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Gaussian estimates for the density of the non-linear stochastic heat equation in any space dimension

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Abstract

In this paper, we establish lower and upper Gaussian bounds for the probability density of the mild solution to the non-linear stochastic heat equation in any space dimension. The driving perturbation is a Gaussian noise which is white in time with some spatially homogeneous covariance. These estimates are obtained using tools of the Malliavin calculus. The most challenging part is the lower bound, which is obtained by adapting a general method developed by Kohatsu-Higa to the underlying spatially homogeneous Gaussian setting. Both lower and upper estimates have the same form: a Gaussian density with a variance which is equal to that of the mild solution of the corresponding linear equation with additive noise.

MSC: 60H15; 60H07

Keywords: Gaussian density estimates; Malliavin calculus; Spatially homogeneous Gaussian noise; Stochastic heat equation

1. Introduction and main result

In this paper, we aim to establish Gaussian lower and upper estimates for the probability density of the solution to the following stochastic heat equation in $\mathbb{R}^d$:

$$\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^d, \tag{1.1}$$

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with initial condition $u(0, x) = u_0(x)$, $x \in \mathbb{R}^d$. Here, $T > 0$ stands for a fixed time horizon, the coefficients $\sigma, b : \mathbb{R} \to \mathbb{R}$ are smooth functions and $u_0 : \mathbb{R}^d \mapsto \mathbb{R}$ is assumed to be measurable and bounded. As far as the driving perturbation is concerned, we will assume that $\dot{W}(t, x)$ is a Gaussian noise which is white in time and has a spatially homogeneous covariance. This can be formally written as:

$$\mathrm{E}\,[\dot{W}(t, x)\dot{W}(s, y)] = \delta_0(t - s)\Lambda(x - y), \tag{1.2}$$

where δ_0 denotes the Dirac delta function at zero and Λ is some tempered distribution on $\mathbb{R}^d$ which is the Fourier transform of a non-negative tempered measure μ on $\mathbb{R}^d$ (the rigorous definition of this Gaussian noise will be given in Section 2.1). The measure μ is usually called the *spectral measure* of the noise W.

The solution to Eq. (1.1) will be understood in the *mild* sense, as follows. Let $(\mathscr{F}_t)_{t\geq 0}$ denote the filtration generated by the spatially homogeneous noise W (see again Section 2.1 for its precise definition). We say that an $\mathscr{F}_t$-adapted process $\{u(t, x),\ (t, x) \in [0, T] \times \mathbb{R}^d\}$ solves (1.1) if it satisfies:

$$\begin{aligned} u(t, x) = (\Gamma(t) * u_0)(x) &+ \int_0^t \int_{\mathbb{R}^d} \Gamma(t - s, x - y)\sigma(u(s, y))W(ds, dy) \\ &+ \int_0^t \int_{\mathbb{R}^d} \Gamma(t - s, x - y)b(u(s, y))\, dyds, \end{aligned} \tag{1.3}$$

where $*$ is the standard convolution product in $\mathbb{R}^d$, and Γ denotes the fundamental solution associated to the heat equation on $\mathbb{R}^d$, that is, the Gaussian kernel of variance $2t$: $\Gamma(t, x) = (4\pi t)^{-\frac{d}{2}} \exp(-\frac{\|x\|^2}{4t})$, for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Note that the stochastic integral on the right-hand side of (1.3) can be understood either in the sense of Walsh [30], or using the further extension of Dalang [4] (see also [21,7] for another equivalent approach). In any case, one needs to assume the following condition:

$$\Phi(T) := \int_0^T \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(t)(\xi)|^2 \mu(d\xi)dt < +\infty. \tag{1.4}$$

Then, [4, Theorem 13] and [7, Theorem 4.3] imply that Eq. (1.3) has a unique solution which is L^2-continuous and satisfies, for all $p \geq 1$:

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E[|u(t, x)|^p] < +\infty.$$

Observe that $\Phi(T)$ measures the variance of the stochastic integral in (1.3) (indeed, it is the variance itself when $\sigma \equiv 1$), therefore it is natural that it will play an important role in the Gaussian lower and upper bounds for the density of the random variable $u(t, x)$. Moreover, it has been proved in [4, Example 2] that condition (1.4) is satisfied if and only if:

$$\int_{\mathbb{R}^d} \frac{1}{1 + \|\xi\|^2}\, \mu(d\xi) < +\infty. \tag{1.5}$$

However, as it will be explained below, in order to prove our main result we shall need a slightly stronger condition than (1.5) above. Namely, we will assume that there exists $\eta \in (0, 1)$ such that:

$$\int_{\mathbb{R}^d} \frac{1}{(1+\|\xi\|^2)^\eta}\,\mu(d\xi) < +\infty. \tag{1.6}$$

We also remark that the stochastic heat equation (1.1) has also been studied in the more abstract framework of Da Prato and Zabczyk [8] and, in this sense, we refer the reader to [24] and references therein. Nevertheless, in the case of our spatially homogeneous noise, the solution in that more abstract setting could be obtained from the solution to Eq. (1.3) (see [7, Sec. 4.5]).

The techniques of the Malliavin calculus have been applied to Eq. (1.3) in the papers [14,21]. Precisely, [21, Theorem 6.2] states that, if the coefficients b and σ are $\mathscr{C}^\infty$-functions with bounded derivatives of order greater than or equal to one, the diffusion coefficient is non-degenerate (i.e. $|\sigma(z)| \geq c > 0$ for all $z \in \mathbb{R}$), and (1.5) is satisfied, then for each $(t,x) \in (0,T] \times \mathbb{R}^d$, the random variable $u(t,x)$ has a $\mathscr{C}^\infty$ density $p_{t,x}$ (see also Theorem 3.3). Moreover, in the recent paper [18], the strict positivity of this density has been established under a $\mathscr{C}^1$-condition on the density and the additional condition of σ being bounded.

Our aim in this paper is to go a step further and prove the following theorem.

Theorem 1.1. *Assume that condition (1.6) is satisfied and $\sigma, b \in \mathscr{C}_b^\infty(\mathbb{R})$ ($\mathscr{C}^\infty$, bounded and bounded derivatives). Moreover, suppose that $|\sigma(z)| \geq c > 0$, for all $z \in \mathbb{R}$. Then, for every $(t,x) \in (0,T] \times \mathbb{R}^d$, the law of the random variable $u(t,x)$ has a $\mathscr{C}^\infty$ density $p_{t,x}$ satisfying, for all $y \in \mathbb{R}$:*

$$C_1 \Phi(t)^{-1/2} \exp\left(-\frac{|y-F_0|^2}{C_2 \Phi(t)}\right) \leq p_{t,x}(y) \leq c_1 \Phi(t)^{-1/2} \exp\left(-\frac{(|y-F_0| - c_3 T)^2}{c_2 \Phi(t)}\right),$$

where $F_0 = (\Gamma(t) * u_0)(x)$ and c_1, c_2, c_3, C_1, C_2 are positive constants that only depend on T, σ and b.

One of the interests of these type of bounds is to understand the behavior of the density when y is large and t is small. In both cases, one obtains the same upper and lower behavior for the density, that is, a Gaussian density with a variance which is equal to that of the stochastic integral term in the mild form of the linear equation. We observe that this variance does not depend on x due to the spatially homogeneous structure of the noise.

In order to prove our main result, we will apply the techniques of the Malliavin calculus, for which we refer the reader to [19,28]. Obtaining lower and upper Gaussian bounds for solutions to non-linear stochastic equations using the Malliavin calculus has been a current subject of research in the past two decades. Precisely, the expression for the density arising from the integration-by-parts formula of the Malliavin calculus provides a direct way for obtaining an upper Gaussian-type bound for the density. Indeed, ones applies Hölder's inequality, and then combines the exponential martingale inequality together with estimates for the Malliavin norms of the derivative and the Malliavin matrix. This is a well-known method that has been applied in many situations (see for instance [9,5]). We will also apply this technique to the density of our stochastic heat equation in order to show the upper bound in Theorem 1.1 (see Section 5).

On the other hand, to obtain Gaussian lower bounds for some classes of Wiener functionals turns out to be a more difficult and challenging issue. In this sense, the pioneering work is the article by Kusuoka and Stroock [12], where the techniques of the Malliavin calculus have been applied to obtain a Gaussian lower estimate for the density of a uniformly hypoelliptic diffusion whose drift is a smooth combination of its diffusion coefficient. Later on, in [10], Kohatsu-Higa took some of Kusuoka and Stroock's ideas and constructed a general method to prove that the density of a multidimensional functional of the Wiener sheet in $[0, T] \times \mathbb{R}^d$ admits a Gaussian-type lower bound. Then, still in [10], the author applies his method to a one-dimensional stochastic heat equation in [0, 1] driven by the space–time white noise, and obtains a lower estimate for the density of the form:

$$C_1\, t^{-\frac{1}{4}} \exp\left(-\frac{|y - F_0|^2}{C_2\, t^{\frac{1}{2}}}\right),$$

where F_0 denotes the contribution of the initial condition. This is the bound we would get if in Eq. (1.1) we let $d = 1$ and $\dot{W}$ be the space–time white noise. Indeed, this case corresponds to take $\Lambda = \delta_0$ in (1.2); therefore, the spectral measure μ is the Lebesgue measure on $\mathbb{R}^d$ and $\Phi(t) = C\, t^{\frac{1}{2}}$. As we will explain below, in the present paper we will use Kohatsu-Higa's method adapted to our spatially homogeneous Gaussian setting. The same author applied his method in [11] to obtain Gaussian lower bounds for the density of a uniformly elliptic non-homogeneous diffusion. Another important case to which the method of [10] has been applied corresponds to a two-dimensional diffusion, which is equivalent to deal with a reduced stochastic wave equation in spatial dimension one, a problem which has been tackled in [6]. Moreover, let us also mention that the ideas of [10] have been further developed by Bally in [1] in order to deal with more general diffusion processes, namely locally elliptic Itô processes, and this has been applied for instance in [9]. Furthermore, in [3], Bally and Kohatsu-Higa have recently combined their ideas in order to obtain lower bounds for the density of a class of hypoelliptic two-dimensional diffusions, with some applications to mathematical finance.

The increasing interest in finding Gaussian lower estimates for Wiener functionals has produced three very recent new approaches, all based again on Malliavin calculus techniques. First, in [17] the authors provide sufficient conditions on a random variable in the Wiener space such that its density exists and admits an explicit formula, from which one can study possible Gaussian lower and upper bounds. This result has been applied in [22,23] to our stochastic heat equation (1.1) in the case where $\sigma \equiv 1$. Precisely, [23, Theorem 1 and Example 8] imply that, if b is of class $\mathscr{C}^1$ with bounded derivative and condition (1.5) is fulfilled, then, for sufficiently small t, $u(t, x)$ has a density $p_{t,x}$ satisfying, for almost all $z \in \mathbb{R}$:

$$\begin{aligned}\frac{E|u(t,x) - M_{t,x}|}{C_2\, \Phi(t)} \exp\left(-\frac{|z - M_{t,x}|^2}{C_1\, \Phi(t)}\right) &\le p_{t,x}(z)\\ &\le \frac{E|u(t,x) - M_{t,x}|}{C_1\, \Phi(t)} \exp\left(-\frac{|z - M_{t,x}|^2}{C_2\, \Phi(t)}\right),\end{aligned}$$

where $M_{t,x} = E(u(t, x))$ (see [22, Theorem 4.4] for a similar result which is valid for all t but is not optimal). Compared to Theorem 1.1, on the one hand, we point out that our result is valid for a general σ, arbitrary time $T > 0$ and our estimates look somehow more Gaussian. On the other hand, the general method that we present in Section 2.2 requires the underlying random variable to be smooth in the Malliavin sense, and this forces to consider a smooth coefficient b. Moreover, we have considered condition (1.6) instead of (1.5). We also remark that, even though the results of [23] are also valid for a more general class of SPDEs with additive noise (such as the stochastic wave equation in space dimension $d \in \{1, 2, 3\}$), Nourdin and Viens' method does not seem to be suitable for multiplicative noise settings.

A second recent method for deriving Gaussian-type lower estimates for multidimensional Wiener functionals has been obtained by Malliavin and Nualart in [13] (see [20] for the one-dimensional counterpart). This technique is based on an exponential moment condition on the divergence of a covering vector field associated to the underlying Wiener functional, and has been applied in [20] to a one-dimensional diffusion.

Last, but not least, in the recent paper [2], Bally and Caramellino develop another method to obtain lower bounds for multidimensional Wiener functionals based on the Riesz transform.

