y-x|}{t})$ belongs to a compact set and therefore if we bound the probability in Proposition 2.5 by 1, we obtain

$$\begin{aligned}f_t(x,y)&\le\frac{C_T}{\sqrt{t}}e^{-|y-x|\sqrt{\ln_+(\frac{|y-x|}{t})}}e^{|y-x|\sqrt{\ln_+(\frac{|y-x|}{t})}}\\&\le\frac{C_T}{\sqrt{t}}e^{-|y-x|\sqrt{\ln_+(\frac{|y-x|}{t})}}e^{T(\xi_1+c_1)\sqrt{\ln_+(\xi_1+c_1)}},\end{aligned}$$

which implies the desired upper bound.

In the case that $\frac{|y-x|}{t}-c_1<0$, the probability of the reverse inequality can be bounded in a similar fashion.

We next prove the lower bound. Using (2.1), the fact that φ is $N(0,\beta)$, and the Gaussian-type lower bound for $p_t(x,y)$ in (1.2), we obtain that for all $t\in(0,T]$ and $x,y\in\mathbb{R}$,

$$\begin{aligned}q_t(x,y)&\ge\frac{1}{A_T\sqrt{a_T}}\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\beta}}e^{-\frac{|y-v|^2}{2\beta}}\frac{1}{\sqrt{2\pi a_T^{-1}t}}e^{-\frac{|v-x|^2}{2a_T^{-1}t}}dv\\&=\frac{1}{A_T\sqrt{a_T}}\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\beta}}e^{-\frac{|v|^2}{2\beta}}\frac{1}{\sqrt{2\pi a_T^{-1}t}}e^{-\frac{|v+(y-x)|^2}{2a_T^{-1}t}}dv\\&=\frac{1}{A_T\sqrt{a_T2\pi(a_T^{-1}t+\beta)}}e^{-\frac{|y-x|^2}{2(a_T^{-1}t+\beta)}},\end{aligned}$$

where in the last equality we have used the fact that the convolution of two Gaussian densities is a Gaussian density with mean equal to the sum of means and variance equal to the sum of variances.

Therefore, using (2.2), we obtain for $r := |y-x|$ and $C_T = \frac{1}{A_T\sqrt{a_T}}$

$$f_t(x,y) \geq e^{-\lambda t} \sum_{n=0}^{\infty} (C_T)^n \frac{C_T}{\sqrt{2\pi(a_T^{-1}t+n\beta)}} e^{-\frac{r^2}{2(a_T^{-1}t+n\beta)}} \frac{(\lambda t)^n}{n!}. \tag{3.1}$$

Observe that it suffices to assume that $\frac{r}{t} \geq e$, otherwise the bound follows trivially, since taking $n=0$ yields to

$$f_t(x,y) \geq e^{-\lambda t} \frac{1}{A_T\sqrt{2\pi t}} e^{-\frac{r^2}{2a_T^{-1}t}} \geq \frac{1}{A_T\sqrt{2\pi T}} e^{-\lambda T} e^{-\frac{Te^2}{2a_T^{-1}}},$$

from which the desired lower bound follows for $r < te$. Observe also that if $r=0$, we get the lower bound

$$f_t(x,y) \geq e^{-\lambda T} \frac{1}{A_T\sqrt{2\pi t}}. \tag{3.2}$$

By Stirling's formula, there exists a constant $K>1$ such that for all $n>K$, it holds that

$$\left| \frac{\sqrt{2\pi} e^{n\ln(n)-n+\frac{1}{2}\ln(n)}}{n!} - 1 \right| < \frac{1}{2}.$$

This implies that for all $n > K$,

$$\frac{(\lambda t)^n}{n!} > \frac{1}{2\sqrt{2\pi}} e^{-n\ln\left(\frac{n}{\lambda t}\right)+n-\frac{1}{2}\ln(n)} > \frac{1}{2\sqrt{2\pi}} e^{-n\ln\left(\frac{n}{\lambda t}\right)-\frac{1}{2}\ln(n)}.$$

Then, substituting this into (3.1), we get that

$$\begin{aligned} f_t(x,y) &\geq \frac{C_T}{4\pi} e^{-\lambda t} \sum_{n>K} e^{n\ln(C_T)-\frac{1}{2}\ln(a_T^{-1}t+n\beta)-\frac{r^2}{2(a_T^{-1}t+n\beta)}-n\ln\left(\frac{n}{\lambda t}\right)-\frac{1}{2}\ln(n)} \\ &\geq \frac{C_T}{4\pi} e^{-\lambda t} \sum_{n>K} e^{-n|\ln(C_T)|-\frac{1}{2}\ln(a_T^{-1}T+n\beta)-\frac{r^2}{2n\beta}-n\ln\left(\frac{n}{\lambda t}\right)-\frac{1}{2}\ln(n)}. \end{aligned} \tag{3.3}$$

We next consider two different cases according to $\frac{r}{\sqrt{\ln(r/t)}} > K+1$ and $\frac{r}{\sqrt{\ln(r/t)}} \leq K+1$.

