On the implied volatility of Asian options under stochastic volatility models

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Abstract

In this paper we study the short-time behavior of the at-the-money implied volatility for arithmetic Asian options with fixed strike price. The asset price is assumed to follow the Black-Scholes model with a general stochastic volatility process. Using techniques of the Malliavin calculus such as the anticipating Itô's formula we first compute the level of the implied volatility of the option when the maturity converges to zero. Then, we find and short maturity asymptotic formula for the skew of the implied volatility that depends on the roughness of the volatility model. We apply our general results to the SABR model and the rough Bergomi model, and provide some numerical simulations that confirm the accurateness of the asymptotic formula for the skew.

Keywords: Stochastic volatility, Asian options, Malliavin calculus, implied volatility

1 Introduction

This paper is devoted to the study of the implied volatility of arithmetic Asian options with a payoff of the form

$$(A_T - K)_+, \tag{1}$$

where A_T is the average asset price in a time interval $[0, T]$ and K denotes the strike price. The behavior of the implied volatility for vanilla options has been the object of several works (see for example Alòs and García-Lorite [6] and the references therein). However, the case of exotic options is less studied. In particular, the asymptotic of Asian implied volatilities has been studied by Pirjol and Zhu [13] in the case of local volatility. In this paper, the authors make use of large deviations techniques to get accurate approximation formulas for the implied volatility. Nevertheless, up to our knowledge, there are no similar results in the case of stochastic volatilities.

The main problem in the study of arithmetic Asian options is that the average asset price A_T does not have a lognormal behaviour. One way to overcome this difficulty is to write $A_T = M_T$, where M is the forward price defined as $M_t = E_t(A_T)$, being $E_t(A)$ the conditional expectation at time t. Under some conditions, we can write the stochastic differential equation satisfied by M_t as $dM_t = \phi_t M_t dW_t$, were W is a standard Brownian motion and ϕ is a stochastic process depending also on the maturity T. Then, the problem reduces to study European options where the underlying is given by a stochastic

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volatility model, with a volatility ϕ depending on T. This methodology allows to adapt the results on vanilla options to options on a non lognormal-type distribution and it can also be applied to other European-type options. See for example Alòs, García-Lorite and Muguruza [7] for an application of this technique to the analysis of the VIX skew.

This approach will allow to study the short-end behavior of the at-the-money implied volatility (ATMI) of Asian options for local, stochastic, and rough volatilities. In particular, we will see that

- The short-end limit of the ATMI is equal to $\frac{\sigma_0}{3}$, where σ_0 denotes the short-end limit of the spot volatility.
- If prices and volatilities are uncorrelated, the short-end skew slope of the ATMI is equal to $\frac{\sqrt{3}\sigma_0}{30}$. In the correlated case, this short-end skew slope depends on the correlation parameter and on the roughness of the volatility process. In particular, it is of the order $H - \frac{1}{2}$ when time to maturity tends to zero, where $H \in (0, 1)$ denotes the Hurst parameter. That is, for rough volatilities, we observe a blow-up that is of the same order as the one we observe in vanilla options. In the case of local volatilities, our results fit the asymptotic analysis in Pirjol and Zhu [13].

Our main tool for the proof of these results will be the Anticipating Itô's formula form the Malliavin calculus (see Theorem 1 below). This formula was also used in Alòs and León [4] to compute the short-time level and skew of the implied volatility of floating strike Asian options under Black-Scholes model with constant volatility.

The paper is organized as follows: in Section 2 we introduce the main elements of the Malliavin calculus needed through the paper, and the main problem and notations are given in Section 3. A Malliavin decomposition of the Asian option price is studied in Section 4. Section 5 is devoted to obtain adequate representations of the at-the-money volatility skew. In Section 6 we derive the main results of the paper, that is, the short-end asymptotics of the level and the skew of the implied volatility. Finally, Section 7 is devoted to the application of the main results to the constant volatility case, the SABR model and the rough Bergomi model, together with some numerical simulations. The Appendix contains some Malliavin derivatives computations needed through the paper.

2 A primer on Malliavin Calculus

We introduce the elementary notions of the Malliavin calculus used in this paper (see for instance in Alòs and García-Lorite [6] and Nualart and Nualart [12]). Let us consider a standard Brownian motion $Z = (Z_t)_{t\in[0,T]}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we done by $\mathcal{F}_t$ the filtration generated by Z_t. Let $\mathcal{S}^Z$ be the set of random variables of the form

$$F = f(Z(h_1), \dots, Z(h_n)), \tag{2}$$

with $h_1, \dots, h_n \in L^2([0,T])$, $Z(h_i)$ denotes the Wiener integral of the function h_i, for $i = 1, .., n$, and $f \in C_b^\infty(\mathbb{R}^n)$ (i.e., f and all its partial derivatives are bounded). Then the Malliavin derivative of F, $D^Z F$, is defined as the stochastic process given by

$$D_s^Z F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(h_1), \dots, W(h_n))h_j(s), \quad s \in [0,T].$$

This operator is closable from $L^p(\Omega)$ to $L^p(\Omega; L^2([0,T]))$, for all $p \geq 1$, and we denote by $\mathbb{D}_Z^{1,p}$ the closure of $\mathcal{S}^Z$ with respect to the norm

$$||F||_{1,p} = \left(\mathbb{E}\,|F|^p + \mathbb{E}||D^Z F||^p_{L^2([0,T])}\right)^{1/p}.$$

We also consider the iterated derivatives $D^{Z,n}$ for all integers $n > 1$ whose domains will be denoted by $\mathbb{D}_Z^{n,p}$, for all $p \geq 1$. We will use the notation $\mathbb{L}_Z^{n,p} := L^p([0,T]; \mathbb{D}_Z^{n,p})$.

Our main tool will be the following change-of-variable formula, see for e.g. Theorem 1 in Alòs [1].

Theorem 1 (Anticipating Itô's Formula)**.** *Consider a process of the form*

$$X_t = X_0 + \int_0^t u_s dZ_s + \int_0^t v_s ds,$$

where X_0 is an $\mathcal{F}_0$-measurable random variable and u and v are $\mathcal{F}_t$-adapted processes in $L^2([0,T] \times \Omega)$.

Consider also a process $Y_t = \int_t^T \theta_s ds$, for some $\theta \in \mathbb{L}_Z^{1,2}$. Let $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be a $C^{1,2}([0,T] \times \mathbb{R}^2)$ function such that there exists a positive constant C such that, for all $t \in [0,T]$, F and its derivatives evaluated in (t, X_t, Y_t) are bounded by C. Then it follows that for all $t \in [0,T]$,

$$\begin{aligned} F(t, X_t, Y_t) = F(0, X_0, Y_0) &+ \int_0^t \partial_s F(s, X_s, Y_s) ds + \int_0^t \partial_x F(s, X_s, Y_s) dX_s \\ &+ \int_0^t \partial_y F(s, X_s, Y_s) dY_s + \int_0^t \partial_{xy}^2 F(s, X_s, Y_s) u_s D^- Y_s ds \\ &+ \frac{1}{2} \int_0^t \partial_{xx}^2 F(s, X_s, Y_s) u_s^2 ds, \end{aligned}$$

where $D^- Y_s = \int_s^T D_s^Z \theta_r dr$.

3 Statement of the problem and notation

In this paper we study fixed strike arithmetic Asian call options under stochastic volatility models. We denote by $(V_t)_{t \in [0,T]}$ the value of such options where T is the maturity. Then, the payoff can be written as

$$V_T = (A_T - K)_+, \qquad A_T = \frac{1}{T} \int_0^T S_t dt,$$

where $(S_t)_{t \in [0,T]}$ is the price of the underlying asset and K is the fixed strike price.

We assume without lost of generality that the interest rate is equal to zero and we consider the following general stochastic volatility model for the underlying asset price

$$dS_t = \sigma_t S_t dW_t, \qquad W_t = \rho W_t' + \sqrt{(1 - \rho^2)} B_t, \tag{3}$$

where $S_0 > 0$ is fixed, W_t, W_t', and B_t are three standard Brownian motions on $[0,T]$ defined on the same complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$. We assume that W_t' and B_t are independent and $\rho \in [-1,1]$ is the correlation coefficient between W_t and W_t'.

We consider the following assumption on the stochastic volatility of the asset price.