These two last methods have not been applied yet to solutions of SPDEs. However, the form of the Gaussian-type lower bounds that one obtains using these other techniques are less explicit than the lower bound that one gets using Kohatsu-Higa's result. This is why in the present paper, we will apply the methodology of Kohatsu-Higa [10]. For this, first we will need to extend the general result [10, Theorem 5] on Gaussian lower bounds for *uniformly elliptic random vectors* from the space–time white noise framework to the case of functionals of our Gaussian spatially homogeneous noise (see Theorem 2.3). This will be done in Section 2, after having precisely described the Gaussian setting which we will work in. The extension of Kohatsu-Higa's result has been done, first, in such a way that the definition of *uniformly elliptic random vector* has been simplified in the following sense.

(i) We have reduced the number of conditions required to a random vector to be uniformly elliptic by adjusting them to the particular application that we have in mind, which is the solution of our stochastic heat equation (see Definition 2.2).
(ii) In particular, we do not need to extend the underlying probability space with a family of Wiener increments independent of the noise W, as it has been done in [10] (see p. 424–425 therein). Moreover, in our Definition 2.2, we only need to consider one approximation sequence F_n, while in [10], a kind of double approximation procedure has to be settled.

Then, once this definition is established, the general result on lower Gaussian bounds can be proved in a rather self-contained way, and the proof turns out to be shorter than the one of [10, Theorem 5]. It should be mentioned, however, that a direct extension of the latter result to our Gaussian setting, though much more complicated to state and with a more involved proof, would be applicable to a larger class of Wiener functionals.

In Section 3, we will recall the main results on differentiability in the Malliavin sense and the existence and smoothness of the density applied to our stochastic heat equation (1.3). Moreover, we will prove a technical and useful result which provides a uniform estimate for the conditional norm of the iterated Malliavin derivative of the solution on a small time interval.

Section 4 is devoted to apply the general result Theorem 2.3 to the stochastic heat equation (1.3), to end up with the lower bound in Theorem 1.1. That is, one needs to show that the solution $u(t,x)$ defines a uniformly elliptic random variable in the sense of Definition 2.2. The fact that this definition, though stated in a rather general setting, has been somehow adapted to the application to our stochastic heat equation, makes the proof's framework slightly simpler than that of [10, Theorem 10]. In particular, we have not been forced to consider a Taylor expansion of the solution in some stochastic integral terms, which would give rise to a collection of cumbersome high-order processes and residues. Nevertheless, in our setting one needs to take care of the more general spatial covariance structure of the underlying Wiener noise.

At a technical level, let us justify now why assuming condition (1.6) does not need to be considered as an important restriction. Precisely, the latter condition implies that

$$\int_0^t \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(s)(\xi)|^2 \mu(d\xi) ds \leq C\, t^{1-\eta}. \tag{1.7}$$

Then, as it will be made explicit in Section 4, this estimate turns out to be crucial in order to prove the lower bound in Theorem 1.1. In fact, the above condition was already important in the proof of [10, Theorem 10] when dealing with the one-dimensional stochastic heat equation driven by the space–time white noise. In this case, condition (1.7) would be written in the form:

$$\int_0^t \int_0^1 \Gamma(s,\xi)^2 \, d\xi ds \leq C\, t^{1/2},$$

which obviously does not require any assumption on the spectral measure (since it would be the Lebesgue measure on $\mathbb{R}^d$). In our setting, however, it seems natural not to get (1.7) for free, since we are dealing with a fairly general spatial covariance.

As mentioned before, the upper bound in Theorem 1.1 will be proved in Section 5. Finally, we have also added an Appendix where we recall some facts concerning Hilbert-space-valued stochastic and pathwise integrals and their conditional moment estimates.

As usual, we shall denote by c, C any positive constants whose dependence will be clear from the context and their values may change from one line to another.

2. General theory on lower bounds for densities

This section is devoted to extend Kohatsu-Higa's result [10, Theorem 5] on lower bounds for the density of a uniformly elliptic random vector to a more general Gaussian space, namely the one determined by a Gaussian random noise on $[0, T] \times \mathbb{R}^d$ which is white in time and has a non-trivial homogeneous structure in space. For this, first we will rigorously introduce the Gaussian noise and the Malliavin calculus framework associated to it and needed in the sequel.

2.1. Gaussian context and Malliavin calculus

Our spatially homogeneous Gaussian noise is described as follows. On a complete probability space $(\Omega, \mathscr{F}, \mathrm{P})$, let $W = \{W(\varphi), \varphi \in \mathscr{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$ be a zero mean Gaussian family of random variables indexed by $\mathscr{C}^\infty$ functions with compact support with covariance functional given by

$$\mathrm{E}[W(\varphi)W(\psi)] = \int_0^\infty dt \int_{\mathbb{R}^d} \Lambda(dx)(\varphi(t,\star) * \tilde{\psi}(t,\star))(x),$$
$$\varphi, \psi \in \mathscr{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d). \tag{2.1}$$

Here, Λ denotes a non-negative and non-negative definite tempered measure on $\mathbb{R}^d$, $*$ stands for the convolution product, the symbol $\star$ denotes the spatial variable and $\tilde{\psi}(t,x) := \psi(t,-x)$. For such a Gaussian process to exist, it is necessary and sufficient that the covariance functional is non-negative definite and this is equivalent to the fact that Λ is the Fourier transform of a non-negative tempered measure μ on $\mathbb{R}^d$ (see [29, Chap. VII, Théorème XVII]). The measure μ is usually called the spectral measure of the noise W. By definition of the Fourier transform of tempered distributions, $\Lambda = \mathscr{F}\mu$ means that, for all ϕ belonging to the space $\mathscr{S}(\mathbb{R}^d)$ of rapidly decreasing $\mathscr{C}^\infty$ functions,

$$\int_{\mathbb{R}^d} \phi(x)\Lambda(dx) = \int_{\mathbb{R}^d} \mathscr{F}\phi(\xi)\mu(d\xi).$$

Moreover, for some integer $m \geq 1$ it holds that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+\|\xi\|^2)^m} < +\infty.$$

Elementary properties of the convolution and Fourier transform show that covariance (2.1) can be written in terms of the measure μ, as follows:

$$\mathrm{E}[W(\varphi)W(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\varphi(t)(\xi)\overline{\mathcal{F}\psi(t)(\xi)}\mu(d\xi)dt.$$

In particular, we obtain that

$$\mathrm{E}[W(\varphi)^2] = \int_0^\infty \int_{\mathbb{R}^d} |\mathcal{F}\varphi(t)(\xi)|^2\mu(d\xi)dt.$$

Example 2.1. Assume that the measure Λ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$ with density f. Then, the covariance functional (2.1) reads

$$\int_0^\infty dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \varphi(t,x) f(x-y)\psi(t,y),$$

which clearly exhibits the spatially homogeneous nature of the noise. The space–time white noise would correspond to the case where f is the Dirac delta at the origin.

We note that the kind of noise above-defined has been widely used as a random perturbation for several classes of SPDEs (see for instance [15,4,24,28]).

At this point, we can describe the Gaussian framework which is naturally associated to our noise W. Precisely, let $\mathcal{H}$ be the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ endowed with the semi-inner product

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} (\phi_1 * \tilde{\phi_2})(x)\, \Lambda(dx) = \int_{\mathbb{R}^d} \mathcal{F}\phi_1(\xi)\overline{\mathcal{F}\phi_2(\xi)}\, \mu(d\xi),$$
$$\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d).$$

Notice that the Hilbert space $\mathcal{H}$ may contain distributions (see [4, Example 6]). Fix $T > 0$ and define $\mathcal{H}_T = L^2([0,T];\mathcal{H})$. Then, the family W can be extended to $\mathcal{H}_T$, so that we end up with a family of centered Gaussian random variables, still denoted by $W = \{W(g), g \in \mathcal{H}_T\}$, satisfying that $\mathrm{E}[W(g_1)W(g_2)] = \langle g_1, g_2 \rangle_{\mathcal{H}_T}$, for all $g_1, g_2 \in \mathcal{H}_T$ (see for instance [7, Lemma 2.4] and the explanation thereafter).

The family W defines an isonormal Gaussian process on the Hilbert space $\mathcal{H}_T$ and we shall use the differential Malliavin calculus based on it (see, for instance, [19,28]). We denote the Malliavin derivative by D, which is a closed and unbounded operator defined in $L^2(\Omega)$ and taking values in $L^2(\Omega;\mathcal{H}_T)$, whose domain is denoted by $\mathbb{D}^{1,2}$. More general, for any $m \geq 1$, the domain of the iterated derivative D^m in $L^p(\Omega)$ is denoted by $\mathbb{D}^{m,p}$, for any $p \geq 2$, and we recall that D^m takes values in $L^p(\Omega;\mathcal{H}_T^{\otimes m})$. As usual, we set $\mathbb{D}^\infty = \cap_{p\geq 1}\cap_{m\geq 1}\mathbb{D}^{m,p}$. The space $\mathbb{D}^{m,p}$ can also be seen as the completion of the set of smooth functionals with respect to the semi-norm

$$\|F\|_{m,p} = \left\{ \mathrm{E}[|F|^p] + \sum_{j=1}^m \mathrm{E}[\|D^j F\|^p_{\mathcal{H}_T^{\otimes j}}] \right\}^{\frac{1}{p}}.$$

For any differentiable random variable F and any $r = (r_1, \ldots, r_m) \in [0, T]^m$, $D^m F(r)$ is an element of $\mathscr{H}^{\otimes m}$ which will be denoted by $D^m_r F$.

We define the Malliavin matrix of a k-dimensional random vector $F \in (\mathbb{D}^{1,2})^k$ by $\gamma_F = (\langle DF_i, DF_j\rangle_{\mathscr{H}_T})_{1\le i,j\le k}$. We will say that a k-dimensional random vector F is smooth if each of its components belongs to $\mathbb{D}^\infty$, and we will say that a smooth random vector F is non-degenerate if $(\det \gamma_F)^{-1} \in \cap_{p\ge 1} L^p(\Omega)$. It is well-known that a non-degenerate random vector has a $\mathscr{C}^\infty$ density (cf. [19, Theorem 2.1.4]).

Let $(\mathscr{F}_t)_{t\ge 0}$ denote the σ-field generated by the random variables $\{W_s(h), h \in \mathscr{H}, 0 \le s \le t\}$ and the P-null sets, where $W_t(h) := W(1_{[0,t]}h)$, for any $t \ge 0$, $h \in \mathscr{H}$. Notice that this family defines a standard cylindrical Wiener process on the Hilbert space $\mathscr{H}$. We define the predictable σ-field as the σ-field in $\Omega \times [0, T]$ generated by the sets $\{(s, t] \times A, 0 \le s < t \le T, A \in \mathscr{F}_s\}$.

As in [10, Section 2], one can define the conditional versions of the above Malliavin norms and spaces (see also [16,6]). For all $0 \le s < t \le T$, we set $\mathscr{H}_{s,t} = L^2([s, t]; \mathscr{H})$ and also $\|\cdot\|_{s,t} := \|\cdot\|_{\mathscr{H}_{s,t}}$. For any integer $m \ge 0$ and $p > 1$, we define the seminorm:

$$\|F\|^{s,t}_{m,p} = \left\{ \mathrm{E}_s[|F|^p] + \sum_{j=1}^m \mathrm{E}_s[\|D^j F\|^p_{\mathscr{H}^{\otimes j}_{s,t}}] \right\}^{\frac{1}{p}},$$

where $\mathrm{E}_s[\cdot] = \mathrm{E}[\cdot|\mathscr{F}_s]$. We will also write $\mathrm{P}_s\{\cdot\} = \mathrm{P}\{\cdot|\mathscr{F}_s\}$. Completing the space of smooth functionals with respect to this seminorm we obtain the space $\mathbb{D}^{m,p}_{s,t}$. We write $L^p_{s,t}(\Omega)$ for $\mathbb{D}^{0,p}_{s,t}$. We say that $F \in \overline{\mathbb{D}}^{m,p}_{s,t}$ if $F \in \mathbb{D}^{m,p}_{s,t}$ and $\|F\|^{s,t}_{m,p} \in \cap_{q\ge 1} L^q(\Omega)$, and we set $\overline{\mathbb{D}}^\infty_{s,t} := \cap_{p\ge 1} \cap_{m\ge 1} \overline{\mathbb{D}}^{m,p}_{s,t}$. Furthermore, we define the conditional Malliavin matrix associated to a k-dimensional random vector F by $\gamma^{s,t}_F := (\langle DF_i, DF_j\rangle_{\mathscr{H}_{s,t}})_{1\le i,j\le k}$.

2.2. The general result

In order to state the main result of this section, we need to define what we understand by a *uniformly elliptic random vector* in our context. Note that, as mentioned in the Introduction, we present here a simpler definition in comparison to that given by Kohatsu-Higa in [10] (see the hypotheses of Theorem 5 therein). Nevertheless, it covers the Gaussian setting described in the previous section and, as it will be made clear in Section 4, it will allow us to deal with the solution to the stochastic heat equation.

