A LOCAL-TIME CORRESPONDENCE
FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. It is frequently the case that a white-noise-driven parabolic and/or
hyperbolic stochastic partial differential equation (SPDE) can have random-
field solutions only in spatial dimension one. Here we show that in many
cases, where the "spatial operator" is the L^2-generator of a Lévy process X, a
linear SPDE has a random-field solution if and only if the symmetrization of
X possesses local times. This result gives a probabilistic reason for the lack of
existence of random-field solutions in dimensions strictly larger than one.

In addition, we prove that the solution to the SPDE is [Hölder] continuous
in its spatial variable if and only if the said local time is [Hölder] continuous in
its spatial variable. We also produce examples where the random-field solution
exists, but is almost surely unbounded in every open subset of space-time. Our
results are based on first establishing a quasi-isometry between the linear L^2-
space of the weak solutions of a family of linear SPDEs, on one hand, and the
Dirichlet space generated by the symmetrization of X, on the other hand.

We mainly study linear equations in order to present the local-time cor-
respondence at a modest technical level. However, some of our work has
consequences for nonlinear SPDEs as well. We demonstrate this assertion
by studying a family of parabolic SPDEs that have additive nonlinearities.
For those equations we prove that if the linearized problem has a random-
field solution, then so does the nonlinear SPDE. Moreover, the solution to the
linearized equation is [Hölder] continuous if and only if the solution to the
nonlinear equation is, and the solutions are bounded and unbounded together
as well. Finally, we prove that in the cases where the solutions are unbounded,
they almost surely blow up at exactly the same points.

1. INTRODUCTION

We consider the *stochastic heat equation* inspired by the fundamental works of
Pardoux [46–48], Krylov and Rozovskii [36–38], and Funaki [31]. Let $\dot{w}$ denote
space-time white noise, t is nonnegative, x is in $\mathbf{R}^d$, and the Laplacian acts on the
variable x. Then, we have:

$$(1.1) \qquad \begin{vmatrix} \partial_t H(t,x) = (\Delta H)(t,x) + \dot{w}(t,x), \\ H(0,x) = 0. \end{vmatrix}$$

Let us also consider the *stochastic wave equation* of Cabaña [12]:

$$(1.2) \qquad \left| \begin{aligned} &\partial_{tt}W(t\,,x) = (\Delta W)(t\,,x) + \dot{w}(t\,,x), \\ &W(0\,,x) = \partial_t W(0\,,x) = 0. \end{aligned} \right.$$

One of the common features of (1.1) and (1.2) is that they suffer from a *curse of dimensionality*. Namely, these equations can have random-field solutions only in dimension one. Moreover, this curse of dimensionality appears to extend beyond the linear parabolic setting of (1.1) or the linear hyperbolic setting of (1.2). For instance, see Perkins [49, Corollary III.4.3] for an example from superprocesses and Walsh [55, Chapter 9] for one from statistical mechanics.

One can informally ascribe this curse of dimensionality to the folklore observation that while the Laplacian smooths, white noise roughens. In one dimension, the roughening effect of white noise turns out to be small relative to the smoothing properties of the Laplacian, and hence we have a random-field solution. However, in dimensions greater than one white noise is much too rough, and the Laplacian cannot smooth the solution enough to yield a random field.

Dalang and Frangos [17] were able to construct a first fully-rigorous explanation of the curse of dimensionality: They replace white noise by a Gaussian noise that is white in time and colored in space; and then they describe precisely the roughening effect of the noise on the solution, viewed as a random generalized function. See also Brzeźniak and van Neerven [11], Dalang and Mueller [20], Millet and Sanz-Solé [41], Peszat [50], and Peszat and Zabczyk [51]. More recently, Dalang and Sanz-Solé [24] fully studied nonlinear stochastic wave equations driven by noises that are white in time and colored in space, and operators that are arbitrary powers of the Laplacian.

In this article we present a different explanation of the curse of dimensionality. Our approach is to accurately describe the smoothing effect of the Laplacian in the presence of white noise. Whereas the answer of Dalang and Frangos [17] is analytic, ours is probabilistic. For instance, we will soon see that (1.1) and (1.2) have solutions only in dimension one because d-dimensional Brownian motion has local times only in dimension one [Theorem 2.1]. This assertion can also be assembled by combining a very general result of Dalang [15, Theorem 11] with the main result of Hawkes [34].

Similarly, when $d = 1$, the solution to (1.1) [and/or (1.2)] is continuous in x because the local time of one-dimensional Brownian motion is continuous in its spatial variable.

The methods that we employ also give us a local-time paradigm that makes precise the claim that the stochastic PDEs (1.1) and (1.2) "have random-field solutions in dimension $d = 2 - \epsilon$ for all $\epsilon \in (0\,,2)$". See Example 7.4 below, where we introduce a family of SPDEs, on fractals, driven by the Laplacian on that same fractal. It is feasible that further analysis of SPDEs on fractals will enrich our understanding of the subject; for instance, by way of producing counterexamples. But we do not pursue this line here.

An outline of this paper follows: In Section 2 we describe a suitable generalization of the stochastic PDEs (1.1) and (1.2) that have sharp local-time correspondences. This section contains the main results of the paper. The existence theorem of Section 2 is proved in Section 3. Section 4 contains the proof of the necessary and sufficient local-time condition for continuity of our SPDEs in their space variables.

Although we are not aware of any interesting connections between local times and temporal regularity of the solutions of SPDEs, we have included a fifth section that contains a sharp analytic condition for temporal continuity of the SPDEs of Section 2. Also, we produce examples of SPDEs that have random-field solutions which are almost surely unbounded in every open space-time set [Example 5.5]; thus, they have very singular discontinuities of the second kind, by default. We are not aware of such general constructions in the existing literature of stochastic partial differential equations. However, special examples exist in the literature of superprocesses; most notably, see Mytnik and Perkins [44].

Section 6 discusses issues of Hölder continuity in either the space or the time variable. In Section 7 we establish a very general [but somewhat weak] connection between Markov processes, their local times, and solutions to various linear SPDEs. The material of that section is strongly motivated by the recent article of Da Prato [13] who studies Kolmogorov SPDEs that are not unlike those studied here, but also have multiplicative nonlinearities. We go on to produce examples where one can make sense of a random-field solution to (1.1) in dimension $2 - \epsilon$ for all $\epsilon \in (0, 2)$. These solutions in fact turn out to be jointly Hölder continuous, but we will not dwell on that here. Finally, in Section 8 we discuss a parabolic version of the SPDEs of Section 2 that have fairly general additive nonlinearities; we prove the existence of solutions and describe exactly when, and where, these solutions blow up.

The chief aim of this paper is to point out some intimate connections between the local-time theory of Markov processes and families of stochastic partial differential equations. In all cases, we have strived to study the simplest SPDEs that best highlight these connections. But it would also be interesting to study much more general equations.

There is an elegant body of research ["isomorphism theorems"] that relates the local times of a symmetric Markov process X to the square of a certain Gaussian process associated to X [10, 29, 30]. The recent book of Marcus and Rosen [40] contains a thorough and wonderful account of that theory. These isomorphism theorems appear to be different from the results presented here. Diaconis and Evans [26] have introduced yet a different isomorphism theorem.

Finally, we conclude by mentioning what we mean by "local times", as there are many [slightly] different versions in the literature. Given a stochastic process $Y := \{Y_t\}_{t \geq 0}$ on $\mathbf{R}^d$, consider the occupation measures,

$$(1.3) \qquad Z(t, \varphi) := \int_0^t \varphi(Y_s) \, \mathrm{d}s \qquad \text{for all } t \geq 0 \text{ and measurable } \varphi : \mathbf{R}^d \to \mathbf{R}_+.$$

We can identify each $Z(t, \bullet)$ with a measure in the usual way: $Z(t, A) \simeq Z(t; \mathbf{1}_A)$. Then, we say that Y has local times when $Z(t, \mathrm{d}x) \ll dx$ for all t. The local times of Y are themselves defined by $Z(t, x) := Z(t, \mathrm{d}x)/\mathrm{d}x$. It follows that if Y has local times, then $Z(t, \varphi) = \int_{\mathbf{R}^d} Z(t, x)\varphi(x) \, \mathrm{d}x$ a.s. for every $t \geq 0$ and all measurable functions $\varphi : \mathbf{R}^d \to \mathbf{R}_+$. The converse also holds.

2. More general equations

In order to describe when (1.1) and (1.2) have random-field solutions, and why, we study more general equations.