In the first case, we set $n = [\frac{r}{\sqrt{\ln(r/t)}}] > K > 1$. Note that $x-1 \leq [x] \leq x$ for any $x \in \mathbb{R}$, then using (3.3), we obtain

$$\begin{aligned} &f_t(x,y) \\ &\geq \frac{C_T}{4\pi} e^{-\lambda t} e^{-\frac{r}{\sqrt{\ln(r/t)}}|\ln(C_T)|-\frac{1}{2}\ln\left(c_T'\frac{r}{\sqrt{\ln(r/t)}}\right)-\frac{r^2}{2\left(\frac{r}{\sqrt{\ln(r/t)}}-1\right)\beta}-\frac{r}{\sqrt{\ln(r/t)}}\ln\left(\frac{r}{\lambda t\sqrt{\ln(r/t)}}\right)-\frac{1}{2}\ln\left(\frac{r}{\sqrt{\ln(r/t)}}\right)} \\ &\geq \frac{C_T}{4\pi} e^{-\lambda T} e^{-c_T r\sqrt{\ln(r/t)}}, \end{aligned}$$

for some constant $c_T > 0$, where $c_T' = a_T^{-1}T + \beta$. In the last inequality we have used the following inequalities

(1) $\dfrac{r^2}{\frac{r}{\sqrt{\ln(r/t)}}-1} \leq \dfrac{K+1}{K} r\sqrt{\ln(r/t)},$

(2) $\dfrac{r}{\sqrt{\ln(r/t)}} \leq r \leq r\sqrt{\ln(r/t)}, \quad \text{since } \dfrac{r}{t} \geq e,$

(3) $\dfrac{r}{\sqrt{\ln(r/t)}} \ln\left(\dfrac{r}{\lambda t\sqrt{\ln(r/t)}}\right) \leq \dfrac{r}{\sqrt{\ln(r/t)}} \ln\left(\dfrac{r/t}{\lambda}\right) \leq r\sqrt{\ln(r/t)}(1+|\ln(\lambda)|).$

For the second case $\frac{r}{\sqrt{\ln(r/t)}} \leq K+1$ the conclusion follows easily since taking $n=0$ in (3.1) yields

$$f_t(x,y) \geq e^{-\lambda t}\frac{1}{A_T\sqrt{2\pi T}}e^{-\frac{r^2}{2a_T^{-1}T}} \geq e^{-\lambda T}\frac{1}{A_T\sqrt{2\pi T}}e^{-(K+1)\frac{r\sqrt{\ln(r/t)}}{2a_T^{-1}T}}.$$

The proof is now completed. □

Proof of Theorem 1.2. We first prove the upper bound. As above it suffices to apply Propositions 2.3 and 2.5. A direct calculation using (1.3) gives that for all $-\mu < u < \mu$,

$$\Psi(u) = \frac{1}{2}u^2c_2^2 + \lambda\left(\frac{\mu}{2}\left(\frac{1}{u+\mu} - \frac{1}{u-\mu}\right) - 1\right).$$

Thus,

$$\xi = \theta(\xi)c_2^2 + \frac{\lambda\mu}{2}\left(\frac{1}{(\theta(\xi)-\mu)^2} - \frac{1}{(\theta(\xi)+\mu)^2}\right), \quad \xi \in (0,\infty).$$

Let us now estimate $\theta(\xi)$. Observe that $\theta(\xi) \uparrow \mu$ as $\xi \uparrow \infty$. This implies that $\xi((\theta(\xi)-\mu)^2 \wedge (\theta(\xi)+\mu)^2)$ converges to $\frac{\lambda\mu}{2}$ as $\xi \uparrow \infty$. Hence, there exists $\xi_1 > 0$ such that for all $\xi > \xi_1$, $\sqrt{\xi}(\mu - \theta(\xi)) - \sqrt{\lambda\mu/2} < 1$, and thus, for all $\xi > \xi_1$, $\theta(\xi) > \mu - \frac{1+\sqrt{\lambda\mu/2}}{\sqrt{\xi}}$. Therefore, by Proposition 2.3, for $\frac{|y-x|}{t} - c_1 > \xi_1$,

$$\mathrm{P}(|X_t^x - x| > |y-x|) \leq 2e^{-t\int_{\xi_1}^{\frac{|y-x|}{t}-c_1}\left(\mu - \frac{1+\sqrt{\lambda\mu/2}}{\sqrt{\xi}}\right)d\xi}.$$