Definition 2.2. Let F be an $\mathscr{F}_t$-measurable non-degenerate k-dimensional random vector. We say that F is *uniformly elliptic* if the following is satisfied.

There exists an element $g \in \mathscr{H}_T$ such that $\|g(s)\|_{\mathscr{H}} > 0$ for almost all s, and an $\epsilon > 0$ such that, for any sequence of partitions $\pi_N = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ whose norm is smaller than ϵ and $\|\pi_N\| = \sup\{t_n - t_{n-1}, n = 1, \ldots, N\}$ converges to zero as $N \to \infty$, there exists a sequence of smooth random vectors $F_0, F_1, \ldots, F_N$ such that $F_N = F$, each F_n is $\mathscr{F}_{t_n}$-measurable and belongs to $\overline{\mathbb{D}}^\infty_{t_{n-1},t_n}$, and for any $n \in \{1, \ldots, N\}$, F_n can be written in the form:

$$F_n = F_{n-1} + I_n(h) + G_n, \tag{2.2}$$

where the random vectors $I_n(h)$ and G_n satisfy the following.

(H1) G_n is an $\mathscr{F}_{t_n}$-measurable random vector that belongs to $\overline{\mathbb{D}}^{\infty}_{t_{n-1},t_n}$, and satisfies that for some $\delta > 0$ and all $m \in \mathbb{N}$ and $p \geq 1$,

$$\|G_n\|^{t_{n-1},t_n}_{m,p} \leq C\, \Delta_{n-1}(g)^{1/2+\delta} \quad \text{a.s.} \tag{2.3}$$

where

$$0 < \Delta_{n-1}(g) := \int_{t_{n-1}}^{t_n} \|g(s)\|^2_{\mathscr{H}} ds < \infty, \quad n = 1, \ldots, N.$$

(H2) $I_n(h)$ denotes a random vector whose components are in the form

$$I_n^{\ell}(h) = \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} h_{\ell}(s, y) W(ds, dy), \quad \ell = 1, \ldots, k,$$

where, for each ℓ, h_ℓ is a smooth $\mathscr{F}_{t_{n-1}}$-predictable process with values in $\mathscr{H}_{t_{n-1},t_n}$ and, for any $m \in \mathbb{N}$ and $p \geq 1$, there exists a constant C such that

$$\|F_n\|_{m,p} + \sup_{\omega \in \Omega} \|h_\ell\|_{t_{n-1},t_n}(\omega) \leq C, \tag{2.4}$$

for any $\ell = 1, \ldots, k$.

(H3) Let $A = (a_{\ell,q})$ denote the $k \times k$ matrix defined by

$$a_{\ell,q} = \Delta_{n-1}(g)^{-1} \int_{t_{n-1}}^{t_n} \langle h_\ell(s), h_q(s) \rangle_{\mathscr{H}} ds.$$

There exist strictly positive constants C_1 and C_2 such that, for all $\zeta \in \mathbb{R}^k$,

$$C_1 \zeta^T \zeta \geq \zeta^T A \zeta \geq C_2 \zeta^T \zeta, \quad \text{a.s.} \tag{2.5}$$

(H4) There is a constant C such that, for any $p > 1$ and all $\rho \in (0, 1]$:

$$\{\mathrm{E}_{t_{n-1}}(\det(\gamma^{t_{n-1},t_n}_{I_n(h)+\rho G_n})^{-p})\}^{1/p} \leq C\, \Delta_{n-1}(g)^{-k} \quad \text{a.s.} \tag{2.6}$$

Note that Hypothesis (2.5) is the ingredient that most directly reflects the *uniformly elliptic* condition for a random vector on the Wiener space.

The next theorem establishes a Gaussian lower bound for the probability density of a uniformly elliptic random vector. Its proof turns out to be shorter and slightly simpler than that of Theorem 5 in [10], where the same type of result has been proved in a Gaussian setting associated to the Hilbert space $L^2([0, T] \times A)$, where $A \subseteq \mathbb{R}^d$ (that is, the space–time white noise). We recall that, in our case, we are dealing with the Gaussian setting associated to the space $\mathscr{H}_T = L^2([0, T]; \mathscr{H})$, as described in the preceding section.

Theorem 2.3. *Let F be a k-dimensional uniformly elliptic random vector and denote by p_F its probability density. Then, there exists a constant $M > 0$ that depends on all the constants of Definition 2.2 such that*

$$p_F(y) \geq M \|g\|^{-k/2}_{\mathscr{H}_t} \exp\left(-\frac{\|y - F_0\|^2}{M \|g\|^2_{\mathscr{H}_t}}\right), \quad \textit{for all } y \in \mathbb{R}^k,$$

where F_0 is the first element in the sequence (2.2).

Proof. The proof is divided in four steps.

Step 1. Observe that the hypotheses on F_n, in particular condition (2.6) with $\rho = 1$, imply that each F_n has a smooth density, conditionally to $\mathscr{F}_{t_{n-1}}$. We will denote this density by $p_n : \mathbb{R}^k \to \mathbb{R}$. Notice that, in particular, $p_F = p_N$.

Set $r_n := \Delta_n(g)^{1/2}$. This part of the proof is devoted to show that if $\|y - F_{n-1}\|^2 \le c\, r_{n-1}^2$ for some constant $c > 0$, then there exist positive constants M, η_0 such that if $r_{n-1}^2 \le \eta_0$, then

$$p_n(y) \ge \frac{1}{M r_{n-1}^k} \quad \text{a.s.} \tag{2.7}$$

The proof of (2.7) follows along the same lines as the one of [10, Theorem 5]. However, the assumptions stated in Definition 2.2 allow us to shorten some of the main parts. Precisely, one first renormalizes the density as follows:

$$p_n(y) = E_{t_{n-1}}(\delta_y(F_n)) = r_{n-1}^{-k}\, E_{t_{n-1}}\left(\delta_{\frac{y-F_{n-1}}{r_{n-1}}}(r_{n-1}^{-1} I_n(h) + r_{n-1}^{-1} G_n)\right).$$

Then, one performs a Taylor expansion of the delta function around the non-degenerate random vector $r_{n-1}^{-1} I_n(h)$. That is, if we set $X := r_{n-1}^{-1} I_n(h)$ and $Y := r_{n-1}^{-1} G_n$, then:

$$p_n(y) = r_{n-1}^{-k}\, E_{t_{n-1}}\left(\delta_{\frac{y-F_{n-1}}{r_{n-1}}}(X)\right) + r_{n-1}^{-k} \int_0^1 \sum_{j=1}^k E_{t_{n-1}}\left(\delta^{(j)}_{\frac{y-F_{n-1}}{r_{n-1}}}(X + \rho Y) Y^j\right) d\rho. \tag{2.8}$$

Next, one applies the integration by parts formula (see [10, p. 324]) in order to get an upper estimate of each of the k terms in the sum of the second term on the right-hand side of (2.8): for any $j \in \{1, \ldots, k\}$,

$$E_{t_{n-1}}\left(\delta^{(j)}_{\frac{y-F_{n-1}}{r_{n-1}}}(X + \rho Y) Y^j\right) = E_{t_{n-1}}(\mathbf{1}_{\{X+\rho Y \ge \frac{y-F_{n-1}}{r_{n-1}}\}} H^j(X + \rho Y, Y^j)),$$

where the latter term can be bounded by

$$(\|X + \rho Y\|_{m_1,p_1}^{t_{n-1},t_n})^{q_1} (\| \det(\psi_{X+\rho Y}^{t_{n-1},t_n})^{-1} \|_{p_2}^{t_{n-1},t_n})^{q_2} \|Y^j\|_{m_3,p_3}^{t_{n-1},t_n}, \tag{2.9}$$

for some parameters p_i, q_i, m_i. Now, one proves that the above product can be bounded, up to some constant, by $r_{n-1}^{2\delta}$, where δ is the parameter of condition (2.3). Indeed, using (H1), (H2) and (H3), one easily proves that the first term in (2.9) is bounded by $C(1 + r_{n-1}^{2\delta q_1})$ while the third one is bounded by $C r_{n-1}^{2\delta}$. On the other hand, owing to (H4), one gets that the second term in (2.9) is bounded by a constant. Hence, for some positive constant C, we conclude that

$$\left| \int_0^1 \sum_{j=1}^k E_{t_{n-1}}\left(\delta^{(j)}_{\frac{y-F_{n-1}}{r_{n-1}}}(X + \rho Y) Y^j\right) d\rho \right| \le C\, r_{n-1}^{2\delta}. \tag{2.10}$$

As far as the first term on the right-hand side of (2.8) is concerned, we observe that, conditioned to $\mathscr{F}_{t_{n-1}}$, the vector X is Gaussian. Thus, its conditional density can be computed explicitly and, due to condition (2.5), the corresponding $\mathscr{F}_{t_{n-1}}$-conditional covariance matrix is invertible. Hence,

$$E_{t_{n-1}}\left(\delta_{\frac{y-F_{n-1}}{r_{n-1}}}(X)\right) \ge \frac{1}{(2\pi)^{k/2} C_1^{k/2}} \exp\left(-\frac{\|y - F_{n-1}\|^2}{C_2 r_{n-1}^2}\right), \quad \text{a.s.} \tag{2.11}$$

Therefore, plugging the estimates (2.10) and (2.11) in (2.8), we end up with

$$p_n(y) \geq r_{n-1}^{-k} \left(\frac{1}{(2\pi)^{k/2} C_1^{k/2}} \exp\left(-\frac{\|y - F_{n-1}\|^2}{C_2 r_{n-1}^2} \right) - C\, r_{n-1}^{2\delta} \right).$$

Now, we have that if $\|y - F_{n-1}\|^2 \leq c\, r_{n-1}^2$ for some constant $c > 0$, then

$$p_n(y) \geq r_{n-1}^{-k} \left(\frac{1}{(2\pi)^{k/2} C_1^{k/2}} \exp(-C_2^{-1} c) - C\, r_{n-1}^{2\delta} \right).$$

We next choose M and $\eta_0 > 0$ as follows. We first choose M such that

$$\frac{1}{(2\pi)^{k/2} C_1^{k/2}} \exp(-C_2^{-1} c) > \frac{1}{M},$$

and then we define

$$\eta_0 := \left(\frac{1}{(2\pi)^{k/2} C_1^{k/2} C} \exp(-C_2^{-1} c) - \frac{1}{MC} \right)^{1/\delta}.$$

Therefore, we conclude that if $r_{n-1}^2 \leq \eta_0$ and $\|y - F_{n-1}\|^2 \leq c\, r_{n-1}^2$, then

$$p_n(y) \geq \frac{1}{M r_{n-1}^k} \quad \text{a.s.}$$

which proves (2.7).

Step 2. This part is devoted to prove that we can choose N and the partition π_N in a suitable way which will be needed in the sequel and such that the condition $r_{n-1}^2 \leq \eta_0$ above makes sense (see also the proof of [10, Theorem 1]). Indeed, we first assume, without any loss of generality, that $\|g\|_{\mathcal{H}_T}^2 \leq M$. Then, for any N, there exists a partition π_N such that

$$r_{n-1}^2 = \frac{\|g\|_{\mathcal{H}_t}^2}{N}. \tag{2.12}$$

Next, one can show that there exists β_0 such that, for any $\beta \leq \beta_0$ and considering N the smallest integer satisfying

$$N \geq \beta^{-1} \left(\frac{\|y - F_0\|^2}{\|g\|_{\mathcal{H}_t}^2} + 1 \right),$$

then we have $\|\pi_N\| < \epsilon$.

Let $\eta < \eta_0$ and assume that $\beta \leq \frac{\eta}{M} \wedge \frac{1}{2} \wedge \frac{c^2}{4}$. Again without any loss of generality, we may suppose that M is big enough so that $\beta \leq \frac{\eta}{M} \leq \beta_0$. Then we have that

$$r_{n-1}^2 \leq \frac{\eta \|g\|_{\mathcal{H}_t}^2}{M} \left(\frac{\|y - F_0\|^2}{\|g\|_{\mathcal{H}_t}^2} + 1 \right)^{-1} \leq \eta < \eta_0.$$

Step 3. We proceed now to obtain a lower bound for the density p_F of F which will be amenable to be transformed to the statement's Gaussian estimate.