Let $\mathscr{L}$ denote the generator of a d-dimensional Lévy process $X := \{X_t\}_{t \geq 0}$ with characteristic exponent Ψ. We can normalize things so that $\mathrm{E} \exp(i\xi \cdot X_t) =$

$\exp(-t\Psi(\xi))$, and consider $\mathscr{L}$ as an L^2-generator with domain

$$(2.1) \qquad \operatorname{Dom} \mathscr{L} := \left\{ f \in L^2(\mathbf{R}^d) : \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2 |\Psi(\xi)| \, \mathrm{d}\xi < \infty \right\}.$$

As usual, $\hat{f}$ denotes the Fourier transform of f; we opt for the normalization

$$(2.2) \qquad \hat{f}(\xi) := \int_{\mathbf{R}^d} e^{i\xi \cdot x} f(x) \, \mathrm{d}x \qquad \text{for all } f \in L^1(\mathbf{R}^d).$$

[The general L^2-theory of Markov processes is described in great depth in Fukushima, Ōshima, and Takeda [33] in the symmetric case, and Ma and Röckner [39] for the general case.] In this way, we can—and will—view $\mathscr{L}$ as a generalized convolution operator with Fourier multiplier $\hat{\mathscr{L}}(\xi) := -\Psi(\xi)$.

We consider two families of stochastic partial differential equations. The first is the stochastic heat equation for $\mathscr{L}$:

$$(2.3) \qquad \left| \begin{aligned} & \partial_t H(t,x) = (\mathscr{L}H)(t,x) + \dot{w}(t,x), \\ & H(0,x) = 0, \end{aligned} \right.$$

where x ranges over $\mathbf{R}^d$ and t over $\mathbf{R}_+ := [0,\infty)$, and $\mathscr{L}$ acts on the variable x. These equations appear naturally in the literature on parabolic SPDEs [15, 25, 35–38, 46–48].

We also consider hyperbolic SPDEs of the following wave type:

$$(2.4) \qquad \left| \begin{aligned} & \partial_{tt} W(t,x) = (\mathscr{L}W)(t,x) + \dot{w}(t,x), \\ & W(0,x) = \partial_t W(0,x) = 0. \end{aligned} \right.$$

As was the case for their parabolic counterparts, this type of hyperbolic equation also appears naturally in the literature [15, 23, 24].

Recall that $X := \{X_t\}_{t \geq 0}$ is a Lévy process whose generator is $\mathscr{L}$. Let X' denote an independent copy of X and define its symmetrization, à la Paul Lévy, by

$$(2.5) \qquad \bar{X}_t := X_t - X'_t \qquad \text{for all } t \geq 0.$$

It is a standard fact that $\bar{X}$ is a symmetric Lévy process with characteristic exponent $2 \operatorname{Re} \Psi$.

The following is our first main result. The terminology will be made precise in Definition 3.1.

Theorem 2.1. *The stochastic heat equation* (2.3) *has random-field solutions if and only if the symmetric Lévy process $\bar{X}$ has local times. The same is true for the stochastic wave equation* (2.4), *provided that the process X is itself symmetric.*

Remark 2.2. 　　(1) Dalang [15, Theorem 11] provides an analytic condition that is necessary as well as sufficient for the existence [and a.s.-uniqueness] of a random-field solution to a large family of SPDEs that include, but are not limited to, (1.1) and (1.2). Theorem 2.1 identifies Dalang's condition with the existence of local times for the symmetrization of X.

(2) We can apply a theorem of Brzeźniak and van Neerven [11] to parabolic equations of the type (2.3) and deduce Dalang's condition in the case that X is symmetric. See also Kotelenez [35]. These references consider more general parabolic equations that are driven by a pseudo-differential operator $\mathscr{L}$. □

It is well known that Lévy processes in $\mathbf{R}^d$ *do not* have local times when $d \geq 2$ [34]. It follows from this that neither (2.3) nor (2.4) [under a symmetry assumption on X] can ever have random-field solutions in dimensions greater than one. But it is possible that there are no random-field solutions, even in dimension one. Here is one such example; many others exist.

Example 2.3. Suppose X is a strictly stable process in $\mathbf{R}$ with stability index $\alpha \in (0, 2]$. It is well known that the symmetric Lévy process $\bar{X}$ has local times if and only if $\alpha > 1$; see, for example, Hawkes [34]. According to Theorem 2.1 the stochastic heat equation (2.3) has a random-field solution if and only if $\alpha > 1$. Debbi and Dozzi [25] have proven that when $\alpha > 1$, fully nonlinear versions of the parabolic equation (2.3) also have solutions, and those solutions are continuous.

If X is itself symmetric, then $\mathscr{L} = -(-\Delta)^{\alpha/2}$ is the α-dimensional fractional Laplacian [54, Chapter V, §1.1], and also the stochastic wave equation (2.4) has a random-field solution if and only if $\alpha > 1$. Dalang and Sanz-Solé [24, Theorem 5] treat the latter symmetric case for fully nonlinear hyperbolic SPDEs and exhibit the existence of solutions that live in suitable Sobolev spaces. $\qquad\square$

The local-time correspondence of Theorem 2.1 is not a mere accident. In fact, the next two theorems suggest far deeper connections between the solutions to the linear SPDEs of this paper and the theory of local times of Markov processes. We emphasize that the next two theorems assume the existence of a random-field solution to at least one of the stochastic PDEs, (2.3) and/or (2.4). Therefore, they are inherently one-dimensional statements.

Theorem 2.4. *Assume that $d = 1$ and the stochastic heat equation* (2.3) *has a random-field solution* $\{H(t, x)\}_{t \geq 0, x \in \mathbf{R}}$*. Then, the following are equivalent:*

(1) *There exists $t > 0$ such that $x \mapsto H(t, x)$ is a.s. continuous.*
(2) *For all $t > 0$, $x \mapsto H(t, x)$ is a.s. continuous.*
(3) *The local times of $\bar{X}$ are a.s. continuous in their spatial variable.*

The same equivalence is true for the solution W to the stochastic wave equation (2.4)*, provided that the process X is itself symmetric.*

Theorem 2.5. *Assume that $d = 1$ and the stochastic heat equation* (2.3) *has a random-field solution* $\{H(t, x)\}_{t \geq 0, x \in \mathbf{R}}$*. Then, the following are equivalent:*

(1) *There exists $t > 0$ such that $x \mapsto H(t, x)$ is a.s. Hölder continuous.*
(2) *For all $t > 0$, $x \mapsto H(t, x)$ is a.s. Hölder continuous.*
(3) *The local times of $\bar{X}$ are a.s. Hölder continuous in their spatial variable.*

If the process X is itself symmetric, then the preceding conditions are also equivalent to the Hölder continuity of the solution W to the stochastic wave equation (2.4) *in the spatial variable. Finally, the critical Hölder indices of $x \mapsto H(t, x)$, $x \mapsto W(t, x)$ and that of the local times of $\bar{X}$ are the same.*

Blumenthal and Getoor [8] have introduced several "indices" that describe various properties of a Lévy process. We recall below their *lower index* β'':

$$(2.6) \qquad \beta'' := \liminf_{|\xi| \to \infty} \frac{\log \operatorname{Re} \Psi(\xi)}{\log |\xi|} = \sup \left\{ \alpha \geq 0 : \lim_{|\xi| \to \infty} \frac{\operatorname{Re} \Psi(\xi)}{|\xi|^\alpha} = \infty \right\}.$$

Theorem 2.6. *If $\beta'' > d$, then the stochastic heat equation* (2.3) *has a random-field solution that is jointly Hölder continuous. The critical Hölder index is $\leq (\beta'' - d)/2$ for the space variable and $\leq (\beta'' - d)/2\beta''$ for the time variable. Furthermore, if*

X is symmetric, then the same assertions hold for the solution to the stochastic wave equation (2.4), *but the critical Hölder exponent is* $\leq (\beta'' - d)/\beta''$ *for the time variable.*

Remark 2.7. If X is in the domain of attraction of Brownian motion on $\mathbf{R}$, then $\beta'' = 2$, and the critical temporal and spatial index bounds of Theorem 2.6 are respectively $1/2$ and $1/4$ for the stochastic heat equation and $1/2$ and $1/2$ for the stochastic wave equation. For (1.1) and (1.2) these numbers are well known to be the optimal Hölder indices. In fact, the relatively simple Hölder-index bounds of Theorem 2.6 are frequently sharp in fair generality; see Example 5.4.   □

It is well known that $0 \leq \beta'' \leq 2$ [8, Theorem 5.1]. Thus, Theorem 2.6 is inherently a one-dimensional result; this is consistent with the conclusions of Theorem 2.1. Alternatively, one can combine Theorems 2.1 and 2.6 to construct a proof of the fact that $\beta'' \leq 2$. [But of course the original proof of Blumenthal and Getoor is simpler.]