Choosing $\frac{1}{\sqrt{\xi_1}} < \frac{\mu}{2(1+\sqrt{\lambda\mu/2})}$, gives

$$\int_{\xi_1}^{\frac{|y-x|}{t}-c_1}\left(\frac{\mu}{1+\sqrt{\lambda\mu/2}} - \frac{1}{\sqrt{\xi}}\right)d\xi > \frac{\mu}{2(1+\sqrt{\lambda\mu/2})}\left(\frac{|y-x|}{t} - c_1 - \xi_1\right).$$

In the case that $\frac{|y-x|}{t} - c_1 \leq \xi_1$, the result follows trivially. This finishes the proof of the upper bound.

Next, we prove the lower bound. Using (2.1) and the lower bound in (1.2), we get that for all $t \in (0,T]$ and $x, y \in \mathbb{R}$,

$$\begin{aligned}
q_t(x,y) &\geq \frac{\mu}{2}\int_{\mathbb{R}} e^{-\mu|z|}\frac{1}{A_T\sqrt{2\pi t}} e^{-\frac{|z-(y-x)|^2}{2a_T^{-1}t}}\,dz \\
&= \frac{\mu}{2A_T\sqrt{2\pi t}} e^{\frac{\mu^2 t a_T^{-1}}{2}}\left(e^{-(y-x)\mu}\int_0^\infty e^{-\frac{|z-(y-x-\mu t a_T^{-1})|^2}{2a_T^{-1}t}}\,dz + e^{(y-x)\mu}\int_{-\infty}^0 e^{-\frac{|z-(y-x+\mu t a_T^{-1})|^2}{2a_T^{-1}t}}\,dz\right) \\
&\geq C_T e^{\frac{\mu^2 t a_T^{-1}}{2}}\left(e^{-(y-x)\mu}\left(\mathbf{1}_{y-x-\mu t a_T^{-1}<0}\frac{1}{2}e^{-\frac{|y-x-\mu t a_T^{-1}|^2}{a_T^{-1}t}} + \frac{1}{2}\mathbf{1}_{y-x-\mu t a_T^{-1}\geq 0}\right)\right. \\
&\qquad \left. + e^{(y-x)\mu}\left(\mathbf{1}_{y-x+\mu t a_T^{-1}\geq 0}\frac{1}{2}e^{-\frac{|y-x+\mu t a_T^{-1}|^2}{a_T^{-1}t}} + \frac{1}{2}\mathbf{1}_{y-x+\mu t a_T^{-1}<0}\right)\right),
\end{aligned}$$

where $C_T = \frac{\mu}{2A_T\sqrt{2a_T}}$. Observe that in order to get the last inequality when $z > 0$, we have used the fact that the integral of a Gaussian density with a non-negative mean on the positive axis is lower bounded by $\frac{1}{2}$. On the other hand, in the case that the mean is negative, we have used the inequality $|a-b|^2 \leq 2(|a|^2 + |b|^2)$, valid for all $a, b \in \mathbb{R}$ and the fact that the integral of a Gaussian density with zero mean on the positive axis equals $\frac{1}{2}$. We have applied a similar argument for the case $z < 0$.