We write, for f a smooth function with compact support,

$$\begin{aligned} \mathrm{E}[f(F_N)] &= \mathrm{E}\left[\int_{\mathbb{R}^k} f(y_N)p_N^{\epsilon}(y_N)dy_N\right] \\ &\geq \mathrm{E}\left[\int_{\mathbb{R}^k} f(y_N)p_N(y_N)\mathbf{1}_{B_{\frac{1}{2}cr_{N-1}}(y_N)}(F_{N-1})dy_N\right], \end{aligned}$$

where, in general, $B_r(z)$ denotes the ball in $\mathbb{R}^k$ of center z and radius r. Then, applying (2.7) we obtain:

$$\begin{aligned} \mathrm{E}[f(F_N)] &\geq \frac{1}{Mr_{N-1}^k}\int_{\mathbb{R}^k} f(y_N)\mathrm{E}[\mathbf{1}_{B_{\frac{1}{2}cr_{N-1}}(y_N)}(F_{N-1})]dy_N \\ &\geq \frac{1}{Mr_{N-1}^k}\int_{\mathbb{R}^k} f(y_N)\frac{1}{Mr_{N-2}^k}\int_{\mathbb{R}^k}\mathbf{1}_{B_{\frac{1}{2}cr_{N-1}}(y_N)}(y_{N-1}) \\ &\quad\times \mathrm{E}[\mathbf{1}_{B_{\frac{1}{2}cr_{N-2}}(y_{N-1})}(F_{N-2})]dy_{N-1}dy_N. \end{aligned} \tag{2.13}$$

Let us consider a sequence of points $x_0, x_1, \dots, x_N$ such that $x_0 = F_0, \dots, x_N = y$, and such that $\|x_n - x_{n-1}\| \leq \frac{1}{6}cr_{n-1}$. Observe that if $y_n \in B_{\frac{1}{6}cr_{n-1}}(x_n)$, then

$$\|y_n - y_{n-1}\| \leq \|y_n - x_n\| + \|x_n - x_{n-1}\| + \|x_{n-1} - y_{n-1}\| \leq \frac{1}{2}cr_{n-1},$$

which implies that $y_{n-1} \in B_{\frac{1}{2}cr_{n-1}}(y_n)$. Using this fact and iterating the expression in (2.13), we end up with:

$$\begin{aligned} \mathrm{E}[f(F_N)] &\geq \frac{1}{Mr_{N-1}^k}\int_{\mathbb{R}^k} f(y_N)\mathbf{1}_{B_{\frac{1}{6}cr_{N-1}}(x_N)}(y_N)dy_N\frac{1}{Mr_{N-2}^k} \\ &\quad\times\int_{\mathbb{R}^k}\mathbf{1}_{B_{\frac{1}{6}cr_{N-2}}(x_{N-1})}(y_{N-1})dy_{N-1} \\ &\quad\cdots\frac{1}{Mr_0^k}\int_{\mathbb{R}^k}\mathbf{1}_{B_{\frac{1}{6}cr_0}(x_1)}(y_1)\mathbf{1}_{B_{\frac{1}{2}cr_0}(y_1)}(F_0)dy_1 \\ &= C(k)\frac{c^{k(N-1)}}{6^{k(N-1)}M^N r_{N-1}^k}\int_{\mathbb{R}^k} f(y_N)\mathbf{1}_{B_{\frac{1}{6}cr_{N-1}}(x_N)}(y_N)dy_N. \end{aligned}$$

Note that in the last equality we have used the facts that $|B_r(z)| = C(k)r^k$ and

$$\|x_0 - y_1\| \leq \|x_0 - x_1\| + \|x_1 - y_1\| \leq \frac{1}{6}cr_0 + \frac{1}{6}cr_0 < \frac{1}{2}cr_0.$$

At this point, we consider f to be an approximation of the Dirac delta function at the point y, so that we can conclude, recalling that $x_N = y$,

$$p_F(y) \geq C(k)\frac{c^{k(N-1)}}{6^{k(N-1)}M^N r_{N-1}^k}. \tag{2.14}$$

Step 4. Let us finally obtain the Gaussian lower bound for $p_F(y)$. Precisely, plugging (2.12) in the above estimate (2.14), we get that

$$p_F(y) \geq \frac{C_k N^{k/2}\exp(-NC^*)}{\|g\|_{\mathcal{H}_t}^k},$$

where $C_k = \frac{6^k}{C(k)c^k}$ and $C^* = \log(M) - \log(C(k)c^k/6^k)$. Note that we can choose M to be sufficiently large such that $C^* > 0$.

Finally, we recall that, thanks to the considerations in Step 2, we have chosen N satisfying that $\beta^{-1} \le N \le \beta^{-1}(\frac{\|y-F_0\|^2}{\|g\|^2_{\mathscr{H}_t}} + 1) + 1$. Hence

$$p(y) \ge \frac{C_k \beta^{-k/2} \exp(-(1+\beta^{-1})C^*) \exp\left(-\beta^{-1} C^* \frac{\|y-F_0\|^2}{\|g\|^2_{\mathscr{H}_t}}\right)}{\|g\|^k_{\mathscr{H}_t}}$$
$$\ge (M'\|g\|_{\mathscr{H}_t})^{-k} \exp\left(-M' \frac{\|y-F_0\|^2}{\|g\|^2_{\mathscr{H}_t}}\right),$$

for some constant $M' = M'(k, c, M, \beta) > 1$. This concludes the proof of the theorem. □

3. The stochastic heat equation

In this section, we will recall some known facts about the stochastic heat equation on $\mathbb{R}^d$ which will be needed in the sequel. We will also prove an estimate involving the iterated Malliavin derivative of the solution which, as far as we know, does not seem to exist in the literature (see Lemma 3.4 below).

We remind that the mild solution to the stochastic heat equation (1.1) is given by the $\mathscr{F}_t$-adapted process $\{u(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ that satisfies:

$$u(t,x) = (\Gamma(t) * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma(u(s,y))W(ds,dy)$$
$$+ \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) b(u(s,y))\, dyds, \tag{3.1}$$

where $\Gamma(t, \star)$ is the Gaussian kernel with variance $2t$ and the following condition is fulfilled:

$$\Phi(T) = \int_0^T \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(t)(\xi)|^2 \mu(d\xi) dt < \infty.$$

As mentioned in the Introduction, this is equivalent to say that

$$\int_{\mathbb{R}^d} \frac{1}{1 + \|\xi\|^2} \mu(d\xi) < \infty. \tag{3.2}$$

Similarly as in [14, Lemma 3.1], one easily proves that, if (3.2) holds, then for all $0 \le \tau_1 < \tau_2 \le T$:

$$C(\tau_2 - \tau_1) \le \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(t)(\xi)|^2 \mu(d\xi) dt, \tag{3.3}$$

for some positive constant C depending on T. Let us also consider the following condition on the spectral measure μ, which turns out to be slightly stronger than (3.2).

(H_η) For some $\eta \in (0,1)$, it holds that:

$$\int_{\mathbb{R}^d} \frac{1}{(1 + \|\xi\|^2)^\eta} \mu(d\xi) < +\infty.$$

Under this hypothesis, it has been proved in [14, Lemma 3.1] that there is a constant C such that, for any $t > 0$:

$$\int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(s)(\xi)|^2 \mu(d\xi) ds \leq C\, t^{1-\eta}. \tag{3.4}$$

We have already commented in the Introduction that imposing to have such an estimate seems to be quite natural. In fact, if the space dimension is one and W is the space–time white noise, Kohatsu-Higa [10] needed to use a bound of the form (3.4) with a term $t^{1/2}$ on the right-hand side (see [10, p. 439]). However, in the latter setting one gets such an estimate for free, while in our general framework we are forced to assume (H_η).

Remark 3.1. In the proof of the lower bound in Theorem 1.1 (see Section 4), the estimate (3.3) will play an important role as well. This has prevented us from proving our main result for other type of SPDEs, such as the stochastic wave equation (see Remark 3.5). Indeed, for the latter SPDE, we do not have a kind of time homogeneous lower bound of the form (3.3) which, for instance, has been a key point in order to conclude the proof of Proposition 4.3 below.

In order to apply the techniques of the Malliavin calculus to the solution of (1.3), let us consider the Gaussian context described in Section 2.1. That is, let $\{W(h),\ h \in \mathcal{H}_T\}$ be the isonormal Gaussian process on the Hilbert space $\mathcal{H}_T = L^2([0,T]; \mathcal{H})$ defined therein. Then, the following result is a direct consequence of [14, Proposition 2.4], [26, Theorem 1] and [21, Proposition 6.1]. For the statement, we will use the following notation: for any $m \in \mathbb{N}$, set $\bar{s} := (s_1, \ldots, s_m) \in [0,T]^m$, $\bar{z} := (z_1, \ldots, z_m) \in (\mathbb{R}^d)^m$, $\bar{s}(j) := (s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_m)$ (resp. $\bar{z}(j)$), and, for any function f and variable X for which it makes sense, set

$$\Delta^m(f, X) := D^m f(X) - f'(X) D^m X.$$

Note that $\Delta^m(f, X) = 0$ for $m = 1$ and, if $m > 1$, it only involves iterated Malliavin derivatives up to order $m - 1$.

Proposition 3.2. *Assume that (3.2) is satisfied and $\sigma, b \in \mathcal{C}^\infty(\mathbb{R})$ and their derivatives of order greater than or equal to one are bounded. Then, for all $(t,x) \in [0,T] \times \mathbb{R}^d$, the random variable $u(t,x)$ belongs to $\mathbb{D}^\infty$. Furthermore, for any $m \in \mathbb{N}$ and $p \geq 1$, the iterated Malliavin derivative $D^m u(t,x)$ satisfies the following equation in $L^p(\Omega; \mathcal{H}_T^{\otimes m})$:*

$$\begin{aligned} D^m u(t,x) = {} & Z^m(t,x) + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)[\Delta^m(\sigma, u(s,y)) \\ & + D^m u(s,y) \sigma'(u(s,y))] W(ds, dy) \\ & + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)[\Delta^m(b, u(s,y)) \\ & + D^m u(s,y) b'(u(s,y))]\, dy ds, \end{aligned} \tag{3.5}$$

where $Z^m(t,x)$ is the element of $L^p(\Omega; \mathcal{H}_T^{\otimes m})$ given by

$$Z^m(t,x)_{\bar{s},\bar{z}} = \sum_{j=1}^m \Gamma(t - s_j, x - z_j) D^{m-1}_{\bar{s}(j),\bar{z}(j)} \sigma(u(s_j, z_j)).$$

We remark that the Hilbert-space-valued stochastic and pathwise integrals in Eq. (3.5) are understood as it has been described in the Appendix.

As far as the existence of a smooth density is concerned, we have the following result (see [21, Theorem 6.2]).

Theorem 3.3. *Assume that (3.2) is satisfied and $\sigma, b \in \mathscr{C}^\infty(\mathbb{R})$ and their derivatives of order greater than or equal to one are bounded. Moreover, suppose that $|\sigma(z)| \geq c > 0$, for all $z \in \mathbb{R}$. Then, for every $(t, x) \in (0, T] \times \mathbb{R}^d$, the law of the random variable $u(t, x)$ has a $\mathscr{C}^\infty$ density.*

The following technical result, which will be used in the proof of Theorem 1.1, exhibits an almost sure estimate for the conditional moment of the iterated Malliavin derivative of u in a small time interval. As it will be explained in Remark 3.5, this result is still valid for a slightly more general class of SPDEs, such as the stochastic wave equation in space dimension $d \in \{1, 2, 3\}$. Nevertheless, for the sake of simplicity, we will focus either the statement and its proof on our stochastic heat equation (1.3).

Lemma 3.4. *Let $0 \leq a < e \leq T$, $m \in \mathbb{N}$ and $p \geq 1$. Assume that condition (3.2) is satisfied and that the coefficients $b, \sigma : \mathbb{R} \to \mathbb{R}$ belong to $\mathscr{C}^\infty(\mathbb{R})$ and all their derivatives of order greater than or equal to one are bounded. Then, there exists a positive constant C, which is independent of a and e, such that, for all $\delta \in (0, e-a]$:*

$$\sup_{(\tau,y)\in[e-\delta,e]\times\mathbb{R}^d} \mathrm{E}_a(\|D^m u(\tau, y)\|^{2p}_{\mathscr{H}^{\otimes m}_{e-\delta,e}}) \leq C\,(\Phi(\delta))^{mp}, \quad a.s.,$$

where we remind that $\mathscr{H}^{\otimes m}_{e-\delta,e}$ denotes the Hilbert space $L^2([e-\delta, e]; \mathscr{H}^{\otimes m})$ and, for all $t \geq 0$,

$$\Phi(t) = \int_0^t \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(s)(\xi)|^2\,\mu(d\xi)ds.$$

Proof. We will proceed by induction with respect to $m \in \mathbb{N}$. First, let us observe that the case $m = 1$ has been proved in [14, Lemma 2.5] (see also [26, Lemma 5]). Suppose now that the statement holds for any $j = 1, \ldots, m-1$, and let us check its veracity for $j = m$.