Hypothesis 1. *The process $\sigma = (\sigma_t)_{t \in [0,T]}$ is adapted to the filtration generated by W', a.s. positive and continuous, and satisfies that for all $t \in [0,T]$,*

$$c_1 \leq \sigma_t \leq c_2,$$

for some positive constants c_1 and c_2.

It is important to observe that the results of Sections 3 and 4 hold in the general case of European call options with value V_t at time t and payoff of the form $V_T = (A_T - K)_+$, where A_T is a random variable which is positive and square-integrable and is adapted to the filtration of W on $[0, T]$, as in Alòs, García-Lorite and Muguruza [7]. In this general setting, we define the forward price as the martingale $M_t = \mathbb{E}_t(A_T)$ and we consider the log-forward price $X_t = \log(M_t)$.

In the particular case of model (3), applying the stochastic Fubini's theorem we get the following expressions for A_T and M_t

$$\begin{aligned}A_T = \frac{1}{T}\int_0^T S_t dt &= \frac{1}{T}\int_0^T \left(S_0 + \int_0^t \sigma_u S_u dW_u\right) dt = \\ &= S_0 + \frac{1}{T}\int_0^T \sigma_u S_u \left(\int_u^T dt\right) dW_u = \\ &= S_0 + \frac{1}{T}\int_0^T (T-u)\sigma_u S_u dW_u,\end{aligned}$$

which implies that

$$dM_t = \frac{\sigma_t S_t (T-t)}{T} dW_t = \phi_t M_t dW_t, \tag{4}$$

where $\phi_t := \frac{\sigma_t S_t (T-t)}{T M_t}$. We observe that $M_0 = S_0 > 0$. Furthermore, the log-forward price satisfies

$$dX_t = \phi_t dW_t - \frac{1}{2}\phi_t^2 dt. \tag{5}$$

Remark 1. *The process ϕ_t corresponds to the stochastic volatility in Alòs, García-Lorite and Muguruza [7] which depends also on T. One can easily check that Hypothesis 1 implies that ϕ_t is positive a.s. and belongs to $L^p([0,T] \times \Omega)$, for all $p \geq 2$. In fact, Hypothesis 1 implies that for all $p \geq 2$, S_t belongs to $L^p([0,T] \times \Omega)$, A_T belongs to $L^p(\Omega)$, and M_t^{-1} belongs to $L^p([0,T] \times \Omega)$. This latter fact corresponds to* **(H2)** *in Alòs, García-Lorite and Muguruza [7].*

The goal of this paper is to study the implied volatility of the call option V_t which is defined as follows. We denote by $BS(t, x, k, \sigma)$ the classical Black-Scholes price of a European call with time to maturity $T - t$, log-stock price x, log-strike price k and volatility σ. That is,

$$\begin{aligned}BS(t, x, k, \sigma) &= e^x N(d_+(k, \sigma)) - e^k N(d_-(k, \sigma)), \\ d_\pm(k, \sigma) &= \frac{x - k}{\sigma\sqrt{T-t}} \pm \frac{\sigma}{2}\sqrt{T-t},\end{aligned}$$

where N is the cumulative distribution function of the standard normal random variable. It is well-known, that the Black-Scholes price satisfies the following PDE

$$\partial_t BS(t, x, k, \sigma) - \frac{1}{2}\sigma^2 \partial_x BS(t, x, k, \sigma) + \frac{1}{2}\sigma^2 \partial_{xx}^2 BS(t, x, k, \sigma) = 0. \tag{6}$$

Next, we observe that, as $BS(T, x, k, \sigma) = (e^x - e^k)_+$ for every $\sigma > 0$, the price of our call option can be written as

$$V_t = \mathbb{E}_t(BS(T, X_T, k, v_T)), \qquad v_t = \sqrt{\frac{1}{T-t}\int_t^T \phi_s^2 ds}. \tag{7}$$

In particular, $V_T = BS(T, X_T, k, v_T)$. Then, we define the implied volatility of the option as $I(t,k) = BS^{-1}(t, X_t, k, V_t)$, and we denote by $I(t, k^*)$ the corresponding at-the-money (ATM) implied volatility which, in the case of zero interest rates, takes the form $BS^{-1}(t, X_t, X_t, V_t)$.

Following Alòs et al. [7], we will apply Malliavin calculus techniques introduced in Section 2 in order to obtain formulas for

$$\lim_{T\to 0} I(0,k^*) \quad \text{and} \quad \lim_{T\to 0} \partial_k I(0,k^*)$$

under the general stochastic volatility model (3).

In our setting, since we have three Brownian motions W, W' and B, if h is a a random variable in $L^2([0,T])$, then we have in view of relation (3) that

$$W(h) = \rho W'(h) + \sqrt{(1-\rho^2)}B(h).$$

Then, a random variable in $\mathbb{D}^{1,2}_{W'} \cap \mathbb{D}^{1,2}_{B}$ is also in $\mathbb{D}^{1,2}_{W}$. In fact, it is easy to see that if X is a random variable in $\mathcal{S}^W$, then

$$D^W X = \rho D^{W'} X + \sqrt{1-\rho^2} D^B X. \tag{8}$$

Thus, we deduce that for all $X \in \mathbb{D}^{1,2}_{W'} \cap \mathbb{D}^{1,2}_{B}$,

$$D^W X = \rho D^{W'} X + \sqrt{1-\rho^2} D^B X. \tag{9}$$

We will need the following additional assumption on the Malliavin differentiability of the stochastic volatility process.

Hypothesis 2. *For $p \geq 2$, $\sigma \in \mathbb{L}^{2,p}_{W'}$.*

Remark 2. *Hypotheses 1 and 2 imply that ϕ_t belongs to $\mathbb{L}^{2,p}_{W}$ and A_T belong to $\mathbb{D}^{2,p}_{W}$ for all $p \geq 2$. This hypothesis on A_T corresponds to* **(H1)** *in Alòs, García-Lorite and Muguruza* [7].

Finally, in order to give the asymptotic skew of the implied volatility as a function of the roughness of the stochastic volatility process we consider the following assumption.

Hypothesis 3. *There exists $H \in (0,1)$ such that for all $0 \leq r \leq t \leq T$*

$$D^{W'}_r \sigma_t \approx c(t-r)^{H-\frac{1}{2}},$$

where c is a constant. The notation $\approx$ means that the difference contains terms of higher order in $L^p(\Omega)$-norm for all $p \geq 1$. We will use this notation throughout all the paper.

4 Decomposition of the option price

In this section we provide a closed form formula for the price of the call option V_t under the stochastic volatility model (3). The main result is the following theorem, which is an extension of Theorem 3 in Alòs [1].

Theorem 2. *Consider the model* (5) *and the process v_t defined in* (7)*, where ϕ_t is positive a.s. and belongs to $\mathbb{L}^{1,p}_{W}$, for all $p \geq 2$. Then, the following relation holds for all $t \in [0,T]$,*

$$V_t = \mathbb{E}_t\left(BS(t, X_t, k, v_t)\right) + \mathbb{E}_t\left(\int_t^T H(s, X_s, k, v_s)\phi_s \left(\int_s^T D^W_s \phi_r^2 dr\right) ds\right),$$

where $H(s, X_s, k, v_s) = \frac{1}{2}(\partial^3_{xxx} BS(s, X_s, k, v_s) - \partial^2_{xx} BS(s, X_s, k, v_s))$.