Expanding the square appearing in the two exponentials yields

$$\begin{aligned}
q_t(x,y) &\geq \frac{C_T}{2}e^{-\frac{\mu^2 t a_T^{-1}}{2}}\left(e^{-\frac{|y-x|^2}{a_T^{-1}t}}\left(e^{(y-x)\mu}\mathbf{1}_{y-x<\mu t a_T^{-1}} + e^{-(y-x)\mu}\mathbf{1}_{y-x\geq -\mu t a_T^{-1}}\right)\right. \\
&\qquad \left. + e^{-(y-x)\mu}\mathbf{1}_{y-x\geq \mu t a_T^{-1}} + e^{(y-x)\mu}\mathbf{1}_{y-x<-\mu t a_T^{-1}}\right) \\
&\geq \frac{C_T}{2}e^{-\frac{\mu^2 t a_T^{-1}}{2}}\left(e^{-\frac{|y-x|^2}{a_T^{-1}t}}e^{-|y-x|\mu}\mathbf{1}_{|y-x|\leq \mu t a_T^{-1}}\right. \\
&\qquad \left. + e^{-|y-x|\mu}\mathbf{1}_{y-x>\mu t a_T^{-1}} + e^{-|y-x|\mu}\mathbf{1}_{y-x<-\mu t a_T^{-1}}\right) \qquad (3.4) \\
&\geq \frac{C_T}{2}e^{-\frac{\mu^2 t a_T^{-1}}{2}}\left(e^{-|y-x|\mu}e^{-|y-x|\mu}\mathbf{1}_{|y-x|\leq \mu t a_T^{-1}}\right. \\
&\qquad \left. + e^{-|y-x|\mu}\mathbf{1}_{y-x>\mu t a_T^{-1}} + e^{-|y-x|\mu}\mathbf{1}_{y-x<-\mu t a_T^{-1}}\right) \\
&\geq \frac{C_T}{2}e^{-\frac{\mu^2 t a_T^{-1}}{2}}e^{-2|y-x|\mu}.
\end{aligned}$$

This implies, using the triangular inequality, that

$$\begin{aligned}(q_{t_1-t_0}\star q_{t_2-t_1})(x,y_2) &\geq \frac{C_T^2}{4}\int_{\mathbb{R}} e^{-\frac{\mu^2 t_1 a_T^{-1}}{2}} e^{-2|y_1-x|\mu} e^{-\frac{\mu^2(t_2-t_1)a_T^{-1}}{2}} e^{-2|y_2-y_1|\mu} dy_1 \\ &= \frac{C_T^2}{4} e^{-\frac{\mu^2 t_2 a_T^{-1}}{2}} \int_{\mathbb{R}} e^{-2|y_1|\mu} e^{-2|y_2-x-y_1|\mu} dy_1 \\ &\geq \frac{C_T^2}{8\mu} e^{-\frac{\mu^2 t_2 a_T^{-1}}{2}} e^{-2|y_2-x|\mu}.\end{aligned}$$

Therefore, iterating the above computation, we obtain that for $n \geq 1$,

$$\begin{aligned}&(q_{t_1-t_0}\star(q_{t_2-t_1}\star\cdots\star(q_{t_n-t_{n-1}}\star p_{t-t_n})\cdots))(x,y) \\ &\quad\geq \tilde{C}_T^n e^{-\frac{\mu^2 t_n a_T^{-1}}{2}} \int_{\mathbb{R}} e^{-2|y_n-x|\mu} p_{t-t_n}(y_n,y) dy_n \\ &\quad\geq \overline{C}_T \tilde{C}_T^n e^{-c_T t} e^{-4|y-x|\mu},\end{aligned}$$

for some constants $\overline{C}_T, \tilde{C}_T, c_T > 0$. Finally, appealing to formula (2.2) yields to

$$f_t(x,y) \geq \overline{C}_T e^{-c_T t} e^{-4|y-x|\mu} e^{-\lambda t} \sum_{n=0}^{\infty} \tilde{C}_T^n \frac{(\lambda t)^n}{n!} \geq \overline{C}_T e^{-(c_T+\lambda)T} e^{-4|y-x|\mu},$$

which proves the lower bound for $x \neq y$. When $x = y$ it suffices to use formula (2.2) for $n = 0$ to obtain the same lower bound as in (3.2). This completes the proof. □

4. Extension to the multidimensional case

The aim of this section is to explain how the results obtained above extend to the multidimensional case. The multidimensional version of equation (1.1) writes as follows:

$$X_t^x = x + \int_0^t \sigma(X_s^x) dB_s + \int_0^t b(X_s^x) ds + \sum_{i=1}^{\infty} Y_i \mathbf{1}_{T_i \leq t}, \quad t \geq 0, \tag{4.1}$$

where $x \in \mathbb{R}^d$, $(B_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion, $Y = (Y_i)_{i\geq 1}$ is a sequence of d-dimensional i.i.d. random variables which have mean zero, finite moments of all orders and probability density function φ. The function $b : \mathbb{R}^d \to \mathbb{R}^d$ and the matrix $\sigma : \mathbb{R}^d \to \mathcal{M}_{d\times d}$ are $\mathcal{C}^\infty$, bounded with bounded partial derivatives of all orders. We also assume that the matrix σ is uniformly elliptic, that is, there exists $\rho > 0$ such that

$$\inf_{\xi \in \mathbb{R}^d : |\xi| = 1} |\sigma(y)\xi|^2 \geq \rho > 0.$$

Under theses conditions it is well-known that there exists a unique càdlàg adapted Markov process $X^x = (X_t^x)_{t\geq 0}$ solution to the integral equation (4.1), see [4]. Moreover, for all $t > 0$ the random vector X_t^x possesses a density $f_t(x,\cdot)$ with respect to the Lebesgue measure on $\mathbb{R}^d$, see [2].