Let $e - \delta \leq t \leq e$ and $x \in \mathbb{R}^d$. Then, the conditioned norm of the Malliavin derivative $D^m u(t, x)$ can be decomposed as follows:

$$E_a(\|D^m u(t, x)\|^{2p}_{\mathscr{H}^{\otimes m}_{e-\delta,e}}) \leq C(B_1 + B_2 + B_3), \quad \text{a.s.}$$

with

$$B_1 = E_a\left(\int_{(e-\delta,e)^m} \left\|\sum_{j=1}^m \Gamma(t-s_j, x-\star) D^{m-1}_{\bar{s}(j)}\sigma(u(s_j,\star))\right\|^2_{\mathscr{H}^{\otimes m}} d\bar{s}\right)^p$$

(here, if we formally denote by $(z_1, \ldots, z_m)$ the variables of $\mathscr{H}^{\otimes m}$, the symbol $\star$ corresponds to z_j),

$$\begin{aligned} B_2 \;=\; E_a\Bigg(\Bigg\|\int_0^t \int_{\mathbb{R}^d} &\Gamma(t-s, x-y)[\Delta^m(\sigma, u(s, y)) \\ &- D^m u(s, y)\sigma'(u(s, y))]W(ds, dy)\Bigg\|^{2p}_{\mathscr{H}^{\otimes m}_{e-\delta,e}}\Bigg), \end{aligned}$$

$$B_3 = E_a\Bigg(\Bigg\|\int_0^t\int_{\mathbb{R}^d}\Gamma(t-s,x-y)[\Delta^m(b,u(s,y)) - D^m u(s,y)b'(u(s,y))]\,dyds\Bigg\|^{2p}_{\mathscr{H}^{\otimes m}_{e-\delta,e}}\Bigg).$$

Let us start with the study of the term B_1. First, note that we must have that $e-\delta \le s_j \le t$, thus

$$B_1 \le C\sum_{j=1}^m E_a\Bigg(\int_{e-\delta}^t ds_j \int_{(e-\delta,e)^{m-1}} d\bar{s}(j) \times \|\Gamma(t-s_j,x-\star)D^{m-1}_{\bar{s}(j)}\sigma(u(s_j,\star))\|^2_{\mathscr{H}^{\otimes m}}\Bigg)^p.$$

At this point, we can proceed as in the proof of [26, Lemma 2] (see p. 173 therein), so that we can infer that

$$B_1 \le C\left(\int_{e-\delta}^t J(t-r)\,dr\right)^{p-1}\int_{e-\delta}^t \sup_{y\in\mathbb{R}^d} E_a(\|D^{m-1}\sigma(u(r,y))\|^{2p}_{\mathscr{H}^{\otimes(m-1)}_{e-\delta,e}})J(t-r)\,dr,$$

where we have used the notation $J(r) = \int_{\mathbb{R}^d}|\mathscr{F}\Gamma(r)(\xi)|^2\,\mu(d\xi)$. Precisely, we have used the fact that Γ is a smooth function, and then applied Hölder's and Cauchy–Schwarz inequalities. Hence, we have that

$$B_1 \le C(\Phi(\delta))^p \sup_{(r,y)\in[e-\delta,e]\times\mathbb{R}^d} E_a(\|D^{m-1}\sigma(u(r,y))\|^{2p}_{\mathscr{H}^{\otimes(m-1)}_{e-\delta,e}}). \tag{3.6}$$

In order to bound the above supremum, one applies the Leibniz rule for the iterated Malliavin derivative (see e.g. [27, Eq. (7)]), the smoothness assumptions on σ, Hölder's inequality and the induction hypothesis, altogether yielding

$$\sup_{(r,y)\in[e-\delta,e]\times\mathbb{R}^d} E_a(\|D^{m-1}\sigma(u(r,y))\|^{2p}_{\mathscr{H}^{\otimes(m-1)}_{e-\delta,e}}) \le C(\Phi(\delta))^{(m-1)p}, \quad \text{a.s.}$$

Plugging this bound in (3.6), we end up with

$$B_1 \le C(\Phi(\delta))^{mp}, \quad \text{a.s.} \tag{3.7}$$

Next, we will deal with the term B_2, which will be essentially bounded by means of Lemma A.1, as follows:

$$B_2 \le C\left(\int_{e-\delta}^t J(t-r)\,dr\right)^{p-1}\int_{e-\delta}^t [\sup_{y\in\mathbb{R}^d} E_a(\|\Delta^m(\sigma,u(s,y))\|^{2p}_{\mathscr{H}^{\otimes m}_{e-\delta,e}}) + \sup_{y\in\mathbb{R}^d} E_a(\|D^m u(s,y)\|^{2p}_{\mathscr{H}^{\otimes m}_{e-\delta,e}})]J(t-s)ds.$$

Owing again to the Leibniz rule for the Malliavin derivative and noting that Δ^m only involves Malliavin derivatives up to order $m-1$, one makes use of the induction hypothesis to infer that

$$\sup_{y\in\mathbb{R}^d} E_a(\|\Delta^m(\sigma,u(s,y))\|^{2p}_{\mathscr{H}^{\otimes m}_{e-\delta,e}}) \le C(\Phi(\delta))^{mp}, \quad \text{a.s.}$$

Hence,

$$B_2 \leq C(\Phi(T))^{p-1} \int_{e-\delta}^{t} [(\Phi(\delta))^{mp} + \sup_{y\in\mathbb{R}^d} E_a(\|D^m u(s,y)\|^{2p}_{\mathscr{H}^{\otimes m}_{e-\delta,e}})] J(t-s) ds$$

$$\leq C_1 \int_{e-\delta}^{t} [(\Phi(\delta))^{mp} + \sup_{(\tau,y)\in[e-\delta,s]\times\mathbb{R}^d} E_a(\|D^m u(\tau,y)\|^{2p}_{\mathscr{H}^{\otimes m}_{e-\delta,e}})] J(t-s) ds, \tag{3.8}$$

almost surely, where C_1 denotes some positive constant.

Furthermore, using similar arguments, we can show that the term B_3 is bounded above by:

$$C \int_{e-\delta}^{t} \int_{\mathbb{R}^d} \Gamma(t-s, x-y) [E_a(\|\Delta^m(b, u(s,y))\|^{2p}_{\mathscr{H}^{\otimes m}_{e-\delta,e}})$$
$$+ E_a(\|D^m u(s,y)\|^{2p}_{\mathscr{H}^{\otimes m}_{e-\delta,e}})] dy ds$$
$$\leq C \int_{e-\delta}^{t} [(\Phi(\delta))^{mp} + \sup_{(\tau,y)\in[e-\delta,s]\times\mathbb{R}^d} E_a(\|D^m u(\tau,y)\|^{2p}_{\mathscr{H}^{\otimes m}_{e-\delta,e}})] ds, \quad \text{a.s.} \tag{3.9}$$

Here, we have also used that $\int_{\mathbb{R}^d} \Gamma(s,y)\, dy$ is uniformly bounded with respect to s.

Set

$$F(t) := \sup_{(s,y)\in[e-\delta,t]\times\mathbb{R}^d} E_a(\|D^m u(s,y)\|^{2p}_{\mathscr{H}^{\otimes n}_{e-\delta,e}}), \quad t \in [e-\delta, e].$$

Then, (3.7)–(3.9) imply that

$$F(t) \leq C_2(\Phi(\delta))^{mp} + C_1 \int_{e-\delta}^{t} [(\Phi(\delta))^{mp} + F(s)](J(t-s)+1) ds, \quad \text{a.s.},$$

where C_1 and C_2 are some positive constants. We conclude the proof by applying Gronwall's lemma [4, Lemma 15]. □

Remark 3.5. Lemma 3.4 still remains valid for a more general class of SPDEs, namely for those that have been considered in the paper [4] (see also [21]). In these references, an SPDE driven by a linear second-order partial differential operator has been considered, where one assumes that the corresponding fundamental solution Γ satisfies the following: for all s, $\Gamma(s)$ is a non-negative measure which defines a distribution with rapid decrease such that condition (1.4) is fulfilled and

$$\sup_{0\leq s\leq T} \Gamma(s, \mathbb{R}^d) < +\infty.$$

As explained in [4, Section 3], together with the stochastic heat equation, the stochastic wave equation in space dimension $d \in \{1,2,3\}$ is another example of such a type of equation. Finally, we point out that the proof of Lemma 3.4 in such a general setting would require a smoothing procedure of Γ in terms of an approximation of the identity, which would make the proof longer and slightly more technical; this argument has been used for instance in [26, Lemma 5].

4. Proof of the lower bound

In this section, we prove the lower bound in the statement of Theorem 1.1. For this, we are going to show that, for any $(t,x) \in (0,T] \times \mathbb{R}^d$, $u(t,x)$ is a uniformly elliptic random variable in the sense of Definition 2.2. Then, an application of Theorem 2.3 will give us the desired lower bound. We recall that we are assuming that condition (H_η) is satisfied, the coefficients b and σ belong to $\mathscr{C}_b^\infty(\mathbb{R})$ and there is a constant $c > 0$ such that $|\sigma(v)| \geq c$, for all $v \in \mathbb{R}$.

To begin with, we fix $(t,x) \in (0,T] \times \mathbb{R}^d$, we consider a partition $0 = t_0 < t_1 < \cdots < t_N = t$ whose norm converges to zero, and define:

$$F_n = (\Gamma(t) * u_0)(x) + \int_0^{t_n} \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma(u(s,y))W(ds,dy) \\ + \int_0^{t_n} \int_{\mathbb{R}^d} \Gamma(t-s, x-y) b(u(s,y)) dy ds.$$

Clearly, F_n is $\mathscr{F}_{t_n}$-measurable, for all $n = 0, \ldots, N$, and note that $F_0 = (\Gamma(t) * u_0)(x)$. Moreover, F_n belongs to $\mathbb{D}^\infty$ and, for all $m \in \mathbb{N}$ and $p \geq 1$, the norm $\|F_n\|_{m,p}$ can be uniformly bounded with respect to $(t,x) \in (0,T] \times \mathbb{R}^d$ (see [14, Proposition 2.4], [26, Theorem 1], and also [21, Proposition 6.1]). Moreover, it is clear that $F_N = u(t,x)$.

The local variance of the random variable $u(t,x)$ will be measured through the function $g(s) := \Gamma(t-s)$. Then, observe that

$$\Delta_{n-1}(g) = \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(t-s)(\xi)|^2 \mu(d\xi) ds,$$

and this quantity is clearly positive for all n (see (1.5)).

Now, we aim to decompose $F_n - F_{n-1}$ in the form $I_n(h) + G_n$ (see (2.2) in Definition 2.2). For this, we observe that:

$$F_n - F_{n-1} = \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma(u(s,y))W(ds,dy) \\ + \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} \Gamma(t-s, x-y) b(u(s,y)) dy ds. \tag{4.1}$$

We then consider the point $u_{n-1}(s,y)$ defined by:

$$u_{n-1}(s,y) = \int_{\mathbb{R}^d} \Gamma(s, y-z) u_0(z) dz + \int_0^{t_{n-1}} \int_{\mathbb{R}^d} \Gamma(s-r, y-z)\sigma(u(r,z))W(dr,dz) \\ + \int_0^{t_{n-1}} \int_{\mathbb{R}^d} \Gamma(s-r, y-z) b(u(r,z))\, dz dr,$$

where $(s,y) \in [t_{n-1}, t_n] \times \mathbb{R}^d$. We clearly have that $u_{n-1}(s,y)$ is $\mathscr{F}_{t_{n-1}}$-measurable and belongs to $\mathbb{D}^\infty$. Moreover, observe that we can write:

$$F_n - F_{n-1} = \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma(u_{n-1}(s,y))W(ds,dy) \\ + \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} \Gamma(t-s, x-y) b(u(s,y))\, dy ds \\ + \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} \Gamma(t-s, x-y)[\sigma(u(s,y)) - \sigma(u_{n-1}(s,y))]W(ds,dy).$$

Hence, we obtain an expression of the form (2.2), where

$$\begin{aligned}
&I_n(h) = \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} h(s, y) W(ds, dy) \quad \text{with } h(s, y) = \Gamma(t-s, x-y)\sigma(u_{n-1}(s, y)) \\
&G_n = G_n^1 + G_n^2 \quad \text{where} \\
&G_n^1 = \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} \Gamma(t-s, x-y) b(u(s, y))\, dy ds, \\
&G_n^2 = \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} \Gamma(t-s, x-y)[\sigma(u(s, y)) - \sigma(u_{n-1}(s, y))] W(ds, dy).
\end{aligned} \tag{4.2}$$

The remaining of the section will be devoted to prove that conditions (H1), (H3) and (H4) in Definition 2.2 are satisfied. Observe that the fact that $\|h\|_{t_{n-1},t_n}$ is bounded in ω will be a direct consequence of the analysis of condition (H3) (see Lemma 4.2 below), which, together with what it has already been commented, shows that all the conditions in (H2) are fulfilled.

To start with, the next lemma shows that (H1) is satisfied.