Proof. The proof follows similar ideas as in Alòs [1]. Applying Theorem 1 to the function $BS(t, X_t, k, v_t)$, we obtain, after an approximation argument, that

$$
\begin{aligned}
BS(T, X_T, k, v_T) &= BS(t, X_t, k, v_t) + \int_t^T \partial_s BS(s, X_s, k, v_s) ds \\
&+ \int_t^T \partial_x BS(s, X_s, k, v_s) \left(-\frac{1}{2}\phi_s^2 ds + \phi_s dW_s \right) \\
&+ \int_t^T \partial_\sigma BS(s, X_s, k, v_s) \left(\frac{v_s^2}{2(T-s)v_s} - \frac{\phi_s^2}{2(T-s)v_s} \right) ds \\
&+ \int_t^T \partial_{\sigma x}^2 BS(s, X_s, k, v_s) \frac{\phi_s}{2(T-s)v_s} \left(\int_s^T D_s^W \phi_r^2 dr \right) ds \\
&+ \frac{1}{2} \int_t^T \partial_{xx}^2 BS(s, X_s, k, v_s) \phi_s^2 ds.
\end{aligned}
$$

By adding and subtracting $\frac{1}{2}\int_t^T v_s^2(\partial_{xx}^2 BS(s, X_s, k, v_s) - \partial_x BS(s, X_s, k, v_s))ds$ to the expression above we get that

$$
\begin{aligned}
& BS(T, X_T, k, v_T) = BS(t, X_t, k, v_t) \\
&+ \int_t^T \left(\partial_s BS(s, X_s, k, v_s) - \frac{1}{2} v_s^2 \partial_x BS(s, X_s, k, v_s) + \frac{1}{2} v_s^2 \partial_{xx}^2 BS(s, X_s, k, v_s) \right) ds \\
&+ \int_t^T \partial_x BS(s, X_s, k, v_s) \phi_s dW_s - \int_t^T \partial_\sigma BS(s, X_s, k, v_s) \frac{\phi_s^2 - v_s^2}{2(T-s)v_s} ds \\
&+ \int_t^T \partial_{\sigma x}^2 BS(s, X_s, k, v_s) \frac{\phi_s}{2(T-s)v_s} \left(\int_s^T D_s^W \phi_r^2 dr \right) ds \\
&+ \frac{1}{2} \int_t^T (\partial_{xx}^2 BS(s, X_s, k, v_s) - \partial_x BS(s, X_s, k, v_s))(\phi_s^2 - v_s^2) ds.
\end{aligned}
$$

The first integral in the above expression is equal to zero due to formula (6). Finally, taking the conditional expectation and using the classical relationship between the Gamma, the Vega and the Delta,

$$
\frac{\partial_\sigma BS(t, x, k, \sigma)}{\sigma(T-t)} = (\partial_{xx}^2 BS(t, x, k, \sigma) - \partial_x BS(t, x, k, \sigma)), \tag{10}
$$

we conclude that

$$
\begin{aligned}
& \mathbb{E}_t \left(BS(T, X_T, k, v_T) \right) = \mathbb{E}_t \left(BS(t, X_t, k, v_t) \right) \\
&+ \mathbb{E}_t \left(\frac{1}{2} \int_t^T (\partial_{xxx}^3 BS(s, X_s, k, v_s) - \partial_{xx}^2 BS(s, X_s, k, v_s)) \phi_s \left(\int_s^T D_s^W \phi_r^2 dr \right) ds \right),
\end{aligned}
$$

which completed the proof. □

5 ATM implied volatility skew

In this section we derive an expression for the ATM implied volatility skew of the call option V_t under the stochastic volatility model (5). Our first result is an adaptation of Theorem 4.2 in Alòs, León and Vives [2].

Theorem 3. *Consider the model* (5) *and the process* v_t *where* ϕ_t *is positive a.s. and belongs to* $\mathbb{L}_W^{1,p}$, *for all* $p \geq 2$. *Then, for every* $t \in [0,T]$ *the following holds*

$$\partial_k I(t,k^*) = \frac{\mathbb{E}_t\left(\int_t^T (\partial_k H(s,X_s,k^*,v_s) - \frac{1}{2}H(s,X_s,k^*,v_s))\Lambda_s ds\right)}{\partial_\sigma BS(t,X_t,k^*,I(t,k^*))},$$

where $H(s,X_s,k,v_s) = \frac{1}{2}(\partial^3_{xxx}BS(s,X_s,k,v_s) - \partial^2_{xx}BS(s,X_s,k,v_s))$ *and* $\Lambda_s = \phi_s \int_s^T D_s^W \phi_r^2 dr$.

Proof. This proof follows the same ideas as in the proof of Theorem 4.2 in Alòs, León and Vives [2]. Since $V_t = BS(t,X_t,k,I(t,k))$, the following equation holds

$$\partial_k V_t = \partial_k BS(t,X_t,k,I(t,k)) + \partial_\sigma BS(t,X_t,k,I(t,k))\partial_k I(t,k).$$

On the other hand, using Theorem 2, we get that

$$\partial_k V_t = \partial_k \mathbb{E}_t\left(BS(t,X_t,k,v_t)\right) + \mathbb{E}_t\left(\int_t^T \partial_k H(s,X_s,k,v_s)\Lambda_s ds\right).$$

Combining both equations, we obtain that that the volatility skew $\partial_k I(t,k)$ is equal to

$$\frac{\mathbb{E}_t\left(\int_t^T \partial_k H(s,X_s,k,v_s)\Lambda_s ds\right) + \mathbb{E}_t\left(\partial_k BS(t,X_t,k,v_t)\right) - \partial_k BS(t,X_t,k,I(t,k))}{\partial_\sigma BS(t,X_t,k,I(t,k))}.$$

Finally, using the fact that

$$\partial_k BS(t,x,k^*,\sigma) = -\frac{1}{2}(e^x - BS(t,x,k^*,\sigma))$$

and Theorem 2 we conclude that

$$\begin{aligned}
&\mathbb{E}_t\left(\partial_k BS(t,X_t,k^*,v_t)\right) - \partial_k BS(t,X_t,k^*,I(t,k^*)) \\
&= \frac{1}{2}\left(\mathbb{E}_t\left(BS(t,X_t,k^*,v_t)\right) - BS(t,X_t,k^*,I(t,k^*))\right) \\
&= \frac{1}{2}\left(\mathbb{E}_t(BS(t,X_t,k^*,v_t)) - V_t\right) = -\frac{1}{2}\mathbb{E}_t\left(\int_t^T H(s,X_s,k^*,v_s)\Lambda_s ds\right),
\end{aligned}$$

which completes the proof. □

The result of Theorem 3 is the first step in the computation of the short time limit of the ATM skew of our call option. The second step consists in applying the Anticipating Itô's Formula in order to get a decomposition for the term that appears in the right hand side of the skew in Theorem 3.

Proposition 1. *Consider the model* (5) *and the process* v_t *defined in* (7), *where* ϕ_t *is positive a.s. and belongs to* $\mathbb{L}_W^{2,p}$, *for all* $p \geq 2$. *Then, for all* $t \leq T$,

$$\begin{aligned}
\mathbb{E}_t\left(\int_t^T G(s,X_s,k,v_s)\Lambda_s ds\right) &= \mathbb{E}_t\left(G(t,X_t,k,v_t)J_t\right) \\
&+ \mathbb{E}_t\left(\frac{1}{2}\int_t^T (\partial^3_{xxx} - \partial^2_{xx})G(s,X_s,k,v_s)J_s\Lambda_s ds\right) \qquad (11) \\
&+ \mathbb{E}_t\left(\int_t^T \partial_x G(s,X_s,k,v_s)\phi_s D^- J_s ds\right),
\end{aligned}$$

where $G(t,X_t,k,v_t) = (\partial_k H(t,X_t,k,v_t) - \frac{1}{2}H(t,X_t,k,v_t))$, $J_t = \int_t^T \Lambda_s ds$, *and* $D^- J_s = \int_s^T D_s^W \Lambda_r dr$.