Several things need to be made clear here; the first being the meaning of a "solution". With this aim in mind, we treat the two equations separately.

We make precise sense of the stochastic heat equation (2.3) in much the same manner as Walsh [55]; see also Brzeźniak and van Neerven [11], Dalang [14, 15], and Da Prato [13]. It is harder to give a precise meaning to stochastic hyperbolic equations of the wave type (2.4), primarily because the Green function is typically not a function. This problem is particularly pronounced when one studies nonlinear equations [15, 16, 18–20, 42, 43, 52]. But a certain amount of the difficulty continues to persist in the present linear setting, since it is not so easy to analyze the Green function for the hyperbolic integral operator $\partial_{tt} - \mathscr{L}$. Thus, as part of the present work, we introduce a simple and direct method that makes rigorous sense of (2.4) and other linear SPDEs of this type. One can combine our work with that of Dalang [15, Theorems 5 and 11] to deduce that the solution that our method produces agrees with that of Dalang [15]. This is valid in all cases where there is a process solution in either case, provided that $\mathscr{L}$ is self-adjoint on $L^2(\mathbf{R})$; equivalently, that the process X is symmetric. [This is how the symmetry assumption on X crops up here for hyperbolic equations.]

2.1. **The parabolic case.** In order to describe the meaning of (2.3) we need to first introduce some notation.

Let $\{P_t\}_{t \geq 0}$ denote the semigroup of the driving Lévy process X; that is,

$$(2.7) \qquad\qquad (P_t f)(x) := \mathrm{E} f(x + X_t)$$

for all bounded Borel-measurable functions $f : \mathbf{R}^d \to \mathbf{R}$ [say], all $x \in \mathbf{R}^d$, and all $t \geq 0$. [As usual, $X_0 := 0$.] Formally speaking, $P_t = \exp(t\mathscr{L})$.

Let $\mathscr{S}(\mathbf{R}^d)$ denote the class of all rapidly-decreasing test functions on $\mathbf{R}^d$, and recall that: (i) $\mathscr{S}(\mathbf{R}^d) \subset \mathrm{Dom}(\mathscr{L}) \cap C^\infty(\mathbf{R}^d) \subset L^2(\mathbf{R}^d) \cap C^\infty(\mathbf{R}^d)$, and (ii) for all $\varphi, \psi \in \mathscr{S}(\mathbf{R}^d)$,

$$(2.8) \qquad\qquad \lim_{h \downarrow 0} \left(\frac{P_{t+h}\varphi - P_t\varphi}{h}, \psi \right) = (\mathscr{L} P_t \varphi, \psi).$$

Needless to say, $(\cdot, \cdot)$ denotes the usual Hilbertian inner product on $L^2(\mathbf{R}^d)$. Thus, $v(t, x) := (P_t \varphi)(x)$ solves the Kolmogorov equation $\partial_t v(t, x) = (\mathscr{L} v)(t, x)$, subject to the initial condition that $v(0, x) = \varphi(x)$. This identifies the Green's function for

$\partial_t - \mathscr{L} = 0$. Hence, we can adapt the Green-function method of Walsh [55, Chapter 3], without any great difficulties, to deduce that a *weak solution* to (2.3) is the Gaussian random field $\{H(t, \varphi); t \geq 0, \varphi \in \mathscr{S}(\mathbf{R}^d)\}$, where

$$(2.9) \qquad H(t, \varphi) := \int_0^t \int_{\mathbf{R}^d} (P_{t-s}\varphi)(y)\, w(\mathrm{d}y\,\mathrm{d}s).$$

The stochastic integral is defined in the sense of Wiener.

Proposition 2.8. *The Gaussian random field $\{H(t, \varphi); t \geq 0, \varphi \in \mathscr{S}(\mathbf{R}^d)\}$ is well defined. Moreover, the process $\varphi \mapsto H(t, \varphi)$ is a.s. linear for each $t \geq 0$.*

Proof. On one hand, the Wiener isometry tells us that

$$(2.10) \qquad \mathrm{E}\left(|H(t, \varphi)|^2\right) = \int_0^t \|P_{t-s}\varphi\|_{L^2(\mathbf{R}^d)}^2\, \mathrm{d}s.$$

On the other hand, it is known that each P_s is a contraction on $L^2(\mathbf{R}^d)$. Here is one way to prove this: If ℓ_d denotes the Lebesgue measure on $\mathbf{R}^d$, then the dual Lévy process $-X$ is ℓ_d-symmetric [33, pp. 27–28]. Therefore, the asserted contraction property of P_s follows from Fukushima, Ōshima, and Takeda [33, eq. (1.4.13), p. 28].

It then follows that $\mathrm{E}(|H(t, \varphi)|^2) \leq t\|\varphi\|_{L^2(\mathbf{R}^d)}^2$, which is finite for all $t \geq 0$ and $\varphi \in \mathscr{S}(\mathbf{R}^d)$. This proves that u is a well-defined Gaussian random field indexed by $\mathbf{R}_+ \times \mathscr{S}(\mathbf{R}^d)$. The proof of the remaining property follows the argument of Dalang [15, Section 4] quite closely, and is omitted. $\square$

2.2. **The nonrandom hyperbolic case.** It has been known for some time that hyperbolic SPDEs can be harder to study, or even to define precisely, than their parabolic counterparts. See, for instance, Dalang and Conus [16] and Dalang and Sanz-Solé [23] for the most recent works on nonlinear stochastic wave equations.

In order to define what the stochastic wave equation (2.4) means precisely, we can try to mimic the original Green-function method of Walsh [55]. But we quickly run into the technical problem of not being able to identify a suitable Green function (or even a measure) for the corresponding integral equation. In order to overcome this obstacle, one could proceed as in Dalang and Sanz-Solé [23], for example, but try to generalize the role of their fractional Laplacian. Instead, we opt for a more direct route that is particularly well suited for studying the *linear* SPDEs that are considered here.

In order to better understand (2.4), we first consider the deterministic integro-differential equation,

$$(2.11) \qquad \left| \begin{array}{l} \partial_{tt}u(t, x) = (\mathscr{L}u)(t, x) + f(t, x), \\ u(0, x) = \partial_t u(0, x) = 0, \end{array} \right.$$

where $f : \mathbf{R}_+ \times \mathbf{R}^d \to \mathbf{R}$ is a "nice" function, and the variables t and x range respectively over $\mathbf{R}_+$ and $\mathbf{R}^d$. We can study this equation only under the following symmetry condition:

$$(2.12) \qquad\qquad \text{The process } X \text{ is symmetric.}$$

Equivalently, we assume that Ψ is *real and nonnegative*.

Recall that "ˆ" denotes the Fourier transform in the variable x, and apply it informally to (2.11) to deduce that it is equivalent to the following: For all $t \geq 0$ and $\xi \in \mathbf{R}^d$,

$$
(2.13) \qquad \left|
\begin{array}{l}
\partial_{tt}\hat{u}(t,\xi) = -\Psi(\xi)\hat{u}(t,\xi) + \hat{f}(t,\xi), \\
\hat{u}(0,\xi) = \partial_t\hat{u}(0,\xi) = 0.
\end{array}
\right.
$$

Since X is symmetric, the function Ψ is real and nonnegative. Thus, the preceding equation is an inhomogeneous second-order real ODE [in t] which can be solved explicitly—via Duhamel's principle—in order to produce the following "formula":

$$
(2.14) \qquad \hat{u}(t,\xi) = \frac{1}{\sqrt{\Psi(\xi)}} \int_0^t \sin\left(\sqrt{\Psi(\xi)}\,(t-s)\right) \hat{f}(s,\xi)\, \mathrm{d}s.
$$

We invert the preceding—informally still—to obtain

$$
(2.15) \qquad u(t,x) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \int_0^t \frac{\sin\left(\sqrt{\Psi(\xi)}\,(t-s)\right)}{\sqrt{\Psi(\xi)}}\, e^{-i\xi\cdot x}\hat{f}(s,\xi)\, \mathrm{d}s\, \mathrm{d}\xi.
$$

We can multiply this by a nice function φ, then integrate $[\mathrm{d}x]$ to arrive at

$$
(2.16) \qquad u(t,\varphi) := \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \int_0^t \frac{\sin\left(\sqrt{\Psi(\xi)}\,(t-s)\right)}{\sqrt{\Psi(\xi)}}\, \widehat{\varphi}(\xi)\, \hat{f}(s,\xi)\, \mathrm{d}s\, \mathrm{d}\xi.
$$

We may think of this as the "weak/distributional solution" to (2.11).