Lemma 4.1. *For all $m \in \mathbb{N}$ and $p > 1$, there exists a constant C such that:*

$$\|G_n\|_{m,p}^{t_{n-1},t_n} \le C\, \Delta_{n-1}(g)^{\frac{2-\eta}{2}} \quad a.s., \tag{4.3}$$

where $\eta \in (0, 1)$ is the parameter of hypothesis (H_η).

Proof. Note that it suffices to show that, for all $p > 1$ and all integer $j \ge 0$, we have:

$$E_{t_{n-1}}(\|D^j G_n\|^p_{\mathscr{H}^{\otimes m}_{t_{n-1},t_n}}) \le C\, \Delta_{n-1}(g)^{\frac{p(2-\eta)}{2}}.$$

According to the above decomposition (4.2), we need to seek upper estimates for the terms

$$E_{t_{n-1}}(\|D^j G_n^1\|^p_{\mathscr{H}^{\otimes m}_{t_{n-1},t_n}}) \quad \text{and} \quad E_{t_{n-1}}(\|D^j G_n^2\|^p_{\mathscr{H}^{\otimes m}_{t_{n-1},t_n}}).$$

For this, we split the proof into three steps.

Step 1. Let us first deal with the term G_n^1. Indeed, by Minkowski's inequality and the fact that Γ is a Gaussian density, we obtain:

$$\begin{aligned}
(E_{t_{n-1}}\|D^j G_n^1\|^p_{\mathscr{H}^{\otimes m}_{t_{n-1},t_n}})^{\frac{1}{p}} &\le \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} \Gamma(t-s, x-y)(E_{t_{n-1}}\|D^j b(u(s, y))\|^p_{\mathscr{H}^{\otimes j}_{t_{n-1},t_n}})^{\frac{1}{p}}\, dyds \\
&\le C \sup_{t_{n-1}\le s\le t_n} \sup_{y\in\mathbb{R}^d} (E_{t_{n-1}}\|D^j b(u(s, y))\|^p_{\mathscr{H}^{\otimes j}_{t_{n-1},t_n}})^{\frac{1}{p}} \\
&\quad \times \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} \Gamma(t-s, x-y) dyds \\
&= C \sup_{t_{n-1}\le s\le t_n} \sup_{y\in\mathbb{R}^d} (E_{t_{n-1}}\|D^j b(u(s, y))\|^p_{\mathscr{H}^{\otimes j}_{t_{n-1},t_n}})^{\frac{1}{p}} (t_{n-1} - t_n).
\end{aligned}$$

The above supremum can be simply bounded by a constant $C(T)$. In fact, one just needs to apply the Leibniz rule for the Malliavin derivative together with the smoothness of b and the estimate given in Lemma 3.4. Altogether, we have that

$$\begin{aligned}(E_{t_{n-1}}\|D^j G_n^1\|^p_{\mathscr{H}^{\otimes m}_{t_{n-1},t_n}})^{\frac{1}{p}} &\le C\,(t_{n-1}-t_n) \le C\,((t-t_{n-1})-(t-t_n))\\ &\le C\,\Delta_{n-1}(g),\end{aligned} \tag{4.4}$$

where the latter bound follows applying (3.3).

Step 2. The remaining of the proof is devoted to analyze the term G_n^2. Precisely, in this step we check that, for all $s \in [t_{n-1}, t_n]$ and $y \in \mathbb{R}^d$, the random variable $X(s, y) := \sigma(u(s, y)) - \sigma(u_{n-1}(s, y))$ satisfies:

$$\sup_{t_{n-1}\le s\le t_n}\sup_{y\in\mathbb{R}^d} E_{t_{n-1}}(\|D^j X(s,y)\|^p_{\mathscr{H}^{\otimes j}_{t_{n-1},t_n}}) \le C\,\Phi(t_n-t_{n-1})^{\frac{p}{2}} \quad \text{a.s.} \tag{4.5}$$

Let us first prove the above statement for $j = 0$, which means that we need to control the moments of $X(s, y)$, conditioned to $\mathscr{F}_{t_{n-1}}$. Indeed, taking into account the equation satisfied by u, the definition of u_{n-1} and the fact that b and σ are assumed to be bounded, we can infer that, up to some positive constant:

$$\begin{aligned}E_{t_{n-1}}(|X(s,y)|^p) &\le E(|u(s,y)-u_{n-1}(s,y)|^p)\\ &\le E_{t_{n-1}}\left(\left|\int_{t_{n-1}}^s\int_{\mathbb{R}^d}\Gamma(s-r,y-z)\sigma(u(r,z))W(dr,dz)\right|^p\right)\\ &\quad + E_{t_{n-1}}\left(\left|\int_{t_{n-1}}^s\int_{\mathbb{R}^d}\Gamma(s-r,y-z)b(u(r,z))\,dzdr\right|^p\right)\\ &\le \Phi(s-t_{n-1})^{\frac{p}{2}} + (s-t_{n-1})^p\\ &\le \Phi(t_n-t_{n-1})^{\frac{p}{2}},\end{aligned}$$

upon recalling again (3.3).

Let us now assume that $j \ge 1$. Then, we simply have:

$$\begin{aligned}E_{t_{n-1}}(\|D^j X(s,y)\|^p_{\mathscr{H}^{\otimes j}_{t_{n-1},t_n}}) &\le C\,(E_{t_{n-1}}(\|D^j\sigma(u(s,y))\|^p_{\mathscr{H}^{\otimes j}_{t_{n-1},t_n}})\\ &\quad + E_{t_{n-1}}(\|D^j\sigma(u_{n-1}(s,y))\|^p_{\mathscr{H}^{\otimes j}_{t_{n-1},t_n}})).\end{aligned}$$

The first term on the right-hand side above can be bounded applying Leibniz rule for the Malliavin derivative and Lemma 3.4. Thus,

$$E_{t_{n-1}}(\|D^j\sigma(u(s,y))\|^p_{\mathscr{H}^{\otimes j}_{t_{n-1},t_n}}) \le C\,\Phi(t_n-t_{n-1})^{\frac{jp}{2}} \le C\,\Phi(t_n-t_{n-1})^{\frac{p}{2}} \quad \text{a.s.} \tag{4.6}$$

On the other hand, by definition of $u_{n-1}(s, y)$, we have that:

$$\begin{aligned}&E_{t_{n-1}}(\|D^j\sigma(u_{n-1}(s,y))\|^p_{\mathscr{H}^{\otimes j}_{t_{n-1},t_n}})\\ &= E_{t_{n-1}}\left(\int_{[t_{n-1},t_n]}\left\|\sum_{i=1}^j\Gamma(t-r_i,x-\star)D^{j-1}_{\bar r(i)}\sigma(u(r_i,\star))\right\|^2_{\mathscr{H}^{\otimes j}}d\bar r\right)^{\frac{p}{2}},\end{aligned} \tag{4.7}$$

where we have used the analogous notation as in the statement of Proposition 3.2. Note that the $\mathscr{H}^{\otimes j}_{t_{n-1},t_n}$-norm of the integral terms that would appear in the expression for $D^j u_{n-1}(s,y)$ vanish; this is because, when defining u_{n-1}, we have used integrals up to t_{n-1} (in order to make the resulting variable $\mathscr{F}_{t_{n-1}}$-measurable). At this point, we observe that the right-hand side of (4.7) corresponds to the term B_1 in the proof of Lemma 3.4 (in the case $a = t_{n-1}$, $e = t_n$, $\delta = t_n - t_{n-1}$ and $m = j$). Therefore, we can infer that (4.7) is bounded by $\Phi(t_n - t_{n-1})^{\frac{p}{2}}$. From this observation, together with (4.6), we conclude that (4.5) is fulfilled.

Step 3. Let us finally prove the lemma's statement. As before, first we tackle the case $j = 0$. Indeed, by standard estimates on the stochastic integral and (4.5) in Step 2, we have:

$$E_{t_{n-1}}(|G_n^2|^p) \le C\, \Delta_{n-1}(g)^{\frac{p}{2}}\, \Phi(t_n - t_{n-1})^{\frac{p}{2}}, \quad \text{a.s.}$$

On the other hand, if $j \ge 1$, we obtain that, up to some positive constant:

$$\begin{aligned} &E_{t_{n-1}}(\|D^j G_n^2\|^p_{\mathscr{H}^{\otimes j}_{t_{n-1},t_n}}) \\ &\le E_{t_{n-1}}\left(\int_{[t_{n-1},t_n]} \left\| \sum_{i=1}^j \Gamma(t-r_i, x-\star) D^{j-1}_{\bar{r}(i)} X(r_i,\star)\right\|^2_{\mathscr{H}^{\otimes j}} d\bar{r}\right)^{\frac{p}{2}} \\ &\quad + E_{t_{n-1}}\left(\left\|\int_{t_{n-1}}^{t_n}\int_{\mathbb{R}^d} \Gamma(t-s,x-y) D^j X(s,y) W(ds,dy)\right\|^p_{\mathscr{H}^{\otimes j}_{t_{n-1},t_n}}\right). \end{aligned}$$

These terms can be estimated using exactly the same method considered in the proof of Lemma 3.4 to deal with B_1 and B_2. Hence, by (4.5), we end up with:

$$E_{t_{n-1}}(\|D^j G_n^2\|^p_{\mathscr{H}^{\otimes j}_{t_{n-1},t_n}}) \le C\, \Delta_{n-1}(g)^{\frac{p}{2}}\, \Phi(t_n - t_{n-1})^{\frac{p}{2}}, \quad \text{a.s.}$$

At this point, we make use of hypothesis (H_η) and then (3.3), so that we have $\Phi(t_n - t_{n-1}) \le C\, \Delta_{n-1}(g)^{1-\eta}$. Thus,

$$E_{t_{n-1}}(\|D^j G_n^2\|^p_{\mathscr{H}^{\otimes j}_{t_{n-1},t_n}}) \le C\, \Delta_{n-1}(g)^{\frac{p(2-\eta)}{2}}, \quad \text{a.s.}$$

From this, together with (4.4) in Step 1, we conclude the proof. □

Condition (H3) in our setting is an immediate consequence of the following result. Recall that here the element h in (H3) is given by $h(s,y) = \Gamma(t-s, x-y)\sigma(u_{n-1}(s,y))$, with $(s,y) \in [t_{n-1}, t_n] \times \mathbb{R}^d$.

Lemma 4.2. *There exist two positive constants C_1 and C_2 such that*

$$C_1 \le \Delta_{n-1}(g)^{-1} \int_{t_{n-1}}^{t_n} \|h(s)\|^2_{\mathscr{H}}\, ds \le C_2, \quad \textit{a.s.}$$

Proof. First, let us observe that, since σ is bounded and Γ is a test function in the space variable, we have:

$$\begin{aligned} &\int_{t_{n-1}}^{t_n} ds \int_{\mathbb{R}^d} \Lambda(dy) \int_{\mathbb{R}^d} dz\, \Gamma(t-s, x-z) \\ &\quad \times \Gamma(t-s, x-z+y)\sigma(u_{n-1}(s,z))\sigma(u_{n-1}(s,z-y)) \end{aligned}$$

$$\leq C \int_{t_{n-1}}^{t_n} ds \int_{\mathbb{R}^d} \Lambda(dy) \int_{\mathbb{R}^d} dz\, \Gamma(t-s, x-z)\Gamma(t-s, x-z+y)$$
$$= C \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(t-s)(\xi)|^2 \mu(\xi) ds = \Delta_{n-1}(g) < +\infty. \tag{4.8}$$

As it has been pointed out in the proof of [7, Proposition 2.6], this implies that the first expression in (4.8) is precisely equal to $\|h\|^2_{t_{n-1},t_n}$. In particular, we obtain the upper bound in the statement.

In order to get the lower estimate, let us simply observe that, in (4.8), we have

$$\sigma(u_{n-1}(s,z))\sigma(u_{n-1}(s,z-y)) \geq c^2,$$

for all s, y, z, where we remind that $c > 0$ is a constant satisfying $|\sigma(v)| \geq c$ for all $v \in \mathbb{R}$. Thus, we conclude the proof. □

Finally, the following result proves condition (H4), which we recall that implies, in particular, that F_n has a smooth conditional density.

Proposition 4.3. *For any $p > 0$, there exists a constant C such that:*

$$\mathrm{E}_{t_{n-1}}(\|D(I_n(h) + \rho G_n)\|^{-2p}_{t_{n-1},t_n}) \leq C\, \Delta_{n-1}(g)^{-p} \quad a.s., \tag{4.9}$$

for all $\rho \in (0, 1]$.