Proof. Applying Theorem 1 to the function $(\partial_k H(t, X_t, k, v_t) - \frac{1}{2} H(t, X_t, k, v_t)) \int_t^T \Lambda_s ds$, we obtain that

$$\begin{aligned}
\int_t^T G(s, X_s, k, v_s)\Lambda_s ds &= G(t, X_t, k, v_t)J_t \\
&+ \int_t^T \left(\partial_s G(s, X_s, k, v_s) + \frac{v_s^2}{2(T-s)v_s}\partial_v G(s, X_s, k, v_s)\right) J_s ds \\
&+ \int_t^T \partial_x G(s, X_s, k, v_s) J_s \left(-\frac{1}{2}\phi_s^2 ds + \phi_s dW_s\right) \\
&- \int_t^T \partial_v G(s, X_s, k, v_s) J_s \frac{\phi_s^2}{2(T-s)v_s} ds + \int_t^T \partial_{vx}^2 G(s, X_s, k, v_s) J_s \Lambda_s \frac{1}{2(T-s)v_s} ds \\
&+ \int_t^T \partial_x G(s, X_s, k, v_s)\phi_s D^- J_s ds + \frac{1}{2}\int_t^T \phi_s^2 \partial_{xx}^2 G(s, X_s, k, v_s) J_s ds.
\end{aligned}$$

By adding and subtracting the term $\frac{1}{2}\int_t^T v_s^2(\partial_{xx}^2 G(s, X_s, k, v_s) - \partial_x G(s, X_s, k, v_s))ds$ to the expression above we get that

$$\begin{aligned}
\int_t^T G(s, X_s, k, v_s)\Lambda_s ds &= G(t, X_t, k, v_t)J_t \\
&+ \int_t^T (\partial_s G(s, X_s, k, v_s) + \frac{1}{2} v_s^2(\partial_{xx}^2 G(s, X_s, k, v_s) - \partial_x G(s, X_s, k, v_s)))J_s ds \\
&+ \int_t^T \frac{1}{2}(\partial_{xx}^2 G(s, X_s, k, v_s) - \partial_x G(s, X_s, k, v_s))(\phi_s^2 - v_s^2) J_s ds \\
&- \int_t^T \partial_v G(s, X_s, k, v_s)\frac{\phi_s^2 - v_s^2}{2(T-s)v_s} J_s ds + \int_t^T \partial_x G(s, X_s, k, v_s) J_s \phi_s dW_s \\
&+ \int_t^T \partial_{vx}^2 G(s, X_s, k, v_s) J_s \Lambda_s \frac{1}{2(T-s)v_s} ds + \int_t^T \partial_x G(s, X_s, k, v_s)\phi_s D^- J_s ds.
\end{aligned}$$

Next, equations (6) and (10) imply that

$$\begin{aligned}
&\partial_s G(s, X_s, k, v_s) - \frac{1}{2} v_s^2 \partial_x G(s, X_s, k, v_s) + \frac{1}{2} v_s^2 \partial_{xx}^2 G(s, X_s, k, v_s) = 0 \text{ and} \\
&\partial_{xx}^2 G(s, X_s, k, v_s) - \partial_x G(s, X_s, k, v_s) = \frac{\partial_v G(s, X_s, k, v_s)}{v_s(T-s)}.
\end{aligned}$$

Finally, taking conditional expectations, we complete the desired proof. □

6 Asymptotics for the ATM implied volatility

In this section we proceed to the key results of this paper. We start showing some approximations for ϕ and the Malliavin derivatives that will be used throughout this section.

We will need the following additional assumption on the continuity of the paths of the volatility process.

Hypothesis 4. *There exits $\gamma > 0$ such that, for all $p \geq 1$ and $0 \leq s \leq r \leq T$,*

$$(\mathbb{E}|\sigma_r - \sigma_s|^p)^{\frac{1}{p}} \leq c(r-s)^\gamma.$$

Lemma 1. *Under Hypotheses 1,2 and 4 the following holds for all* $0 \le s \le r \le T$,

$$\phi_r \approx \frac{\sigma_0(T-r)}{T}, \tag{12}$$

$$\phi_r^2 \approx \frac{\sigma_0^2(T-r)^2}{T^2}, \tag{13}$$

$$D_s^W S_r \approx S_0\sigma_0,$$

$$D_s^W M_r \approx \frac{\sigma_0 S_0(T-s)}{T},$$

$$D_s^W \phi_r \approx \frac{\rho(T-r)D_s^{W'}\sigma_r}{T} + \frac{(T-r)s\sigma_0^2}{T^2}, \tag{14}$$

$$D_s^W \phi_r^2 \approx \frac{\sigma_0\rho(T-r)^2 D_s^{W'}\sigma_r}{T^2} + \frac{(T-r)^2 s\sigma_0^3}{T^3}, \tag{15}$$

where we recall that the notation $\approx$ *means that the difference contains terms of higher order in* $L^p(\Omega)$*-norm for all* $p \ge 1$.

Proof. The decomposition for ϕ_r follows by applying Itô's lemma to the function

$$F(S_s, M_s) := \frac{\sigma_0 S_s(T-r)}{TM_s}, \quad 0 \le s \le r.$$

Observe that

$$\phi_r = F(S_r, M_r) + (\sigma_r - \sigma_0)\frac{S_r(T-r)}{TM_r}.$$

Then, we get

$$F(S_r, M_r) = \frac{\sigma_0(T-r)}{T} + \frac{(T-r)}{T}\left\{\int_0^r \frac{\sigma_0}{M_s}dS_s - \int_0^r \frac{\sigma_0 S_s}{M_s^2}dM_s + \int_0^r \frac{\sigma_0^3 S_s^3}{M_s^3}\frac{(T-s)^2}{T^2}ds\right\}.$$

Then, using Hypothesis 4, we conclude (12). Similarly, applying Itô's formula to the function $F^2(S_s, M_s)$ and using Hypothesis 4, we obtain (13).

The other expressions follow from the Malliavin derivatives (24) and (25) computed in the Appendix. □

6.1 ATM implied volatility level

In this subsection we compute the at-the-money level of the implied volatility of an Asian call option under the stochastic volatility model (3) when the maturity converges to zero. The main tool in our analysis is the following result (Theorem 6 in Alòs et al. [7]) which uses the general framework detailed in Section 2.

Theorem 4. *Assume that for all* $p > 1$, $A_T \in \mathbb{D}_W^{2,p}$, $M_t^{-1} \in L^p([0,T]\times\Omega)$, *and*

$$\lim_{T\to 0}\mathbb{E}\int_0^T \frac{\Lambda_s}{v_s^2(T-s)}ds = 0, \tag{16}$$

$$\lim_{T\to 0}\frac{1}{v_0 T^2}\mathbb{E}\int_0^T\left(\int_s^T D_s^W\phi_r^2 dr\right)^2 ds = 0, \tag{17}$$

where $\Lambda_s = \phi_s\int_s^T D_s^W\phi_r^2 dr$. *Then,*

$$\lim_{T\to 0}\left(I(0,k^*) - \mathbb{E}\sqrt{\frac{1}{T}\int_0^T \phi_s^2 ds}\right) = 0.$$

We next present the main result of this subsection.

Theorem 5. *Under Hypotheses 1, 2, 3, and 4, the ATM implied volatility of an Asian call option under the stochastic volatility model* (3) *satisfies that*

$$\lim_{T\to 0} I(0,k^*) = \frac{\sigma_0}{\sqrt{3}}.$$

Proof. It is easy to check that Hypotheses 1 and 2 imply the first two hypotheses of Theorem 4 (see Remarks 1 and 2). Moreover, using Lemma 1 one can show that hypotheses (16) and (17) are also satisfied. In fact,

$$\int_0^T \frac{\Lambda_s}{v_s^2(T-s)}ds \approx \int_0^T \frac{3T}{\sigma_0(T-s)^2}\int_s^T \left(\frac{\sigma_0\rho(T-r)^2 D_s^{W'}\sigma_r}{T^2} + \frac{(T-r)^2 s\sigma_0^3}{T^3}\right) drds.$$

The second term converges to 0 as $T \to 0$. Using Hypothesis 3, the first term can be approximated by

$$\int_0^T \frac{3}{(T-s)^2}\int_s^T \frac{\rho(T-r)^2 c(r-s)^{H-\frac{1}{2}}}{T}drds,$$

which converges to 0 as $T \to 0$ for all $H \in (0,1)$. This shows (16).

Moreover,

$$\begin{aligned}&\int_0^T \left(\int_s^T D_s^W \phi_r^2 dr\right)^2 ds\\ &\quad\approx \int_0^T \left(\int_s^T \left(\frac{\sigma_0\rho(T-r)^2 c(r-s)^{H-\frac{1}{2}}}{T^2} + \frac{(T-r)^2 s\sigma_0^3}{T^3}\right) dr\right)^2 ds,\end{aligned}$$

and (17) holds for all $H \in (0,1)$.

Using the convexity correction given in (4.1.22) of Alòs and García-Lorite [6] together with Theorem 4, we obtain that

$$\lim_{T\to 0}\mathbb{E}\sqrt{\frac{1}{T}\int_0^T \phi_s^2 ds} = \frac{\sigma_0}{\sqrt{3}} - \lim_{T\to 0}\frac{1}{8\sqrt{T}}\mathbb{E}\int_0^T \frac{\left(\mathbb{E}_r\int_r^T D_r^W\phi_s^2 ds\right)^3}{\left(\sqrt{\mathbb{E}_r\int_0^T\sigma_s^2 ds}\right)^3}dr.$$

Finally, using Lemma 1 it is easy to check that the second term is equal to 0, which concludes the desired result. □

6.2 ATM implied volatility skew

The next result will allow us to find the limiting behaviour of the at-the-money skew of an Asian option under the stochastic volatility.