As in the one-dimensional case, we denote by Z_t^x the solution to equation (4.1) with $Y \equiv 0$ and by $p_t(x,\cdot)$ its probability density function. Then, it is well-known that for all $T > 0$, there exist constants $A_T, a_T > 1$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,

$$\frac{1}{A_T(2\pi t)^{d/2}} e^{-\frac{a_T|y-x|^2}{2t}} \leq p_t(x,y) \leq \frac{A_T}{(2\pi t)^{d/2}} e^{-\frac{|y-x|^2}{2a_T t}}. \tag{4.2}$$

We also denote by $q_t(x,\cdot)$ the probability density function of $Z_t^x + Y$ and we observe that for all $x, y \in \mathbb{R}^d$,

$$q_t(x,y) = (p_t * \varphi)(x,y) = \int_{\mathbb{R}^d} \varphi(y-v) p_t(x,v) dv. \tag{4.3}$$

The expression for the density obtained in Proposition 2.1 can be easily extended in this multidimensional setting as follows. The proof follows exactly as in the one-dimensional case.

Proposition 4.1. *For any $t > 0$ and $x, y \in \mathbb{R}^d$,*

$$\begin{aligned} f_t(x,y) &= p_t(x,y)e^{-\lambda t} \\ &\quad + \sum_{n=1}^{\infty} \int_{t_1 < \cdots < t_n < t < t_{n+1}} (q_{t_1-t_0} \star (\cdots \star (q_{t_n - t_{n-1}} \star p_{t-t_n}) \cdots))(x,y) \lambda^{n+1} e^{-\lambda t_{n+1}} dt_1 \cdots dt_{n+1}, \end{aligned}$$

where given two measurable functions $g(x,y)$ and $k(x,y)$ in $\mathbb{R}^d \times \mathbb{R}^d$ we define the product

$$(g \star k)(x,y) = \int_{\mathbb{R}^d} g(x,v) k(v,y) dv.$$

The lower bounds of Theorems 1.1 and 1.2 also extended to equation (4.1) as follows.

Theorem 4.2. *Assume that φ is the centered d-dimensional Gaussian density with covariance matrix Σ. Then for all $T > 0$ there exist constants $C_T, c_T > 1$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,*

$$f_t(x,y) \geq C_T^{-1} \left(e^{-c_T |y-x| \sqrt{\ln_+(\frac{|y-x|}{t})}} + \frac{\mathbf{1}_{x=y}}{t^{d/2}} \right).$$

Proof. Using (4.3) and convolution properties for Gaussian densities with the lower bound in (4.2) yields, for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,

$$q_t(x,y) \geq \frac{1}{A_T (2\pi a_T)^{d/2} \sqrt{|\det(\Sigma + a_T^{-1} t I)|}} e^{-\frac{(y-x)^T (\Sigma + a_T^{-1} t I)^{-1} (y-x)}{2}},$$

where I denotes the identity matrix of order $d \times d$. Therefore, using Proposition 4.1, we obtain in a similar manner that

$$\begin{aligned} f_t(x,y) &\geq e^{-\lambda t} \sum_{n=0}^{\infty} C_T^n \frac{C_T}{(2\pi)^{d/2} \sqrt{|\det(n\Sigma + a_T^{-1} t I)|}} e^{-\frac{(y-x)^T (n\Sigma + a_T^{-1} t I)^{-1} (y-x)}{2}} \frac{(\lambda t)^n}{n!} \\ &\geq e^{-\lambda t} \sum_{n=1}^{\infty} C_T^n \frac{C_T}{(2\pi)^{d/2} n^{d/2} \sqrt{|\det(\Sigma + a_T^{-1} T I)|}} e^{-\frac{r^2 \|(\Sigma)^{-1}\|}{2n}} \frac{(\lambda t)^n}{n!}, \end{aligned}$$

where $r := |y-x|$, $C_T = \frac{1}{A_T (a_T)^{d/2}}$, and $\|(\Sigma)^{-1}\| = \sup_{z \in \mathbb{R}^d : z \neq 0} \frac{|(\Sigma)^{-1} z|}{|z|}$.