2.3. The random hyperbolic case.
Let us follow standard terminology and identify the white noise $\dot{w}$ with the iso-Gaussian process $\{w(h)\}_{h\in L^2(\mathbf{R}_+\times\mathbf{R}^d)}$ as follows:

$$
(2.17) \qquad w(h) := \int_{\mathbf{R}^d} \int_0^\infty h(s,x)\, w(\mathrm{d}s\, \mathrm{d}x).
$$

Next, we define the *Fourier transform* $\hat{w}$ of white noise:

$$
(2.18) \qquad \hat{w}(h) := \frac{w(\hat{h})}{(2\pi)^{d/2}} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \int_0^\infty \hat{h}(s,\xi)\, w(\mathrm{d}s\, \mathrm{d}\xi),
$$

all the time remembering that "$\hat{h}$" refers to the Fourier transform of h in its spatial variable. Suppose $h(s,x) = \varphi_1(s)\varphi_2(x)$ for $t \geq 0$ and $x \in \mathbf{R}^d$, where $\varphi_1 \in L^2(\mathbf{R}_+)$ and $\varphi_2 \in L^2(\mathbf{R}^d)$. Then, it follows from the Wiener isometry and Plancherel's theorem that

$$
(2.19) \qquad \|\hat{w}(h)\|^2_{L^2(\mathbf{R}_+\times\mathbf{R}^d)} = \|h\|^2_{L^2(\mathbf{R}_+\times\mathbf{R}^d)}.
$$

Because $L^2(\mathbf{R}_+) \otimes L^2(\mathbf{R}^d)$ is dense in $L^2(\mathbf{R}_+ \times \mathbf{R}^d)$, this proves that $\hat{w}$ is defined continuously on all of $L^2(\mathbf{R}_+ \times \mathbf{R}^d)$. Moreover, $\hat{w}$ corresponds to a white noise which is correlated with $\dot{w}$, as described by the following formula:

$$
(2.20) \qquad \mathrm{E}\left[w(h_1) \cdot \overline{\hat{w}(h_2)}\right] = \frac{1}{(2\pi)^{d/2}} \int_0^\infty \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}\xi\, \overline{\hat{h}_1(s,\xi)}\, h_2(s,\xi),
$$

valid for all $h_1, h_2 \in L^2(\mathbf{R}_+ \times \mathbf{R}^d)$.

Equation (2.16) justifies the following definition.

Definition 2.9. We define the [Fourier-analytic] *weak solution* to the stochastic wave equation (2.4) as the Wiener integral

$$(2.21) \qquad W(t\,,\varphi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \int_0^t \frac{\sin\left(\sqrt{\Psi(\xi)}\,(t-s)\right)}{\sqrt{\Psi(\xi)}}\, \hat{\varphi}(\xi)\, \hat{w}(\mathrm{d}s\,\mathrm{d}\xi).$$

One can define a corresponding [Fourier-analytic] weak solution to the stochastic heat equation (2.3). But one can verify that it produces a formulation that is equivalent to the Walsh-type solution (2.9). However, in the present setting, the Fourier-analytic definition of a weak solution saves us from having to describe the existence [and regularity] of the Green function for the integral equation (2.11).

Proposition 2.10. *If the symmetry condition* (2.12) *holds, then the stochastic wave equation* (2.4) *has a weak solution W for all $\varphi \in \mathscr{S}(\mathbf{R}^d)$. Moreover, $\{W(t\,,\varphi);$ $t \geq 0,\ \varphi \in \mathscr{S}(\mathbf{R}^d)\}$ is a well-defined Gaussian random field, and $\varphi \mapsto W(t\,,\varphi)$ is a.s. linear for all $t \geq 0$.*

Proof. Thanks to (2.12), Ψ is real and nonnegative. We apply the Wiener isometry to obtain

$$(2.22) \qquad \mathrm{E}\left(|W(t\,,\varphi)|^2\right) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{\varphi}(\xi)|^2 \mathrm{d}\xi \int_0^t \frac{\sin^2\left(\sqrt{\Psi(\xi)}\,(t-s)\right)}{\Psi(\xi)}\, \mathrm{d}s.$$

Because $|\sin\theta/\theta| \leq 1$ for all real numbers θ,

$$
\begin{aligned}
(2.23) \qquad \mathrm{E}\left(|W(t\,,\varphi)|^2\right) &\leq \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{\varphi}(\xi)|^2 \,\mathrm{d}\xi \int_0^t (t-s)^2 \,\mathrm{d}s \\
&= \frac{t^3}{3} \|\varphi\|_{L^2(\mathbf{R}^d)}^2,
\end{aligned}
$$

thanks to Plancherel's theorem. It follows immediately from this that $\{W(t\,,\varphi);\ t \geq 0,\ \varphi \in \mathscr{S}(\mathbf{R}^d)\}$ is a well-defined Gaussian random field. The remainder of the proposition is standard. $\qquad\square$

Remark 2.11. If (2.12) were to fail, then Ψ would in general be complex-valued and the preceding argument would not be valid. This is one of the many instances where (2.12) is used for the stochastic wave equation in this paper. $\qquad\square$

3. EXISTENCE OF RANDOM-FIELD SOLUTIONS: PROOF OF THEOREM 2.1

Let $u := \{u(t\,,\varphi);\ t \geq 0,\ \varphi \in \mathscr{S}(\mathbf{R}^d)\}$ denote the weak solution to either one of the two SPDEs (2.3) or (2.4). In the latter hyperbolic case, by "weak solution" we mean "Fourier-analytic weak solution" in the sense of Definition 2.9. Our present goal is to extend uniquely the Gaussian random field u to a Gaussian random field indexed by $\mathbf{R}_+ \times M$, where M is a maximal subset of $\mathscr{S}'(\mathbf{R}^d)$—the space of all tempered distributions on $\mathbf{R}^d$. Such an M exists thanks solely to functional-analytic facts: Define, temporarily,

$$(3.1) \qquad d_t(\varphi) := \sqrt{\mathrm{E}\left(|u(t\,,\varphi)|^2\right)} \qquad \text{for all } \varphi \in \mathscr{S}(\mathbf{R}^d) \text{ and } t \geq 0.$$

Then, the linearity of u in φ shows that $(\varphi\,,\psi) \mapsto d_t(\varphi - \psi)$ defines a metric for each $t \geq 0$. Let M_t denote the completion of $\mathscr{S}(\mathbf{R}^d)$ with respect to the metric induced by d_t, and define $M := \bigcap_{t \geq 0} M_t$. We identify M with a subset of $\mathscr{S}'(\mathbf{R}^d)$

in the usual way, and think of M as the largest possible family of candidate "test functions" for weak solutions to either the stochastic heat equation (2.3) or the stochastic wave equation (2.4), although M typically contains many distributions other than functions. Standard heuristics from PDEs suggest the following.

Definition 3.1. We say that (2.3) [respectively (2.4)] has a *random-field solution* if and only if $\delta_x \in M$ for all $x \in \mathbf{R}^d$.

When $\delta_x \in M$ we write $u(t,x)$ in place of $u(t,\delta_x)$.

Definition 3.1 is a variation on the following, which is due to Dalang [15, Theorem 11].

Definition 3.2 (Dalang, 1999)**.** We say that (2.3) [respectively (2.4)] has a *D-random-field solution* if and only if (2.3) [respectively (2.4)] has a mild solution $\{u(t,x)\}_{t\geq 0, x\in\mathbf{R}^d}$, defined by Equation (43) in [15] that is continuous [equivalently, locally bounded] in $L^2(\mathrm{P})$.

Definitions 3.1 and 3.2 turn out to be effectively equivalent. Indeed, Theorem 2.1, Propositions 3.3, 3.8, 5.6, and 5.7, and Theorem 11 of Dalang [15] together prove the following:

- The stochastic heat equation (1.1) has a D-random-field solution if and only if it has a random-field solution. Also, the two notions agree up to evanescence.
- If the Lévy process X is symmetric, then the stochastic wave equation (1.2) has a D-random-field solution if and only if it has a random-field solution. Also, the two notions agree up to evanescence.

In order to prove Theorem 2.1 we will need some *a priori* estimates on the weak solutions of both of the stochastic equations (2.3) and (2.4). We proceed by identifying M with generalized Sobolev spaces that arise in the potential theory of symmetric Lévy processes. Now let us begin by studying the parabolic case.