Lemma 2. *Under Hypotheses 1, 2, and 4, the following holds for all* $0 \le s \le T$,

$$\begin{aligned}\int_s^T D_s^W \Lambda_r dr \approx{}& \frac{\sigma(T-s)^5\left(6\rho^2T^2\left((D_r^{W'}\sigma_u)^2 + (D_s^{W'}D_r^{W'}\sigma_u)^2\right) + 6\rho^2\sigma_0 T^2 D_s^{W'}D_r^{W'}\sigma_u\right)}{45T^5}\\ &- \frac{\rho\sigma^3(s-T)^5\left(3D_r^{W'}\sigma_u\left(9s+\left(2\rho^2-1\right)T\right) + D_s^{W'}D_r^{W'}\sigma_u(11s+T)\right)}{45T^4}\\ &+ \frac{\sigma^5(s-T)^5\left(\rho^4T(T-s)+\rho^2T(s-T)+2s(T-10s)\right)}{45T^5},\end{aligned}$$

where $\Lambda_r = \phi_r\int_r^T D_r^W\phi_u^2 du$.

Proof. By the definition of Λ_s, we have

$$\int_s^T D_s^W \Lambda_r dr = \int_s^T D_s^W \left(\phi_r \int_r^T D_r^W \phi_u^2 du\right) dr$$
$$= \int_s^T \left((D_s^W \phi_r) \int_r^T D_r^W \phi_u^2 du + \phi_r \int_r^T D_s^W D_r^W \phi_u^2 du \right) dr,$$

where

$$D_s^W D_r^W \phi_u^2 = 2(D_s^W \phi_u D_r^W \phi_u + \phi_u D_s^W D_r^W \phi_u).$$

We next find an approximation for $D_s^W D_r^W \phi_u$. Using the Malliavin derivatives computed in (26) and (27), we obtain the following approximations for $0 \leq s \leq r \leq u \leq T$,

$$\begin{aligned}
D_s^B D_r^{W'} S_u &\approx \rho\sqrt{1-\rho^2} S_0 \sigma_0^2, \\
D_s^{W'} D_r^{W'} S_u &\approx \rho^2 S_0 \sigma_0^2 + \rho S_0 D_s^{W'} \sigma_r, \\
D_s^{W'} D_r^B S_u &\approx \sqrt{1-\rho^2} S_0 D_s^{W'} \sigma_r + \rho\sqrt{1-\rho^2} S_0 \sigma_0^2, \\
D_s^{W'} D_r^{W'} M_u &\approx \frac{\rho(T-r)(\sigma_0^2 \rho S_0 + S_0 D_s^{W'} \sigma_r)}{T}, \\
D_s^B D_r^{W'} M_u &\approx \frac{\rho\sqrt{1-\rho^2}(T-r) S_0 \sigma_0^2}{T}, \\
D_s^{W'} D_r^B M_u &\approx \frac{\sqrt{1-\rho^2}(T-r)(\sigma_0^2 \rho S_0 + S_0 D_s^{W'} \sigma_r)}{T}, \\
D_s^B D_r^B M_u &\approx \frac{(1-\rho^2)(T-r)\sigma_0^2 S_0}{T},
\end{aligned}$$

and

$$D_s^W D_r^W \phi_u \approx \frac{(T-u)\Big(\rho\sigma_0 T \left(2r + s + \left(\rho^2 - 1\right) T\right) D_r^{W'} \sigma_u + \rho^2 T^2 D_s^{W'} D_r^{W'} \sigma_u}{T^3}$$
$$+ \frac{\sigma_0^3 \left(\rho^4 T(s-r) + \rho^2 T(r-s) + s(2r-T)\right)\Big)}{T^3}.$$

This, together with (14) gives

$$D_s^W D_r^W \phi_u^2 \approx \frac{2(T-u)^2 \left((D_r^{W'} \sigma_u)^2 \rho^2 T^2 + \rho\sigma^2 T \left(3r + 2s + \left(\rho^2 - 1\right) T\right) D_r^{W'} \sigma_u \right)}{T^4}$$
$$+ \frac{2(T-u)^2 \left(\rho^2 \sigma T^2 D_s^{W'} D_r^{W'} \sigma_u + \sigma^4 \left(\rho^4 T(s-r) + \rho^2 T(r-s) + s(3r-T)\right) \right)}{T^4}.$$

Finally, using also (15), we conclude the proof. □

The next lemma analyses the limiting behaviour of the last two terms on the right hand side of the equation (11).

Lemma 3. *Under Hypotheses 1,2, 3, and 4 the following holds*

$$\lim_{T \to 0} \frac{1}{\sqrt{T}} \mathbb{E}_t \left(\frac{1}{2} \int_t^T (\partial_{xxx}^3 - \partial_{xx}^2) G(s, X_s, k, v_s) J_s \Lambda_s ds \right) = 0$$

and

$$\lim_{T \to 0} \frac{1}{\sqrt{T}} \mathbb{E}_t \left(\int_t^T \partial_x G(s, X_s, k, v_s) \phi_s D^- J_s ds \right) = 0.$$

Proof. Direct differentiation gives the following approximations for $s \leq T$

$$(\partial^3_{xxx} - \partial^2_{xx})G(s, X_s, k, v_s) \approx \frac{3\sqrt{\frac{3}{2\pi}}\left(\sigma_0^2(s-T) - 36\right)e^{\frac{1}{24}\sigma_0^2(s-T)+X_0}}{8\sigma_0^5(T-s)^{5/2}},$$

$$\partial_x G(s, X_s, k, v_s) \approx \frac{3\sqrt{\frac{3}{2\pi}}e^{\frac{1}{24}\sigma_0^2(s-T)+X_0}}{2\sigma_0^3(T-s)^{3/2}},$$

where we have used that for small s and x, $X_s \approx X_0$, $v_s \approx v_0 \approx \frac{\sigma_0}{\sqrt{3}}$, and $e^{-x} \approx (1-x)$. Moreover,

$$\Lambda_s = \phi_s \int_s^T D_s^W \phi_r^2 dr \approx \frac{2\rho\sigma_0^2(T-s)}{T^3}\int_s^T (T-r)^2 D_s^{W'}\sigma_r dr + \frac{2\sigma^4 s(T-s)^4}{3T^4},$$

and

$$J_s = \int_s^T \Lambda_r dr \approx \int_s^T \left(\frac{2\rho\sigma_0^2(T-r)}{T^3}\int_r^T (T-u)^2 D_r^{W'}\sigma_u du\right) dr + \frac{\sigma^4(T-s)^5(5s+T)}{45T^4}. \tag{18}$$

Using Lemma 2 and substituting all the approximations to the last two terms on the right hand side of the equation (11) yields to

$$\frac{1}{\sqrt{T}}\mathbb{E}_t\left(\frac{1}{2}\int_t^T (\partial^3_{xxx} - \partial^2_{xx})G(s, X_s, k, v_s)J_s\Lambda_s ds\right) \approx O(T) + O(T^2),$$

$$\frac{1}{\sqrt{T}}\mathbb{E}_t\left(\int_t^T \partial_x G(s, X_s, k, v_s)\phi_s D^- J_s ds\right) \approx O(T) + O(T^2),$$

from which the desired result follows. □

We finally provide the main result of this paper, which is the value of the skew for the at-the-money Asian call option under stochastic volatility.