Proof. As it has been similarly done in the proof [21, Theorem 6.2], we note that it suffices to show that, for any $q > 2$, there exists $\epsilon_0 = \epsilon_0(q) > 0$ such that, for all $\epsilon \leq \epsilon_0$:

$$\mathrm{P}_{t_{n-1}}\{\Delta^{-1}_{n-1}(g)\, \|D(I_n(h) + \rho G_n)\|^2_{\mathscr{H}_{t_{n-1},t_n}} < \epsilon\} \leq C\epsilon^q \quad \text{a.s.} \tag{4.10}$$

Indeed, if we set $X := \Delta^{-1}_{n-1}(g)\, \|D(I_n(h) + \rho G_n)\|^2_{\mathscr{H}_{t_{n-1},t_n}}$, then we have:

$$\mathrm{E}_{t_{n-1}}(X^{-p}) = \int_0^\infty p y^{p-1} \mathrm{P}_{t_{n-1}}\left\{X < \frac{1}{y}\right\} dy \quad \text{a.s.}$$

Choosing q sufficiently large in (4.10) (namely $q > p$), we conclude that (4.9) is fulfilled, and hence the statement of hypothesis (H4).

So the rest of the proof is devoted to prove (4.10). First, we observe that the term $I_n(h) + \rho G_n$ can be split as follows:

$$I_n(h) + \rho G_n = \rho(I_n(h) + G_n) + (1-\rho)I_n(h) = \rho(F_n - F_{n-1}) + (1-\rho)I_n(h).$$

Thus, by definition of $I_n(h)$ and using the expression (4.1), we have, for all $r \in [t_{n-1}, t_n]$:

$$\begin{aligned} D_r(I_n(h) + \rho G_n) = \rho \Big\{ & \Gamma(t-r, x-\star)\sigma(u(r,\star)) \\ & + \int_r^{t_n} \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma'(u(s,y)) D_r u(s,y) W(ds,dy) \\ & + \int_r^{t_n} \int_{\mathbb{R}^d} \Gamma(t-s, x-y) b'(u(s,y)) D_r u(s,y)\, dy ds \Big\} \\ & + (1-\rho) \Big\{ \Gamma(t-r, x-\star)\sigma(u_{n-1}(r,\star)) \end{aligned}$$

$$+\int_r^{t_n}\int_{\mathbb{R}^d}\Gamma(t-s,x-y)\sigma'(u_{n-1}(s,y))$$
$$\left.\times\ D_r u_{n-1}(s,y)W(ds,dy)\right\}. \tag{4.11}$$

At this point we assume, without any loss of generality, that σ is a positive function, and we write $\sigma_0 := \inf_{v\in\mathbb{R}}\sigma(v)$. Then, for any small $\delta>0$, we can argue as follows:

$$\begin{aligned}
\|\Gamma(t-\cdot,x-\star)\sigma_0\|^2_{t_n-\delta,t_n} &= \|\rho\Gamma(t-\cdot,x-\star)\sigma_0+(1-\rho)\Gamma(t-\cdot,x-\star)\sigma_0\|^2_{t_n-\delta,t_n}\\
&= \int_{t_n-\delta}^{t_n}dr\int_{\mathbb{R}^d}\Lambda(dz)\int_{\mathbb{R}^d}dy[\rho\Gamma(t-r,x-y)\sigma_0\\
&\quad +(1-\rho)\Gamma(t-r,x-y)\sigma_0]\\
&\quad \times[\rho\Gamma(t-r,x-y+z)\sigma_0\\
&\quad +(1-\rho)\Gamma(t-r,x-y+z)\sigma_0]\\
&\le \int_{t_n-\delta}^{t_n}dr\int_{\mathbb{R}^d}\Lambda(dz)\int_{\mathbb{R}^d}dy[\rho\Gamma(t-r,x-y)\sigma(u(r,y))\\
&\quad +(1-\rho)\Gamma(t-r,x-y)\sigma(u_{n-1}(r,y))]\\
&\quad \times[\rho\Gamma(t-r,x-y+z)\sigma(u(r,y))\\
&\quad +(1-\rho)\Gamma(t-r,x-y+z)\sigma(u_{n-1}(r,y))]\\
&= \|\rho\Gamma(t-\cdot,x-\star)\sigma(u(\cdot,\star))\\
&\quad +(1-\rho)\Gamma(t-\cdot,x-\star)\sigma(u_{n-1}(\cdot,\star))\|^2_{t_n-\delta,t_n}.
\end{aligned}$$

The latter equality follows because the random field inside the norm is non-negative and a well-defined element of $\mathscr{H}_T$, a.s. (see e.g. [7, Proposition 2.6]). On the other hand, by (4.11), we have that:

$$\begin{aligned}
&\|\rho\Gamma(t-\cdot,x-\star)\sigma(u(\cdot,\star))+(1-\rho)\Gamma(t-\cdot,x-\star)\sigma(u_{n-1}(\cdot,\star))\|^2_{t_n-\delta,t_n}\\
&\quad\le C_1\|D(I_n(h)+\rho G_n)\|^2_{t_n-\delta,t_n}+C_2\sum_{i=1}^3\|R_i\|^2_{t_n-\delta,t_n},
\end{aligned}$$

for some positive constants C_1, C_2 independent of ρ, where:

$$\begin{aligned}
R_1 &:= \int_\cdot^{t_n}\int_{\mathbb{R}^d}\Gamma(t-s,x-y)\sigma'(u(s,y))Du(s,y)W(ds,dy),\\
R_2 &:= \int_\cdot^{t_n}\int_{\mathbb{R}^d}\Gamma(t-s,x-y)b'(u(s,y))Du(s,y)dyds,\\
R_3 &:= \int_\cdot^{t_n}\int_{\mathbb{R}^d}\Gamma(t-s,x-y)\sigma'(u_{n-1}(s,y))Du_{n-1}(s,y)W(ds,dy).
\end{aligned}$$

Hence, we have seen that, for any $\delta\in(0,t_n-t_{n-1})$:

$$\|D(I_n(h)+\rho G_n)\|^2_{t_n-\delta,t_n}\ge C_1 I_0(\delta)-C_2\sum_{i=1}^3\|R_i\|^2_{t_n-\delta,t_n}\quad \text{a.s.}, \tag{4.12}$$

where we have used the notation $I_0(\delta)=\int_{t_n-\delta}^{t_n}\int_{\mathbb{R}^d}|\mathscr{F}\Gamma(t-r)(\xi)|^2\mu(d\xi)dr$. Moreover, we have applied that there is a constant $c>0$ such that $|\sigma(v)|\ge c$, for all $v\in\mathbb{R}$.

In order to get suitable estimates of the probability in (4.10), we will make use of (4.12), so that we need to bound the pth moments of $\|R_i\|^2_{t_n-\delta,t_n}$ for $p > 1$. First, applying Lemmas A.1 and 3.4, we get:

$$\begin{aligned}
&\mathrm{E}_{t_{n-1}}(\|R_1\|^{2p}_{t_n-\delta,t_n}) \qquad (4.13)\\
&\quad = \mathrm{E}_{t_{n-1}}\left(\left\|\int_{t_n-\delta}^{t_n}\int_{\mathbb{R}^d}\Gamma(t-s,x-z)\sigma'(u(s,z))Du(s,z)\,W(ds,dz)\right\|^{2p}_{t_n-\delta,t_n}\right)\\
&\quad \le C\,I_0(\delta)^{p-1}\int_{t_n-\delta}^{t_n}\sup_{z\in\mathbb{R}^d}\mathrm{E}_{t_{n-1}}(\|Du(s,z)\|^{2p}_{t_n-\delta,t_n})\int_{\mathbb{R}^d}|\mathcal{F}\Gamma(t-s)(\xi)|^2\mu(d\xi)\,ds\\
&\quad \le C\,I_0(\delta)^{p}\sup_{t_n-\delta\le s\le t_n}\sup_{z\in\mathbb{R}^d}\mathrm{E}_{t_{n-1}}(\|Du(s,z)\|^{2p}_{t_n-\delta,t_n})\\
&\quad \le C\,I_0(\delta)^{p}\,\Phi(\delta)^{p}. \qquad (4.14)
\end{aligned}$$

Similarly, appealing to Lemma A.2 and again Lemma 3.4, we obtain:

$$\begin{aligned}
&\mathrm{E}_{t_{n-1}}(\|R_2\|^{2p}_{t_n-\delta,t_n})\\
&\quad = \mathrm{E}_{t_{n-1}}\left(\left\|\int_{t_n-\delta}^{t_n}\int_{\mathbb{R}^d}\Gamma(t-s,x-z)b'(u(s,z))Du(s,z)\,dzds\right\|^{2p}_{t_n-\delta,t_n}\right)\\
&\quad \le C\,\bar{I}_0(\delta)^{p-1}\int_{t_n-\delta}^{t_n}\sup_{z\in\mathbb{R}^d}\mathrm{E}_{t_{n-1}}(\|Du(s,z)\|^{2p}_{t_n-\delta,t_n})\int_{\mathbb{R}^d}\Gamma(t-s,z)\,dzds\\
&\quad \le C\,\bar{I}_0(\delta)^{p}\,\Phi(\delta)^{p}, \qquad (4.15)
\end{aligned}$$

where

$$\bar{I}_0(\delta) := \int_{t_n-\delta}^{t_n}\int_{\mathbb{R}^d}\Gamma(t-s,z)\,dzds \le C\,\delta, \qquad (4.16)$$

upon recalling that Γ is a Gaussian density.

The analysis of the term R_3 is very similar to that of R_1. However, one needs to observe first that, by the very definition on u_{n-1}, for all $r \in [t_{n-1}, t_n]$ we have:

$$D_r u_{n-1}(s,y) = \Gamma(s-r, y-\star)\sigma(u(r,\star)), \quad (s,y)\in[r,t_n]\times\mathbb{R}^d.$$

Thus, using that σ is assumed to be bounded, one simply has that, for all $s \in [t_n-\delta, t_n]$ and $y \in \mathbb{R}^d$:

$$\begin{aligned}
\|Du_{n-1}(s,y)\|^2_{t_n-\delta,t_n} &\le C\int_{t_n-\delta}^{t_n}\int_{\mathbb{R}^d}|\mathcal{F}\Gamma(s-r)(\xi)|^2\mu(d\xi)dr\\
&\le C\,\Phi(\delta).
\end{aligned}$$

From this, we conclude that

$$\mathrm{E}_{t_{n-1}}(\|R_3\|^{2p}_{t_n-\delta,t_n}) \le C\,I_0(\delta)^p\,\Phi(\delta)^p. \qquad (4.17)$$

Therefore, putting together the estimates (4.14)–(4.17), we have that:

$$\sum_{i=1}^{3}\mathrm{E}_{t_{n-1}}(\|R_i\|^{2p}_{t_n-\delta,t_n}) \le C\,\Phi(\delta)^p(I_0(\delta)^p + \bar{I}_0(\delta)^p) \quad \text{a.s.}$$

At this point, taking into account this estimate and (4.12), we apply the (conditional) Chebyshev's inequality, yielding:

$$\begin{aligned} &\mathrm{P}_{t_{n-1}}\{\Delta_{n-1}^{-1}(g)\,\|D(I_n(h)+\rho G_n)\|^2_{t_{n-1},t_n}<\epsilon\} \\ &\quad\le \mathrm{P}_{t_{n-1}}\{\Delta_{n-1}^{-1}(g)\,\|D(I_n(h)+\rho G_n)\|^2_{t_n-\delta,t_n}<\epsilon\} \\ &\quad\le C\left(C_1\,\frac{I_0(\delta)}{\Delta_{n-1}(g)}-\epsilon\right)^{-p}(\Delta_{n-1}(g))^{-p}\,\Phi(\delta)^p(I_0(\delta)^p+\bar{I}_0(\delta)^p)\quad \text{a.s.} \end{aligned}$$

Now, taking a small enough ϵ_0 if necessary, we choose $\delta=\delta(\epsilon)$ in such a way that $\frac{C_1}{2}\,\frac{I_0(\delta)}{\Delta_{n-1}(g)}=\epsilon$. In particular, observe that condition (3.3) implies that $\epsilon\ge C\,\frac{\delta}{\Delta_{n-1}(g)}$. Thus, since $t_n\le t$, we have:

$$\begin{aligned} \delta &\le C\,\epsilon\,\Delta_{n-1}(g)=C\,\epsilon\int_{t_{n-1}}^{t_n}\int_{\mathbb{R}^d}|\mathscr{F}\Gamma(t-s)(\xi)|^2\mu(d\xi)ds \\ &\le C\,\epsilon\int_0^t\int_{\mathbb{R}^d}|\mathscr{F}\Gamma(t-s)(\xi)|^2\mu(d\xi)ds\le C\,\Phi(T)\epsilon\le C\,\epsilon, \end{aligned}$$

where we recall that we are assuming $\phi(T)<+\infty$. Therefore, by hypothesis (H_η) and conditions (3.3) and (4.16), we can infer that:

$$\begin{aligned} \mathrm{P}_{t_{n-1}}\{\Delta_{n-1}^{-1}(g)\,\|D(I_n(h)+\rho G_n)\|^2_{t_{n-1},t_n}<\epsilon\} &\le C\,\Phi(\delta)^p I_0(\delta)^{-p}(I_0(\delta)^p+\bar{I}_0(\delta)^p) \\ &\le C\,\delta^{p(1-\eta)}\le C\,\epsilon^{p(1-\eta)}. \end{aligned}$$

In order to obtain (4.10), it suffices to choose p sufficiently large such that $p(1-\eta)\ge q$. The proof of (4.9) is now complete. □

We can conclude that, in view of Definition 2.2, the random variable $F=u(t,x)$ is uniformly elliptic. Therefore, by Theorem 2.3, we have proved the lower bound in Theorem 1.1.