3.1. The parabolic case.

Proposition 3.3. *Let H denote the weak solution* (2.9) *to the stochastic heat equation* (2.3)*. Then, for all $\varphi \in \mathscr{S}(\mathbf{R}^d)$, $\lambda > 0$, and $t \geq 0$,*

$$(3.2) \qquad \frac{1 - e^{-2t/\lambda}}{2}\mathscr{E}(\lambda\,;\varphi) \leq \mathrm{E}\left(|H(t,\varphi)|^2\right) \leq \frac{e^{2t/\lambda}}{2}\mathscr{E}(\lambda\,;\varphi),$$

where

$$(3.3) \qquad \mathscr{E}(\lambda\,;\varphi) := \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}\frac{|\hat{\varphi}(\xi)|^2}{(1/\lambda) + \mathrm{Re}\Psi(\xi)}\,\mathrm{d}\xi.$$

Next we record the following immediate but useful corollary; it follows from Proposition 3.3 by simply setting $\lambda := t$.

Corollary 3.4. *If H denotes the weak solution to the stochastic heat equation* (2.3)*, then for all $\varphi \in \mathscr{S}(\mathbf{R}^d)$ and $t \geq 0$,*

$$(3.4) \qquad \tfrac{1}{3}\mathscr{E}(t\,;\varphi) \leq \mathrm{E}\left(|H(t,\varphi)|^2\right) \leq 4\mathscr{E}(t\,;\varphi).$$

The key step of the proof of Proposition 3.3 is an elementary real-variable result which we prove next.

Lemma 3.5. *If* $g : \mathbf{R}_+ \to \mathbf{R}_+$ *is Borel measurable and nonincreasing, then for all* $t, \lambda > 0$,

$$(3.5) \quad \left(1 - e^{-2t/\lambda}\right) \int_0^\infty e^{-2s/\lambda} g(s) \, \mathrm{d}s \le \int_0^t g(s) \, \mathrm{d}s \le e^{2t/\lambda} \int_0^\infty e^{-2s/\lambda} g(s) \, \mathrm{d}s.$$

Monotonicity is not needed for the upper bound on $\int_0^t g(s) \, \mathrm{d}s$.

Proof of Lemma 3.5. The upper bound on $\int_0^t g(s) \, \mathrm{d}s$ follows simply because $e^{2(t-s)/\lambda} \ge 1$ whenever $t \ge s$. In order to derive the lower bound we write

$$(3.6) \quad \begin{aligned} \int_0^\infty e^{-2s/\lambda} g(s) \, \mathrm{d}s &= \sum_{n=0}^\infty \int_{nt}^{(n+1)t} e^{-2s/\lambda} g(s) \, \mathrm{d}s \\ &\le \sum_{n=0}^\infty e^{-2nt/\lambda} \int_0^t g(s + nt) \, \mathrm{d}s. \end{aligned}$$

Because g is nonincreasing we can write $g(s + nt) \le g(s)$ to conclude the proof. $\square$

Proof of Proposition 3.3. We know from (2.9) and the Wiener isometry that

$$(3.7) \quad \mathrm{E}\left(|H(t, \varphi)|^2\right) = \int_0^t \|P_s \varphi\|_{L^2(\mathbf{R}^d)}^2 \, \mathrm{d}s.$$

Since the Fourier multiplier of P_s is $\exp(-s\Psi)$, we can apply the Plancherel theorem and deduce the following formula:

$$(3.8) \quad \|P_s \varphi\|_{L^2(\mathbf{R}^d)}^2 = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-2s\mathrm{Re}\Psi(\xi)} |\hat{\varphi}(\xi)|^2 \, \mathrm{d}\xi.$$

Because $\mathrm{Re}\Psi(\xi) \ge 0$, Lemma 3.5 readily proves the proposition. $\square$

In (3.3) we defined $\mathscr{E}(\lambda ; \varphi)$ for all $\varphi \in \mathscr{S}(\mathbf{R}^d)$. But in fact, (3.3) defines $\mathscr{E}(\lambda ; \varphi)$ for all tempered distributions φ whose Fourier transform is a function. Moreover, it is possible to verify directly that $\varphi \mapsto \mathscr{E}(\lambda ; \varphi)^{1/2}$ defines a norm on the subset of $\mathscr{S}'(\mathbf{R}^d)$ whose Fourier transforms are functions. In all but uninteresting cases, $\mathscr{S}(\mathbf{R}^d)$ is *not* complete in the norm $\varphi \mapsto \mathscr{E}(\lambda ; \varphi)^{1/2}$. Let $L_{\mathscr{L}}^2(\mathbf{R}^d)$ denote the completion of $\mathscr{S}(\mathbf{R}^d)$ in the norm $\mathscr{E}(\lambda ; \bullet)^{1/2}$. According to Proposition 3.3, the Hilbert space $L_{\mathscr{L}}^2(\mathbf{R}^d)$ can be identified with M. The following is a result about the potential theory of symmetric Lévy processes, but we present a self-contained proof that does not depend on that deep theory.

Lemma 3.6. *The space* $L_{\mathscr{L}}^2(\mathbf{R}^d)$ *does not depend on the value of* λ. *Moreover,* $L_{\mathscr{L}}^2(\mathbf{R}^d)$ *is a Hilbert space in norm* $\mathscr{E}(\lambda ; \bullet)^{1/2}$ *for each fixed* $\lambda > 0$. *Finally, the quasi-isometry* (3.2) *is valid for all* $t \ge 0$, $\lambda > 0$, *and* $\varphi \in L_{\mathscr{L}}^2(\mathbf{R}^d)$.

Proof. We write, temporarily, $L_{\mathscr{L}, \lambda}^2(\mathbf{R}^d)$ for $L_{\mathscr{L}}^2(\mathbf{R}^d)$, and seek to prove that it is independent of the choice of λ.

Define for all distributions φ and ψ,

$$(3.9) \quad \mathscr{E}(\lambda ; \varphi, \psi) := \frac{1}{2(2\pi)^d} \left[\int_{\mathbf{R}^d} \frac{\hat{\varphi}(\xi) \, \overline{\hat{\psi}(\xi)}}{(1/\lambda) + \mathrm{Re}\Psi(\xi)} \, \mathrm{d}\xi + \int_{\mathbf{R}^d} \frac{\hat{\psi}(\xi) \, \overline{\hat{\varphi}(\xi)}}{(1/\lambda) + \mathrm{Re}\Psi(\xi)} \, \mathrm{d}\xi \right].$$

For each $\lambda > 0$ fixed, $(\varphi, \psi) \mapsto \mathscr{E}(\lambda ; \varphi, \psi)$ is a pre-Hilbertian inner product on $\mathscr{S}(\mathbf{R}^d)$, and $\mathscr{E}(\lambda ; \varphi) = \mathscr{E}(\lambda ; \varphi, \varphi)$.

Thanks to Proposition 3.3, for all $\alpha > 0$ there exists a finite and positive constant $c = c_{\alpha,\lambda}$ such that $c^{-1}\mathscr{E}(\alpha\,;\varphi) \le \mathscr{E}(\lambda\,;\varphi) \le c\mathscr{E}(\alpha\,;\varphi)$ for all $\varphi \in \mathscr{S}(\mathbf{R}^d)$. This proves that $L^2_{\mathscr{L},\lambda}(\mathbf{R}^d) = L^2_{\mathscr{L},\alpha}(\mathbf{R}^d)$, whence follows the independence of $L^2_{\mathscr{L}}(\mathbf{R}^d)$ from the value of λ. The remainder of the lemma is elementary. $\qquad\square$

The space $L^2_{\mathscr{L}}(\mathbf{R}^d)$ is a generalized Sobolev space and contains many classical spaces of Bessel potentials, as the following example shows.

Example 3.7. Suppose $\mathscr{L} = -(-\Delta)^{s/2}$ for $s \in (0\,,2]$. Then, $\mathscr{L}$ is the generator of an isotropic stable-s Lévy process, and $L^2_{\mathscr{L}}(\mathbf{R}^d)$ is the space $H_{-s/2}(\mathbf{R}^d)$ of Bessel potentials. For a nice pedagogic treatment see the book of Folland [32, Chapter 6]. $\qquad\square$

3.2. **The hyperbolic case.** The main result of this section is the following quasi-isometry; it is the wave-equation analogue of Proposition 3.3.