Theorem 6. *Under Hypotheses 1, 2, 3, and 4 the following holds*

$$\lim_{T\to 0} \partial_k I(0, k^*) = \begin{cases} \frac{\sqrt{3}\sigma_0}{30} & \text{if} \quad H > \frac{1}{2} \\ \frac{\sqrt{3}\rho c}{\sigma_0 5} + \frac{\sqrt{3}\sigma_0}{30} & \text{if} \quad H = \frac{1}{2} \\ \pm\infty & \text{if} \quad H < \frac{1}{2}. \end{cases}$$

Proof. Straightforward differentiation gives that

$$\partial_\sigma BS(0, X_0, k^*, \sigma) = \frac{\sqrt{T}e^{X_0 - \frac{\sigma_0^2 T}{8}}}{\sqrt{2\pi}}$$

and

$$G(0, X_0, k, v_0) \approx \frac{3\sqrt{\frac{3}{2\pi}}e^{X_0 - \frac{\sigma_0^2 T}{24}}}{2\sigma_0^3 T^{3/2}}.$$

Therefore, together with (18), allows us to conclude that

$$\lim_{T\to 0}\frac{1}{\partial_\sigma BS(0, X_0, k^*, I(0, k^*))}\mathbb{E}\left(G(0, X_0, k, v_0)\int_0^T \Lambda_s ds\right)$$
$$= \lim_{T\to 0}\frac{3\sqrt{3}\rho}{\sigma_0 T^5}\int_0^T \left((T-r)\int_r^T (T-u)^2 D_r^{W'}\sigma_u du\right) dr + \frac{\sqrt{3}\sigma_0}{30}.$$

Applying Theorem 3, Proposition 1, and Hypothesis 3, we finally conclude that

$$\lim_{T\to 0}\partial_k I(0,k^*) = \lim_{T\to 0}\frac{3\sqrt{3}\rho}{\sigma_0 T^5}\int_0^T \left((T-r)\int_r^T (T-u)^2 c(u-r)^{H-\frac{1}{2}}du\right)dr + \frac{\sqrt{3}\sigma_0}{30},$$

from which the desired result follows.

□

7 Numerical analysis

In this section we present numerical evidence of the adequacy of Theorems 5 and 6 in different settings.

7.1 The Black-Scholes model under constant volatility

We consider the Black-Scholes model (3) under constant volatility $\sigma > 0$, that is,

$$dS_t = \sigma S_t dW_t, \qquad S_t = S_0 e^{\sigma W_t - \frac{\sigma^2}{2}t}.$$

Appealing to Theorems 5 and 6 we conclude that the level and the skew of the at-the-money implied volatility satisfy that

$$\lim_{T\to 0} I(0,k^*) = \frac{\sigma}{\sqrt{3}} \quad \text{and} \quad \lim_{T\to 0}\partial_k I(0,k^*) = \frac{\sigma\sqrt{3}}{30}.$$

Notice that these results coincide with the ones obtained in Pirjol and Zhu [13], see Section 7.2 below.

We next proceed with numerical simulations that will confirm the presented results. The parameters of the simulation are the following $S_0 = 10$, $T = \frac{1}{252}$ and $\sigma \in [0.1, 0.2, \ldots, 1.4]$.

We use Control Variates method in order to get estimates of an Asian call option price. As a control variate we use geometric Asian call option which price is calculated as follows

$$\begin{aligned}
BS_{gasian} &= e^{-\frac{1}{4}\sigma_G^2 T} S_0 N(d_1) - KN(d_2),\\
d_1 &= \frac{\log\frac{S_0}{K} + \frac{1}{4}\sigma_G^2 T}{\sigma_G\sqrt{T}},\\
d_2 &= d_1 - \sigma_G\sqrt{T},\\
\sigma_G &= \frac{\sigma}{\sqrt{3}}.
\end{aligned} \tag{19}$$

Finally, Asian call option price estimator has the following form

$$\begin{aligned}
\hat{BS}_{asian} &= \frac{1}{N}\sum_{i=1}^N V_T^i - c^*\frac{1}{N}\sum_{i=1}^N (\hat{BS}^i_{gasian} - BS_{gasian}),\\
c^* &= \frac{\sum_{i=1}^N (V_T^i - \frac{1}{N}\sum_{i=1}^N A_T^i)(\hat{BS}^i_{gasian} - BS_{gasian})}{\sum_{i=1}^N (\hat{BS}^i_{gasian} - BS_{gasian})^2},\\
\hat{BS}^i_{gasian} &= \max(\sqrt{S_0^i S_1^i \ldots S_m^i} - K, 0),
\end{aligned} \tag{20}$$

where $N = 2000000$, $m = 50$, $V_T^i = \max(A_T^i - K, 0)$ and sub index i indicates quantity estimated on a realisation of a path from Monte Carlo simulation.

For the skew estimation we use finite difference approximation of the following form

$$\partial_k \hat{I}(0,k^*) = \frac{\hat{I}(0, k^* \log(1+dk)) - \hat{I}(0, \frac{k^*}{\log(1+dk)})}{2\log(1+dk)}, \tag{21}$$

where $dk = 0.001$.

In order to retrieve implied volatility we use algorithm presented in Jäckel [11]. Then, the at-the-money level and the skew of the implied volatility are presented on Figures 1 and 2, respectively. We conclude that the results of the numerical simulation are in line with the presented theoretical formulas.

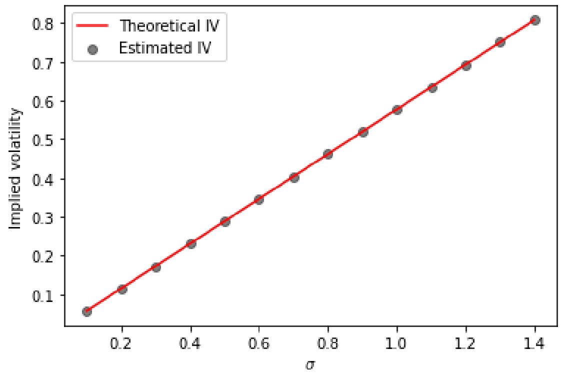


Figure 1: At-the-money Level of the Implied under the Black-Scholes.

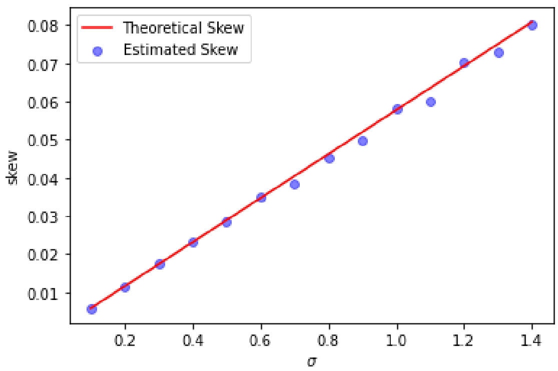


Figure 2: At-the-money Skew of the Implied Volatility under the Black-Scholes.

7.2 Local volatility model

In Pirjol and Zhu [13], the authors consider a local volatility of the form $\sigma_t = \sigma(S_t)$, where $\sigma(.)$ is a twice differentiable function. Then, in their Proposition 19, they show that the small maturity asymptotic of the implied volatility for an Asian option has the following expansion in $x = \log(\frac{K}{S_0})$ around the ATM point

$$\lim_{T\to 0} I(0,k^*) = \frac{\sigma(S_0)}{\sqrt{3}}\left(1 + \left(\frac{1}{10} + \frac{3\sigma'(S_0)}{5\sigma(S_0)}S_0\right)x + O(x^2)\right).$$

In particular,

$$\lim_{T\to 0} \partial_k I(0,k^*) = \frac{1}{\sqrt{3}}\left(\frac{1}{10}\sigma(S_0) + \frac{3}{5}S_0\sigma'(S_0)\right). \tag{22}$$

On the other hand, we apply our Theorem 6 in the case of local volatility, that is, we assume that $\rho = 1$ and

$$dS_t = \sigma(S_t)S_t dW_t,$$

where $\sigma \in C_b^3(\mathbb{R})$ (bounded with bounded derivatives) and $\sigma(x) \geq c > 0$, for all $x \in \mathbb{R}$. In this case, for $r \leq u$, we have that

$$D_r\sigma(S_u) \approx S_0\sigma'(S_0)\sigma(S_0),$$

so Hypothesis 3 is satisfied with $c = S_0\sigma'(S_0)\sigma(S_0)$ and $H = \frac{1}{2}$. Thus, applying Theorem 6, we obtain

$$\lim_{T\to 0} \partial_k I(0, k^*) = \frac{\sqrt{3}}{5} S_0\sigma'(S_0) + \frac{\sqrt{3}\sigma(S_0)}{30},$$

which is the same as in (22). This serves as one more evidence of the validity of Theorem 6.