5. Proof of the upper bound

This section is devoted to prove the upper bound of Theorem 1.1. For this, we will follow a standard procedure based on the density formula provided by the integration-by-parts formula of the Malliavin calculus and the exponential martingale inequality applied to the martingale part of our random variable $u(t,x)$ for $(t,x)\in(0,T]\times\mathbb{R}^d$ (this method has been used for instance in [19, Proposition 2.1.3] and [9,5]). We remind that we are assuming that the coefficients b and σ belong to $\mathscr{C}_b^\infty(\mathbb{R})$. Moreover, we have that:

$$\begin{aligned} u(t,x) = F_0 &+ \int_0^t\int_{\mathbb{R}^d}\Gamma(t-s,x-y)\sigma(u(s,y))W(ds,dy) \\ &+\int_0^t\int_{\mathbb{R}^d}\Gamma(t-s,x-y)b(u(s,y))\,dyds,\quad \text{a.s.,} \end{aligned}\tag{5.1}$$

where $F_0=(\Gamma(t)*u_0)(x)$.

To begin with, we consider the continuous one-parameter martingale $\{M_a,\mathscr{F}_a,\,0\le a\le t\}$ defined by

$$M_a=\int_0^a\int_{\mathbb{R}^d}\Gamma(t-s,x-y)\sigma(u(s,y))\,W(ds,dy),$$

where the filtration $\{\mathscr{F}_a,\ 0 \le a \le t\}$ is the one generated by W. Notice that $M_0 = 0$ and one has that

$$\langle M \rangle_t = \|\Gamma(t-\cdot, x-\star)\sigma(u(\cdot,\star))\|_{\mathscr{H}_t}.$$

Since σ is bounded, we clearly get that $\langle M \rangle_t \le c_2 \Phi(t)$, a.s. for some positive constant c_2 (see for instance [4, Theorem 2]).

On the other hand, since the drift b is also assumed to be bounded and $\Gamma(s)$ defines a probability density, we can directly estimate the drift term in (5.1) as follows:

$$\left| \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) b(u(s,y))\, dy ds \right| \le c_3 T, \quad \text{a.s.} \tag{5.2}$$

We next consider the expression for the density of a non-degenerate random variable that follows from the integration-by-parts formula of the Malliavin calculus. Precisely, we apply [19, Proposition 2.1.1] so that we end up with the following expression for the density $p_{t,x}$ of $u(t,x)$:

$$p_{t,x}(y) = E[\mathbf{1}_{\{u(t,x)>y\}} \delta(Du(t,x)\, \|Du(t,x)\|^{-2}_{\mathscr{H}_T})], \quad y \in \mathbb{R},$$

where δ denotes the divergence operator or Skorohod integral, that is the adjoint of the Malliavin derivative operator (see [19, Ch. 1]). Taking into account that the Skorohod integral above has mean zero, one can also check that:

$$p_{t,x}(y) = -E[\mathbf{1}_{\{u(t,x)<y\}} \delta(Du(t,x)\, \|Du(t,x)\|^{-2}_{\mathscr{H}_T})], \quad y \in \mathbb{R}.$$

Then, owing to (5.1), [19, Proposition 2.1.2] and the estimate (5.2), we can infer that:

$$\begin{aligned} p_{t,x}(y) &\le c_{\alpha,\beta,q} \mathrm{P}\{|M_t| > |y - F_0| - c_3 T\}^{1/q} \\ &\quad \times (\mathrm{E}[\|Du(t,x)\|^{-1}_{\mathscr{H}_t}] + \|D^2 u(t,x)\|_{L^\alpha(\Omega;\mathscr{H}_t^{\otimes 2})} \| \|Du(t,x)\|^{-2}_{\mathscr{H}_t} \|_{L^\beta(\Omega)}), \end{aligned} \tag{5.3}$$

where α, β, q are any positive real numbers satisfying $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{q} = 1$. Thus, we proceed to bound all the terms on the right-hand side of (5.3).

First, by the exponential martingale inequality (see for instance [19, Section A2]) and the fact that $\langle M \rangle_t \le c_2 \Phi(t)$, we obtain:

$$\mathrm{P}\{|M_t| > |y - F_0| - c_3 T\} \le 2 \exp\left(- \frac{(|y - F_0| - c_3 T)^2}{c_2 \Phi(t)} \right). \tag{5.4}$$

Second, we observe that the following estimate is satisfied: for all $p > 0$, there exists a constant C, depending also on T, such that:

$$E(\|Du(t,x)\|^{-2p}_{\mathscr{H}_t}) \le C\, \Phi(t)^{-p}. \tag{5.5}$$

Indeed, this is an immediate consequence of Proposition 4.3: one just needs to consider, in (4.9), $\rho = 1$, $n = N$ and replace t_{n-1} by 0. Then, letting $p = \frac{1}{2}$ and $p = \beta$ in (5.5), we have, respectively:

$$\mathrm{E}[\|Du(t,x)\|^{-1}_{\mathscr{H}_t}] \le C\, \Phi(t)^{-1/2} \quad \text{and} \quad \| \|Du(t,x)\|^{-2}_{\mathscr{H}_t} \|_{L^\beta(\Omega)} \le C\, \Phi(t)^{-1}. \tag{5.6}$$

Finally, Lemma 3.4 implies that

$$\|D^2 u(t,x)\|_{L^\alpha(\Omega;\mathscr{H}_t^{\otimes 2})} \le C\, \Phi(t) \le C\, \Phi(t)^{\frac{1}{2}}, \tag{5.7}$$

where the latter constant C depends on T. Hence, plugging estimates (5.4)–(5.7) into (5.3) we end up with:

$$p_{t,x}(y) \le c_1 \, \Phi(t)^{-1/2} \exp\left(-\frac{(|y - F_0| - c_3 T)^2}{c_2 \Phi(t)}\right),$$

where the constants c_i do not depend on (t, x). This concludes the proof of Theorem 1.1.

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Appendix

This section is devoted to recall the construction of the Hilbert-space-valued stochastic and pathwise integrals used throughout the paper, as well as to establish the corresponding conditional L^p-bounds for them. This is an important point in order to consider the linear stochastic equation satisfied by the iterated Malliavin derivative of the solution of many SPDEs (for a more detailed exposition, see [27, Section 2] and [28]).

More precisely, let $\mathscr{A}$ be a separable Hilbert space and $\{K(t, x),\ (t, x) \in [0, T] \times \mathbb{R}^d\}$ an $\mathscr{A}$-valued predictable process satisfying the following condition:

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(\|K(t, x)\|^p_{\mathscr{A}}) < +\infty, \tag{A.1}$$

where $p \ge 2$. We aim to define the $\mathscr{A}$-valued stochastic integral

$$G \cdot W_t = \int_0^t \int_{\mathbb{R}^d} G(s, y) W(ds, dy), \quad t \in [0, T],$$

for integrands of the form $G = \Gamma(s, dy)K(s, y)$; here we assume that Γ is as described in Remark 3.5. In particular, Γ satisfies condition (1.4), that is:

$$\int_0^T \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(t)(\xi)|^2 \mu(d\xi) dt < +\infty.$$

Note that these assumptions imply that G is a well-defined element of $L^2(\Omega \times [0, T]; \mathscr{H} \otimes \mathscr{A})$. Recall that we denote by $\{\mathscr{F}_t,\ t \ge 0\}$ the (completed) filtration generated by W. Then, the stochastic integral of G with respect to W can be defined componentwise, as follows: let $\{e_j,\ j \in \mathbb{N}\}$ be a complete orthonormal basis of $\mathscr{A}$ and set $G^j := \Gamma(s, dy)K^j(s, y)$, where $K^j(s, y) := \langle K(s, y), e_j\rangle_{\mathscr{A}}$, $j \in \mathbb{N}$. We define

$$G \cdot W_t := \sum_{j\in\mathbb{N}} G^j \cdot W_t,$$

where $G^j \cdot W_t = \int_0^t \int_{\mathbb{R}^d} \Gamma(s, y)K^j(s, y)W(ds, dy)$ is a well-defined real-valued stochastic integral (see [27, Remark 1]). By (A.1), one proves that the above series is convergent in

$L^2(\Omega; \mathscr{A})$ and the limit does not depend on the orthonormal basis. Moreover, $\{G \cdot W_t, \mathscr{F}_t,\ t \in [0, T]\}$ is a continuous square-integrable martingale such that

$$E(\|G \cdot W_T\|_{\mathscr{A}}^2) = E(\|G\|_{\mathscr{H}_T \otimes \mathscr{A}}^2).$$

We also have the following estimate for the pth moment of $G \cdot W_t$ (see [27, Theorem 1]): for all $t \in [0, T]$,

$$E(\|G \cdot W_t\|_{\mathscr{A}}^p) \le C_p \Phi(t)^{\frac{p}{2}-1} \int_0^t \sup_{x \in \mathbb{R}^d} E(\|K(s,x)\|_{\mathscr{A}}^p) J(s)\, ds, \tag{A.2}$$

where we remind that

$$\Phi(t) = \int_0^t \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(s)(\xi)|^2\, \mu(d\xi) ds \quad \text{and} \quad J(s) = \int_{\mathbb{R}^d} |\mathscr{F}\Gamma(s)(\xi)|^2\, \mu(d\xi).$$

Next, we consider a conditional version of (A.2).

Lemma A.1. *For all $p \ge 2$ and $0 \le a < b \le T$, we have:*

$$\begin{aligned} &E[\|G \cdot W_b - G \cdot W_a\|_{\mathscr{A}}^p | \mathscr{F}_a] \\ &\le C_p (\Phi(b) - \Phi(a))^{\frac{p}{2}-1} \int_a^b \sup_{x \in \mathbb{R}^d} E[\|K(s,x)\|_{\mathscr{A}}^p | \mathscr{F}_a]\, J(s)\, ds, \quad \textit{a.s.} \end{aligned}$$

The proof of this result is essentially the same as its non-conditioned counterpart (A.2), except of the use of a conditional Burkholder–Davis–Gundy type inequality for Hilbert-space-valued martingales.

Let us now recall how we define the Hilbert-space-valued pathwise integrals involved in the stochastic equations satisfied by the Malliavin derivative of the solution. Namely, as before, we consider a Hilbert space $\mathscr{A}$, a complete orthonormal system $\{e_j,\ \ j \in \mathbb{N}\}$, and an $\mathscr{A}$-valued stochastic process $\{Y(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ such that, for $p \ge 2$,

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} E(\|Y(t,x)\|_{\mathscr{A}}^p) < +\infty. \tag{A.3}$$

Then, we define the following pathwise integral, with values in $L^2(\Omega; \mathscr{A})$:

$$\mathscr{I}_t := \int_0^t \int_{\mathbb{R}^d} Y(s,y)\, \Gamma(s,dy) ds := \sum_{j \in \mathbb{N}} \left(\int_0^t \int_{\mathbb{R}^d} \langle Y(s,y), e_j \rangle_{\mathscr{A}}\, \Gamma(s,dy) ds \right) e_j, \quad t \in [0,T],$$

where Γ is again as general as described in Remark 3.5. Moreover, a direct consequence of the considerations in [25, p. 24] is that:

$$E(\|\mathscr{I}_t\|_{\mathscr{A}}^p) \le \left(\int_0^t \Gamma(s, \mathbb{R}^d)\, ds \right)^{p-1} \int_0^t \sup_{z \in \mathbb{R}^d} E(\|Y(s,z)\|_{\mathscr{A}}^p)\, \Gamma(s, \mathbb{R}^d)\, ds. \tag{A.4}$$

In the paper, we need the following straightforward conditional version of the above estimate (A.4).

Lemma A.2. *Let $p \geq 2$. Then, for any σ-field $\mathscr{G}$, we have:*

$$E[\|\mathscr{I}_t\|_{\mathscr{A}}^p|\mathscr{G}] \leq \left(\int_0^t \Gamma(s, \mathbb{R}^d)\, ds\right)^{p-1} \int_0^t \sup_{z\in\mathbb{R}^d} E[\|Y(s,z)\|_{\mathscr{A}}^p|\mathscr{G}]\, \Gamma(s, \mathbb{R}^d)\, ds, \quad a.s.$$

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