Proposition 3.8. *Suppose the symmetry condition* (2.12) *holds, and let* $W := \{W(t\,,\varphi); t \ge 0, \varphi \in \mathscr{S}(\mathbf{R}^d)\}$ *denote the weak solution to the stochastic wave equation* (2.4). *Then,*

$$(3.10) \qquad \tfrac{1}{4}t\mathscr{E}\left(t^2;\varphi\right) \le \mathrm{E}\left(|W(t\,,\varphi)|^2\right) \le 2t\mathscr{E}\left(t^2;\varphi\right)$$

for all $t \ge 0$ *and* $\varphi \in \mathscr{S}(\mathbf{R}^d)$. *Moreover, we can extend* W *by density so that the preceding display continues to remain valid when* $t \ge 0$ *and* $\varphi \in L^2_{\mathscr{L}}(\mathbf{R}^d)$.

Proof. Although Lemma 3.5 is not applicable, we can proceed in a similar manner as we did when we proved the earlier quasi-isometry result for the heat equation (Proposition 3.3). Namely, we begin by observing that since Ψ is real and nonnegative [equivalently, (2.12) holds],

$$(3.11) \qquad \mathrm{E}\left(|W(t\,,\varphi)|^2\right) = \frac{1}{(2\pi)^d}\int_0^t \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}\xi\, \frac{\sin^2\left(\sqrt{\Psi(\xi)}\,s\right)}{\Psi(\xi)}\,|\hat{\varphi}(\xi)|^2.$$

See (2.21). If $\theta > 0$, then $\sin\theta \le \min(1\,,\theta)$, whence

$$(3.12) \qquad \begin{aligned} \mathrm{E}\left(|W(t\,,\varphi)|^2\right) &\le \frac{1}{(2\pi)^d}\int_0^t \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}\xi\, \left(s^2 \wedge \frac{1}{\Psi(\xi)}\right)\,|\hat{\varphi}(\xi)|^2 \\ &\le \frac{t}{(2\pi)^d}\int_{\mathbf{R}^d}\left(t^2 \wedge \frac{1}{\Psi(\xi)}\right)\,|\hat{\varphi}(\xi)|^2\,\mathrm{d}\xi. \end{aligned}$$

The upper bound follows from this and the elementary inequality

$$t^2 \wedge z^{-1} \le 2/(t^{-2}+z),$$

valid for all $z \ge 0$.

In order to derive the [slightly] harder lower bound we first rewrite (3.11) as follows:

$$(3.13) \qquad \mathrm{E}\left(|W(t\,,\varphi)|^2\right) = \frac{t}{2(2\pi)^d}\int_{\mathbf{R}^d}\left(1 - \frac{\sin\left(2\sqrt{\Psi(\xi)}\,t\right)}{2\sqrt{\Psi(\xi)}\,t}\right)\frac{|\hat{\varphi}(\xi)|^2}{\Psi(\xi)}\,\mathrm{d}\xi.$$

We shall analyze the integral by splitting it according to whether or not $\Psi \le 1/t^2$.

Taylor's expansion [with remainder] reveals that if θ is nonnegative, then $\sin\theta$ is at most $\theta - (\theta^3/6) + (\theta^5/120)$. This and a little algebra together show that

$$(3.14) \qquad 1 - \frac{\sin\theta}{\theta} \geq \frac{2\theta^2}{15} \qquad \text{if } 0 \leq \theta \leq 2.$$

Consequently,

$$(3.15)$$

$$\int_{\{\Psi \leq 1/t^2\}} \left(1 - \frac{\sin\left(2\sqrt{\Psi(\xi)}\,t\right)}{2\sqrt{\Psi(\xi)}\,t}\right) \frac{|\hat{\varphi}(\xi)|^2}{\Psi(\xi)}\,\mathrm{d}\xi \geq \frac{8t^2}{15} \int_{\{\Psi \leq 1/t^2\}} |\hat{\varphi}(\xi)|^2\,\mathrm{d}\xi$$

$$\geq \frac{1}{2} \int_{\{\Psi \leq 1/t^2\}} \left(t^2 \wedge \frac{1}{\Psi(\xi)}\right) |\hat{\varphi}(\xi)|^2\,\mathrm{d}\xi.$$

For the remaining integral we use the elementary bound $1 - (\sin\theta/\theta) \geq 1/2$, valid for all $\theta > 2$. This leads to the following inequalities:

$$(3.16)$$

$$\int_{\{\Psi > 1/t^2\}} \left(1 - \frac{\sin\left(2\sqrt{\Psi(\xi)}\,t\right)}{2\sqrt{\Psi(\xi)}\,t}\right) \frac{|\hat{\varphi}(\xi)|^2}{\Psi(\xi)}\,\mathrm{d}\xi \geq \frac{1}{2} \int_{\{\Psi > 1/t^2\}} \frac{|\hat{\varphi}(\xi)|^2}{\Psi(\xi)}\,\mathrm{d}\xi$$

$$= \frac{1}{2} \int_{\{\Psi > 1/t^2\}} \left(t^2 \wedge \frac{1}{\Psi(\xi)}\right) |\hat{\varphi}(\xi)|^2\,\mathrm{d}\xi.$$

The proof concludes from summing up equations (3.15) and (3.16), and then plugging the end result into (3.13). □

We now give a proof of Theorem 2.1.

Proof of Theorem 2.1. Let us begin with (2.3). Proposition 3.3 describes a quasi-isometry between $L^2_{\mathscr{L}}(\mathbf{R}^d)$ and the maximal space M of test functions for weak solutions of (2.3). Thus, M can be identified with the Hilbert space $L^2_{\mathscr{L}}(\mathbf{R}^d)$, and hence (2.3) has random-field solutions if and only if $\delta_x \in L^2_{\mathscr{L}}(\mathbf{R}^d)$ for all $x \in \mathbf{R}^d$. Thanks to Lemma 3.6, the stochastic heat equation (2.3) has random-field solutions if and only if

$$(3.17) \qquad \int_{\mathbf{R}^d} \frac{\mathrm{d}\xi}{\vartheta + \mathrm{Re}\Psi(\xi)} < \infty \qquad \text{for some, and hence all, } \vartheta > 0.$$

The first part of the proof is concluded since condition (3.17) is known to be necessary as well as sufficient for $\bar{X}$ to have local times [34, Theorem 1]. The remaining portion of the proof follows from Proposition 3.8 in much the same way as the first portion was deduced from Proposition 3.3. We emphasize that the symmetry condition (2.12) is only needed here as it is used in the proof of Proposition 3.8; confer with Remark 2.11. □

4. Spatial continuity: Proof of Theorem 2.4

Proof. Since (2.3) has a random-field solution $H(t,x)$, $\bar{X}$ has local times (Theorem 2.1). Let $\varphi := \delta_x - \delta_y$, and note that $|\hat{\varphi}(\xi)|^2 = 2(1 - \cos(\xi(x-y)))$ is a function of $x - y$. Because $H(t,\varphi) = H(t,x) - H(t,y)$, equations (3.7) and (3.8) imply that $z \mapsto H(t,z)$ is a centered Gaussian process with stationary increments for each fixed $t \geq 0$.

Consider the function

$$(4.1) \qquad h(r) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \cos(r\xi)}{1 + \operatorname{Re} \Psi(\xi)} \, d\xi, \qquad \text{defined for all } r \geq 0.$$

According to Lemma 3.6, Proposition 3.3 holds for all tempered distributions $\varphi \in L^2_{\mathscr{L}}(\mathbf{R}^d)$. The existence of random-field solutions is equivalent to the condition that $\delta_x \in L^2_{\mathscr{L}}(\mathbf{R}^d)$ for all $x \in \mathbf{R}$. We apply Proposition 3.3 to $\varphi := \delta_x - \delta_y$, with $\lambda := 1$ [say], and find that

$$(4.2) \qquad \left(1 - e^{-2t}\right) h(|x - y|) \leq \mathrm{E}\left(|H(t, x) - H(t, y)|^2\right) \leq e^{2t} h(|x - y|).$$

Define $\bar{h}$ to be the Hardy–Littlewood nondecreasing rearrangement of h. That is,

$$(4.3) \quad \bar{h}(r) := \inf\{y \geq 0 : g(y) > r\}, \quad \text{where} \quad g(y) := \operatorname{meas}\{r \geq 0 : h(r) \leq y\}.$$

Then according to the proof of Corollary 6.4.4 of Marcus and Rosen [40, p. 274], the stationary-increments Gaussian process $x \mapsto H(t, x)$ has a continuous modification if and only if

$$(4.4) \qquad \int_{0^+} \frac{\bar{h}(r)}{r |\log r|^{1/2}} \, dr < \infty.$$

Next we claim that (4.4) is equivalent to the continuity of the local times of the symmetrized Lévy process $\bar{X}$. We recall that the characteristic exponent of $\bar{X}$ is $2\operatorname{Re}\Psi$, and hence by the Lévy–Khintchine formula it can be written as

$$(4.5) \qquad 2\operatorname{Re}\Psi(\xi) = \sigma^2 \xi^2 + \int_{-\infty}^{\infty} (1 - \cos(\xi x)) \, \nu(dx),$$

where ν is a σ-finite Borel measure on $\mathbf{R}$ with $\int_{-\infty}^{\infty} (1 \wedge x^2) \, \nu(dx) < \infty$. See, for example, Bertoin [5, Theorem 1, p. 13].