7.3 The SABR model

In this section we consider the SABR stochastic volatility model with skewness parameter 1 which is the most common case from a practical point of view, see Section 2.3.2 in Alòs and García-Lorite [6]. This corresponds to equation (3), where S_t denotes the forward price of the underlying asset and

$$d\sigma_t = \alpha\sigma_t dW_t', \qquad \sigma_t = \sigma_0 e^{\alpha W_t' - \frac{\alpha^2}{2}t}.$$

where $\alpha > 0$ is the volatility of volatility.

Notice that this model does not satisfy Hypothesis 1, so a truncation argument as in Alòs and Shiraya [5] is needed in order to apply the results in the previous sections.

In this case for $r \leq t$, we have that $D_r^{W'}\sigma_t = \alpha\sigma_t$ which implies that Hypothesis 3 holds with $c = \alpha\sigma_0$ and $H = \frac{1}{2}$. Moreover, Hypothesis 4 holds with $\gamma < 1/2$. Therefore, appealing to Theorem 6 we conclude that

$$\lim_{T\to 0} \partial_k I(0, k^*) = \frac{\sqrt{3}\rho\alpha}{5} + \frac{\sqrt{3}\sigma_0}{30}$$

The parameters of a Monte Carlo simulation are the following

$$S_0 = 10,\ T = \frac{1}{252},\ dt = \frac{T}{50},\ \alpha = 0.5,\ \rho = -0.3,\ \sigma_0 = (0.1, 0.2, \ldots, 1.4).$$

In order to get estimates of an Asian call option we use Antithetic Variates. The estimate of the price is defined as follows

$$\hat{V}_{sabr} = \frac{\frac{1}{N}\sum_{i=1}^N V_T^i + \frac{1}{N}\sum_{i=1}^N V_T^{i,A}}{2}, \tag{23}$$

where $N = 2000000$, sub index A denotes the value of an Asian call option calculated on the antithetic trajectory of Monte Carlo path.

We use equation (21) in order to get estimates of the skew.

In Figure 3 we present the results of a Monte Carlo simulation which aims to evaluate numerically the level of the at-the-money Asian call option implied volatility under the SABR model. In Figure 4 we present a Monte Carlo estimate of the implied volatility skew of the at-the-money Asian call option.

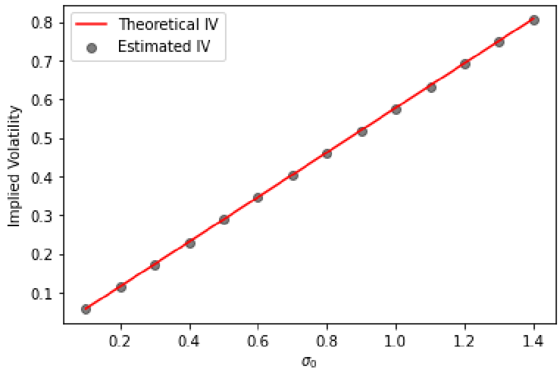


Figure 3: At-the-money Level of the Implied Volatility under the SABR.

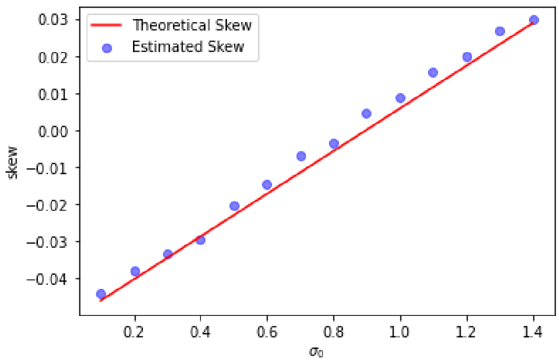


Figure 4: At-the-money Skew of the Implied Volatility under the SABR.

7.4 The rough Bergomi model

The rough Bergomi stochastic volatility model asssumes equation (3) with

$$\sigma_t^2 = \sigma_0^2 e^{v\sqrt{2H}Z_t - \frac{1}{2}v^2 t^{2H}}, \qquad Z_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s',$$

where $H \in (0,1)$ and $v > 0$, see Example 2.5.1 in Alòs and García-Lorite [6].

As for the SABR model, a truncation argument as in Alòs and Shiraya [5] is needed in order to apply the results in the previous sections, as Hypothesis 1 is not satisfied. Moreover, for $r \leq t$, we have

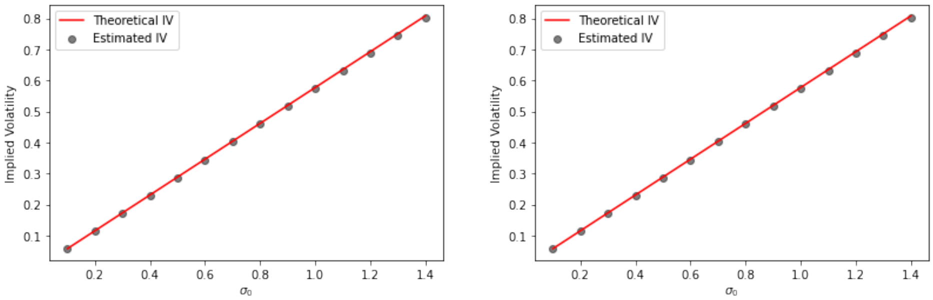
$$D_r^{W'} \sigma_t = \frac{1}{2} \sigma_t v \sqrt{2H} (t-r)^{H-\frac{1}{2}}.$$

Thus, Hypothesis 3 holds with $c = \frac{1}{2}\sigma_0^2 v\sqrt{2H}$ and Hypothesis 4 holds for any $\gamma < H$. The parameters of a Monte Carlo simulation are the following

$$S_0 = 10,\ T = 0.001,\ dt = \frac{T}{50},\ H = (0.4, 0.7),\ v = 0.5,\ \rho = -0.3,\ \sigma_0 = (0.1, 0.2, \ldots, 1.4).$$

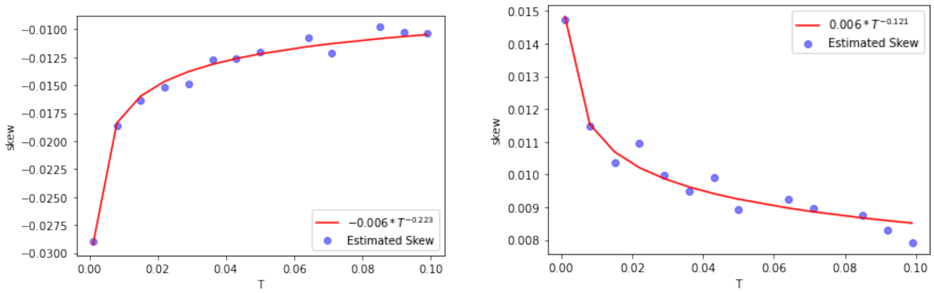
In order to get estimates of the price of an Asian call option under Rough Bergomi model we use a combination of Antithetic and Control Variates presented in equations (20) and (23).

The level of at-the-money implied volatility of an Asian call option is presented on Figures 5. One can see that the result is independent of H.



(a) H=0.4 (b) H=0.7

Figure 5: At-the-money Level of the Implied Volatility under the Rough Bergomi

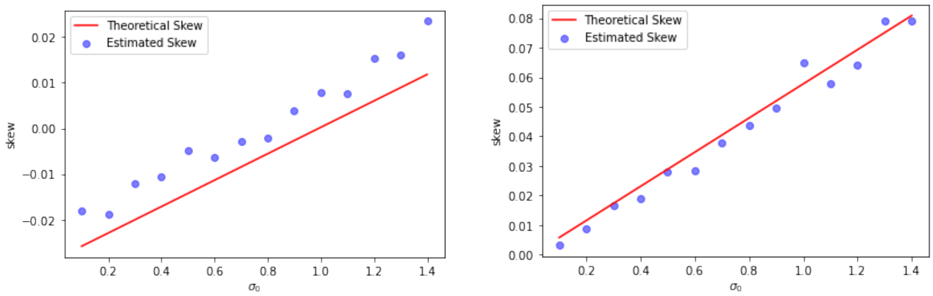


(a) H=0.4 (b) H=0.7

Figure 6: ATM Implied Volatility Skew as a function of T under Rough Bergomi

On the Figure 6 we present ATM implied volatility skew as a function of maturity of an option for two different values of H.