The rest of the proof follows exactly along the same lines as in the one-dimensional case. ∎

Theorem 4.3. *Assume that φ is the multivariate centered Laplace density with $\mu = (\mu_1, \dots, \mu_d)$, $\mu_i > 0$, given by*

$$\varphi(z) = \prod_{i=1}^{d} \frac{1}{2} \mu_i e^{-\mu_i |z_i|}, \quad z = (z_1, \dots, z_d).$$

Then for all $T > 0$ there exist constants $C_T, c_T > 1$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,

$$f_t(x, y) \geq C_T^{-1} \left(e^{-c_T |y-x|} + \frac{\mathbf{1}_{x=y}}{t^{d/2}} \right).$$

Proof. We start proving a lower bound for $q_t(x, y)$. Using (4.3) and the lower bound in (4.2), we get that for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,

$$q_t(x, y) \geq \frac{\prod_{i=1}^{d} \mu_i}{2^d A_T} \int_{\mathbb{R}^d} e^{-\sum_{i=1}^{d} \mu_i |z_i|} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|z-(y-x)|^2}{2a_T^{-1} t}} dz.$$

We next show by induction on $d \geq 1$ that for $x \neq y$

$$\int_{\mathbb{R}^d} e^{-\sum_{i=1}^{d} \mu_i |z_i|} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|z-(y-x)|^2}{2a_T^{-1} t}} dz \geq \left(\frac{1}{2\sqrt{2a_T}} \right)^d e^{-\frac{a_T^{-1} t |\mu|^2}{2}} e^{-2\sum_{i=1}^{d} |y_i - x_i| \mu_i}. \tag{4.4}$$

When $d = 1$, it is shown in (3.4) that (4.4) holds. Moreover, the left hand side in (4.4) is equal to

$$\int_{\mathbb{R}^{d-1}} e^{-\sum_{i=1}^{d-1} \mu_i |z_i|} \frac{1}{(2\pi t)^{(d-1)/2}} e^{-\frac{\sum_{i=1}^{d-1} (z_i - (y_i - x_i))^2}{2a_T^{-1} t}} dz_1 \cdots dz_{d-1} \int_{\mathbb{R}} e^{-\mu_d |z_d|} \frac{1}{(2\pi t)^{1/2}} e^{-\frac{(z_d - (y_d - x_d))^2}{2a_T^{-1} t}} dz_d.$$

Thus, using the induction hypothesis for the first integral and the one-dimensional case for the second, we conclude that (4.4) holds true.

Therefore, we have shown that

$$q_t(x, y) \geq C_T e^{-\frac{a_T^{-1} t |\mu|^2}{2}} e^{-2\sum_{i=1}^{d} |y_i - x_i| \mu_i},$$

where $C_T := \frac{\prod_{i=1}^{d} \mu_i}{2^d A_T} \left(\frac{1}{2\sqrt{2a_T}} \right)^d$.

The rest of the proof follows exactly as in the proof of the one-dimensional case. That is, appealing to Proposition 4.1 we obtain that for some positive constants $\overline{C}_T$ and c_T,

$$f_t(x, y) \geq \overline{C}_T e^{-(c_T + \lambda) t} e^{-4\sum_{i=1}^{d} |y_i - x_i| \mu_i} \geq \overline{C}_T e^{-(c_T + \lambda) T} e^{-4|y-x||\mu|}.$$

This completes the proof. $\square$

Concerning upper tail bounds, we observe that although Proposition 2.5 can be easily extended, Proposition 2.3 uses a one-dimensional argument which cannot be easily extended to the multidimensional setting. Thus, we leave it for further work.

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Arturo Kohatsu-Higa, Department of Mathematical Sciences, Ritsumeikan University, 1-1-1 Nojihigashi, Kusatsu, Shiga, 525-8577, Japan

Email address: `khts00@fc.ritsumei.ac.jp`

Eulalia Nualart, Department of Economics and Business, Universitat Pompeu Fabra and Barcelona Graduate School of Economics, Ramón Trias Fargas 25-27, 08005 Barcelona, Spain.

Email address: `eulalia.nualart@upf.edu`

Ngoc Khue Tran, Department of Natural Science Education, Pham Van Dong University, 509 Phan Dinh Phung, Quang Ngai City, Quang Ngai, Vietnam

Email address: `tnkhue@pdu.edu.vn`