Suppose, first, that

$$(4.6) \qquad \text{either } \sigma^2 > 0 \ \text{ or } \int_{-\infty}^{\infty} (1 \wedge |x|) \, \nu(dx) = \infty.$$

Then, (4.4) is also necessary and sufficient for the [joint] continuity of the local times of the symmetrized process $\bar{X}$; confer with Barlow [2, Theorems B and 1]. On the other hand, if (4.6) fails to hold, then $\bar{X}$ is a compound Poisson process. Because $\bar{X}$ is also a symmetric process, the Lévy–Khintchine formula tells us that it has zero drift. That is, $\bar{X}_t = \sum_{j=1}^{\Pi(t)} Z_i$, where $\{Z_i\}_{i=1}^{\infty}$ are i.i.d. and symmetric, and Π is an independent Poisson process. It follows immediately from this that the range of $\bar{X}$ is a.s. countable in that case. This proves that the occupation measure for $\bar{X}$ is a.s. singular with respect to Lebesgue measure, and therefore $\bar{X}$ cannot possess local times. But that contradicts the fact that $\bar{X}$ has local times; confer with Theorem 2.1. Consequently, (4.4) implies (4.6), and is equivalent to the spatial continuity of [a modification of] local times of $\bar{X}$. This proves the theorem in the parabolic case.

Let us assume further the symmetry condition (2.12). Because $\mathscr{E}(t^2; \varphi)/\mathscr{E}(1; \varphi)$ is bounded above and below by positive finite constants that depend only on $t > 0$ (Proposition 3.3), we can apply the very same argument to the stochastic wave equation (2.4), but use Proposition 3.8 in place of Proposition 3.3. This completes our proof of Theorem 2.4. □

5. AN ASIDE ON TEMPORAL CONTINUITY

In this section we spend a few pages discussing matters of temporal continuity—especially temporal Hölder continuity—of weak solutions of the stochastic heat equation (2.3), as well as the stochastic wave equation (2.4).

Definition 5.1. We call a function $g(s)$ a *gauge function* if the following are satisfied:

    (1) $g(s)$ is an increasing function;
    (2) $g(s)$ is a slowly varying function at infinity;
    (3) $g(s)$ satisfies the integrability condition,

$$(5.1) \qquad \int_{0^+} \frac{\mathrm{d}s}{s \log(1/s) g(1/s)} < \infty.$$

Next, we quote a useful property of slowly varying functions [6, p. 27].

Proposition 5.2. *If g is a slowly varying function and $\alpha > 1$, then the integral $\int_x^\infty t^{-\alpha} g(t)\, \mathrm{d}t$ converges for every $x > 0$, and*

$$(5.2) \qquad \int_x^\infty \frac{g(t)}{t^\alpha}\, \mathrm{d}t \sim \frac{g(x)}{(\alpha-1)x^{\alpha-1}} \qquad \text{as } x \to \infty.$$

We can now state the main theorem of this section. It gives a criterion for the temporal continuity of the weak solutions of our stochastic equations.

Theorem 5.3. *Let H denote the weak solution to the stochastic heat equation (2.3). Let g be a gauge function in the sense of Definition 5.1. Choose and fix $\varphi \in L^2_{\mathscr{L}}(\mathbf{R}^d)$. Then, $t \mapsto H(t,\varphi)$ has a continuous modification if the following is satisfied:*

$$(5.3) \qquad \int_{\mathbf{R}^d} \frac{\log(1 + |\Psi(\xi)|) g(1 + |\Psi(\xi)|)}{1 + \mathrm{Re}\,\Psi(\xi)} |\hat{\varphi}(\xi)|^2\, \mathrm{d}\xi < \infty.$$

Moreover, the critical Hölder exponent of $t \mapsto H(t,\varphi)$ is precisely $\frac{1}{2}\underline{\mathrm{ind}}\,\mathscr{E}(\bullet\,;\varphi)$, where

$$(5.4) \qquad \underline{\mathrm{ind}}\,\mathscr{E}(\bullet\,;\varphi) := \liminf_{\epsilon \downarrow 0} \frac{\log \mathscr{E}(\epsilon\,;\varphi)}{\log(1/\epsilon)}.$$

Consequently, $t \mapsto H(t,\varphi)$ has a Hölder-continuous modification (a.s.) if and only if $\underline{\mathrm{ind}}\,\mathscr{E}(\bullet\,;\varphi) > 0$.

If the symmetry condition (2.12) holds, then (5.3) guarantees the existence of a continuous modification of $t \mapsto W(t,\varphi)$, where W denotes the weak solution to the stochastic wave equation. Furthermore, (5.4) implies the temporal Hölder continuity of $t \mapsto W(t,\varphi)$ of any order $< \underline{\mathrm{ind}}\,\mathscr{E}(\bullet\,;\varphi)$.

We now give two examples. The first one is about the temporal Hölder exponent, while the second one provides a family of random-field solutions which are almost surely unbounded in every open space-time set.

Example 5.4. Suppose $d = 1$ and $\mathscr{L} = -(-\Delta)^{\alpha/2}$ for some $\alpha \in (1, 2]$. Choose and fix $x \in \mathbf{R}$. According to Theorem 2.1 and Example 2.3, $\delta_x \in L^2_{\mathscr{L}}(\mathbf{R})$, so we can apply Theorem 5.3 with $\varphi := \delta_x$. In this case,

$$(5.5) \qquad \begin{aligned} \mathscr{E}(\epsilon\,;\delta_x) &= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{\mathrm{d}\xi}{(1/\epsilon) + |\xi|^\alpha} \\ &= \mathrm{const} \cdot \epsilon^{1-(1/\alpha)}. \end{aligned}$$

In particular, $\underline{\mathrm{ind}}\,\mathscr{E}(\bullet\,;\delta_x) = 1 - (1/\alpha)$, whence it follows that the critical Hölder exponent of $t \mapsto H(t\,,x)$ is precisely $\frac{1}{2} - (2\alpha)^{-1}$. When $\mathscr{L} = \Delta$, we have $\alpha = 2$ and the critical temporal exponent is $1/4$, which agrees with a well-known folklore theorem. For example, Corollary 3.4 of Walsh [55, pp. 318–320] and its proof contain this statement for the closely-related stochastic cable equation. In particular, see the last two lines on page 319 of Walsh's lectures (*loc. cit.*). $\square$

Example 5.5. Choose and fix $\alpha \in \mathbf{R}$. According to the Lévy–Khintchine formula; see, for example, Bertoin [5, Theorem 1, p. 13], we can find a symmetric Lévy process whose characteristic exponents satisfies

$$(5.6) \qquad \lim_{\xi \to \infty} \frac{\Psi(\xi)}{\xi(\log \xi)^\alpha} = 1.$$

[To construct such an example, one can consider a symmetric process with no Gaussian component and Lévy measure with density $\nu(\mathrm{d}x)/\mathrm{d}x := C|x|^{-2}(\log_+|x|)^\alpha$ for a suitable constant $C \in (0\,,\infty)$.] Throughout, we assume that $\alpha > 1$ in (5.6). This ensures that condition (3.17) is in place (i.e., $(1+\Psi)^{-1} \in L^1(\mathbf{R})$); equivalently, that both SPDEs (2.3) and (2.4) have random-field solutions.