Due to the blow up of the at-the-money implied volatility skew of an Asian call option when $H < \frac{1}{2}$ we present $T^{\frac{1}{2}-H}\partial_k \hat{I}(0, k^*)$ for $H = 0.4$ and $\partial_k \hat{I}(0, k^*)$ for $H = 0.7$ on Figure 7.



(a) H=0.4 (b) H=0.7

Figure 7: ATM Implied Volatility Skew as a function of σ_0 under Rough Bergomi

A Computation of Malliavin derivatives

In this section we provide the computations of the first and second Malliavin derivatives of the processes S_t, M_t and ϕ_t defined in Section 2.

Using the fact that σ_t is adapted to the filtration of W' and the formula for the derivative of a stochastic integral (see for example (3.6) in Nualart and Nualart [12]), we get that, for $0 \le s \le r \le T$,

$$
\begin{aligned}
D_s^{W'} S_r &= S_r \left(\rho\sigma_s - \frac{1}{2}\int_s^r D_s^{W'}\sigma_u^2 du + \int_s^r D_s^{W'}\sigma_u dW_u \right), \\
D_s^B S_r &= S_r \sigma_s \sqrt{1-\rho^2}, \\
D_s^{W'} M_r &= \frac{\rho\sigma_s S_s (T-s)}{T} + \int_s^r \frac{(T-u) D_s^{W'}(\sigma_u S_u)}{T} dW_u, \\
D_s^B M_r &= \frac{\sqrt{1-\rho^2}\sigma_s S_s (T-s)}{T} + \int_s^r \frac{(T-u)\sigma_u D_s^B(S_u)}{T} dW_u.
\end{aligned}
$$

Moreover, appealing to (9), we find that

$$
\begin{aligned}
D_s^{W} S_r &= \rho S_r \left(-\frac{1}{2}\int_s^r D_s^{W'}\sigma_u^2 du + \int_s^r D_s^{W'}\sigma_u dW_u \right) + S_r\sigma_s, \\
D_s^{W} M_r &= \frac{\sigma_s S_s (T-s)}{T} + \rho \int_s^r \frac{(T-u) D_s^{W'}(\sigma_u S_u)}{T} dW_u \\
&\quad + \sqrt{1-\rho^2} \int_s^r \frac{(T-u) D_s^{B}(\sigma_u S_u)}{T} dW_u.
\end{aligned} \tag{24}
$$

Finally, from the definition of ϕ_t, we conclude that

$$
\begin{aligned}
D_s^W \phi_r &= \frac{\rho(T-r) D_s^{W'}(\sigma_r S_r)}{T M_r} - \frac{\rho(T-r) S_r \sigma_r D_s^{W'} M_r}{T M_r^2} \\
&\quad + \frac{\sqrt{1-\rho^2}(T-r) D_s^{B}(\sigma_r S_r)}{T M_r} - \frac{\sqrt{1-\rho^2}(T-r) S_r \sigma_r D_s^{B} M_r}{T M_r^2}.
\end{aligned} \tag{25}
$$

We next compute the second Malliavin derivatives. Similarly as before, using the fact that we can differentiate Lebesgue integrals of stochastic processes (see for example Proposition 3.4.3 in Nualart and Nualart [12]), we get that, for $0 \le s \le r \le u \le T$,

$$
\begin{aligned}
D_s^B D_r^{W'} S_u &= S_u \sigma_s \sqrt{1-\rho^2} \left(\rho\sigma_r - \frac{1}{2}\int_r^u D_r^{W'}\sigma_v^2 dv + \int_r^u D_r^{W'}\sigma_v dW_v \right), \\
D_s^{W'} D_r^{W'} S_u &= S_u \left(\rho\sigma_s - \frac{1}{2}\int_s^u D_s^{W'}\sigma_v^2 dv + \int_s^u D_s^{W'}\sigma_v dW_v \right) \\
&\qquad \times \left(\rho\sigma_r - \frac{1}{2}\int_r^u D_r^{W'}\sigma_v^2 dv + \int_r^u D_r^{W'}\sigma_v dW_v \right) \\
&\quad + S_u \left(\rho D_s^{W'}\sigma_r - \frac{1}{2}\int_r^u D_s^{W'} D_r^{W'}\sigma_v^2 dv + \int_r^u D_s^{W'} D_r^{W'}\sigma_v dW_v \right), \\
D_s^{W'} D_r^B S_u &= \sqrt{1-\rho^2} S_u D_s^{W'}\sigma_r + \sqrt{1-\rho^2}\sigma_r D_s^{W'} S_u, \\
D_s^{W'} D_r^{W'} M_u &= \frac{\rho(T-r) D_s^{W'}(\sigma_r S_r)}{T} + \int_r^u \frac{(T-v) D_s^{W'} D_r^{W'}(\sigma_v S_v)}{T} dW_v, \\
D_s^{B} D_r^{W'} M_u &= \frac{\rho(T-r)\sigma_r D_s^B S_r}{T} + \int_r^u \frac{(T-v) D_s^{B} D_r^{W'}(\sigma_v S_v)}{T} dW_v, \\
D_s^{W'} D_r^{B} M_u &= \frac{\sqrt{1-\rho^2}(T-r) D_s^{W'}(\sigma_r S_r)}{T} + \int_r^u \frac{(T-v) D_s^{W'}(\sigma_v D_s^B S_v)}{T} dW_v, \\
D_s^{B} D_r^{B} M_u &= \frac{\sqrt{1-\rho^2}(T-r)\sigma_r D_s^{B}(S_r)}{T} + \int_r^u \frac{(T-v)\sigma_v D_s^B D_r^B S_v}{T} dW_v,
\end{aligned} \tag{26}
$$

and

$$
\begin{aligned}
D_s^W D_r^W \phi_u &= \frac{\rho^2 (T-u) D_s^{W'} D_r^{W'} (\sigma_u S_u)}{T M_u} - \frac{\rho^2 (T-u) D_s^{W'} (\sigma_u S_u) D_s^{W'} M_u}{T M_u^2} \\
&- \frac{\rho^2 (T-u) D_s^{W'} \left(\sigma_u S_u D_r^{W'} M_u \right)}{T M_u^2} + \frac{2\rho^2 (T-u) \sigma_u S_u D_s^{W'} M_u D_r^{W'} M_u}{T M_u^3} \\
&+ \frac{\rho\sqrt{1-\rho^2} (T-u) D_s^{W'} D_r^{B} (\sigma_u S_u)}{T M_u} - \frac{\rho\sqrt{1-\rho^2} (T-u) D_r^{B} (\sigma_u S_u) D_s^{W'} M_u}{T M_u^2} \\
&- \frac{\rho\sqrt{1-\rho^2} (T-u) D_s^{W'} \left(\sigma_u S_u D_r^{B} M_u \right)}{T M_u^2} + \frac{2\rho\sqrt{1-\rho^2} (T-u) \sigma_u S_u D_r^{B} M_u D_s^{W'} M_u}{T M_u^3} \\
&+ \frac{\rho\sqrt{1-\rho^2} (T-u) D_s^{B} D_r^{W'} (\sigma_u S_u)}{T M_u} - \frac{\rho\sqrt{1-\rho^2} (T-u) D_r^{W'} (\sigma_u S_u) D_s^{B} M_u}{T M_u^2} \\
&- \frac{\rho\sqrt{1-\rho^2} (T-u) D_s^{B} \left(\sigma_u S_u D_r^{W'} M_u \right)}{T M_u^2} + \frac{\rho\sqrt{1-\rho^2} (T-u) \sigma_u S_u D_r^{W'} M_u D_s^{B} M_u}{T M_u^3} \\
&+ \frac{(1-\rho^2)(T-u) D_s^{B} D_r^{B} (\sigma_u S_u)}{T M_u} - \frac{(1-\rho^2)(T-u) D_r^{B} (\sigma_u S_u) D_s^{B} M_u}{T M_u^2} \\
&- \frac{(1-\rho^2)(T-u) D_s^{B} \left(S_u \sigma_u D_r^{B} M_u \right)}{T M_u^2} + \frac{2(1-\rho^2)(T-u) S_u \sigma_u D_r^{B} M_u D_s^{B} M_u}{T M_u^3}.
\end{aligned} \tag{27}
$$

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