We note that

$$
(5.7) \qquad
\begin{aligned}
\mathscr{E}(\epsilon\,;\delta_x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{(1/\epsilon) + \Psi(\xi)} \\
&= \frac{1}{\pi} \int_0^{\infty} \frac{\mathrm{d}z}{1 + \epsilon\Psi(z/\epsilon)}.
\end{aligned}
$$

We consider the integral separately over two regions: Choose and fix $c_0 \in (1\,,\infty)$ so that $\frac{1}{2}\xi(\log \xi)^\alpha \le \Psi(\xi) \le 2\xi(\log \xi)^\alpha$ whenever $\xi > c_0$. Then write

$$(5.8) \qquad \mathscr{E}(\epsilon\,;\delta_x) = \frac{1}{2\pi}(T_1 + T_2),$$

where

$$(5.9) \qquad T_1 := \int_0^{c_0\epsilon} \frac{\mathrm{d}z}{1 + \epsilon\Psi(z/\epsilon)} \quad \text{and} \quad T_2 := \int_{c_0\epsilon}^{\infty} \frac{\mathrm{d}z}{1 + \epsilon\Psi(z/\epsilon)}.$$

Because Ψ is nonnegative,

$$(5.10) \qquad T_1 = O(\epsilon) \quad \text{as } \epsilon \to 0.$$

Also, it is easy to see from the defining property of c_0 that

$$(5.11) \qquad \frac{1}{2}T_3 \le T_2 \le 2T_3,$$

where

$$(5.12) \qquad T_3 := \int_{c_0\epsilon}^{\infty} \frac{\mathrm{d}z}{1 + z\left(\log(z/\epsilon)\right)^\alpha} = \int_{c_0}^{\infty} \frac{\mathrm{d}t}{(1/\epsilon) + t(\log t)^\alpha}.$$

There exists $c_1 \in (1\,,\infty)$ such that for all $\epsilon > 0$ sufficiently small, $t(\log t)^\alpha \ge \epsilon^{-1}$ as soon as $t \ge c_1/(\epsilon|\log(1/\epsilon)|^\alpha)$. Therefore, for all ϵ sufficiently small,

$$
(5.13) \qquad
\begin{aligned}
T_3 &\ge \int_{c_1/\epsilon|\log(1/\epsilon)|^\alpha}^{\infty} \frac{\mathrm{d}t}{(1/\epsilon) + t(\log t)^\alpha} \\
&\ge \frac{1}{2} \int_{c_1/\epsilon|\log(1/\epsilon)|^\alpha}^{\infty} \frac{\mathrm{d}t}{t(\log t)^\alpha} \\
&= (\text{const} + o(1))\,|\log(1/\epsilon)|^{-\alpha+1} \qquad \text{as } \epsilon \downarrow 0.
\end{aligned}
$$

Similarly,

$$T_3 = \int_{c_0}^{c_1/\epsilon|\log(1/\epsilon)|^\alpha} \frac{\mathrm{d}t}{(1/\epsilon) + t(\log t)^\alpha} + \int_{c_1/\epsilon|\log(1/\epsilon)|^\alpha}^\infty \frac{\mathrm{d}t}{(1/\epsilon) + t(\log t)^\alpha}$$

(5.14)
$$\leq \frac{c_1}{|\log(1/\epsilon)|^\alpha} + \int_{c_1/\epsilon|\log(1/\epsilon)|^\alpha}^\infty \frac{\mathrm{d}t}{t(\log t)^\alpha}$$

$$= O\left(|\log(1/\epsilon)|^{-\alpha+1}\right) \qquad \text{as } \epsilon \downarrow 0.$$

Consequently, for all $x \in \mathbf{R}$,

$$(5.15) \qquad\qquad \mathscr{E}(\epsilon\,;\delta_x) \asymp |\log(1/\epsilon)|^{-\alpha+1}\,,$$

where $a(\epsilon) \asymp b(\epsilon)$ means that $a(\epsilon)/b(\epsilon)$ is bounded above and below by absolute constants, uniformly for all $\epsilon > 0$ sufficiently small and $x \in \mathbf{R}$. Next, consider

$$(5.16) \qquad\qquad d(s\,,t) := \sqrt{\mathrm{E}\left(|H(s\,,x) - H(t\,,x)|^2\right)}.$$

According to Proposition 5.6 below, (5.15) implies that

$$(5.17) \qquad\qquad d(s\,,t) \asymp \left|\log\left(\frac{1}{|s-t|}\right)\right|^{(1-\alpha)/2},$$

uniformly for all s and t in a fixed compact subset $[0\,,T]$ of $\mathbf{R}_+$, say. Let N denote the *metric entropy* of $[0\,,T]$ in the [pseudo-] metric d; confer with Dudley [28]. That is, for all $\epsilon > 0$, we define $N(\epsilon)$ to be the minimum number of d-balls of radius ϵ needed to cover $[0\,,T]$. Then, it is easy to deduce from the previous display that $\log N(\epsilon) \asymp \epsilon^{-2/(\alpha-1)}$, and hence

$$(5.18) \qquad\qquad \lim_{\epsilon \downarrow 0} \epsilon \sqrt{\log N(\epsilon)} = 0 \quad \text{if and only if} \quad \alpha > 2.$$

Therefore, if $1 < \alpha \leq 2$, then Sudakov minorization [40, p. 250] tells us that the process $t \mapsto H(t\,,x)$—and also $t \mapsto W(t\,,x)$—does *not* have any continuous modifications, almost surely. On the other hand, if $\alpha > 2$, then the integrability condition (5.3) holds manifestly, and hence $t \mapsto H(t\,,x)$ and $t \mapsto W(t\,,x)$ both have continuous modifications. Thus, the sufficiency condition of Theorem 5.3 is also necessary for the present example. In addition, when $\alpha \leq 2$, the random-field solutions H and W are both unbounded a.s. in every open set. This assertion follows from general facts about Gaussian processes. For example apply equation (6.33) of Marcus and Rosen [40, p. 250] with the set T of Marcus and Rosen replaced by an arbitrary closed subinterval of our set $[0\,,T]$ here. $\qquad\square$

5.1. Estimate in the parabolic case.

Proposition 5.6. *For every $t, \epsilon \geq 0$ and $\varphi \in L^2_{\mathscr{L}}(\mathbf{R}^d)$,*

$$(5.19) \qquad \frac{1}{4}\mathscr{E}(\epsilon\,;\varphi) \leq \mathrm{E}\left(|H(t+\epsilon\,,\varphi) - H(t\,,\varphi)|^2\right) \leq 2\mathscr{E}(\epsilon\,;\varphi) + e^{2t}\mathscr{F}(\epsilon\,;\varphi),$$

where

$$(5.20) \qquad\qquad \mathscr{F}(\epsilon\,;\varphi) := \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \left(1 \wedge \epsilon^2 |\Psi(\xi)|^2\right) \frac{|\hat{\varphi}(\xi)|^2}{1 + \mathrm{Re}\,\Psi(\xi)}\,\mathrm{d}\xi.$$

Proof. Because of density it suffices to prove that the proposition holds for all functions $\varphi \in \mathscr{S}(\mathbf{R}^d)$ of rapid decrease. By (2.9) and Wiener's isometry,

(5.21)

$$\mathrm{E}\left(|H(t+\epsilon,\varphi) - H(t,\varphi)|^2\right) = \int_0^t \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}y\, |(P_{t-s+\epsilon}\varphi)(y) - (P_{t-s}\varphi)(y)|^2$$
$$+ \int_t^{t+\epsilon} \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}y\, |(P_{t-s+\epsilon}\varphi)(y)|^2.$$

We apply Plancherel's theorem to find that

(5.22)

$$\mathrm{E}\left(|H(t+\epsilon,\varphi) - H(t,\varphi)|^2\right)$$
$$= \frac{1}{(2\pi)^d} \int_0^t \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}\xi \left|e^{-(s+\epsilon)\Psi(-\xi)} - e^{-s\Psi(-\xi)}\right|^2 |\hat{\varphi}(\xi)|^2$$
$$+ \frac{1}{(2\pi)^d} \int_0^\epsilon \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}\xi\, e^{-2s\mathrm{Re}\,\Psi(\xi)} |\hat{\varphi}(\xi)|^2.$$

Thus, we can write

(5.23)

$$\mathrm{E}\left(|H(t+\epsilon,\varphi) - H(t,\varphi)|^2\right) := \frac{T_1 + T_2}{(2\pi)^d},$$

where

(5.24)

$$T_1 := \int_0^t \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}\xi\, e^{-2s\mathrm{Re}\,\Psi(\xi)} \left|1 - e^{-\epsilon\Psi(\xi)}\right|^2 |\hat{\varphi}(\xi)|^2$$

and

(5.25)

$$T_2 := \int_0^\epsilon \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}\xi\, e^{-2s\mathrm{Re}\,\Psi(\xi)} |\hat{\varphi}(\xi)|^2.$$

First we estimate T_2, viz.,

(5.26)

$$\int_0^\epsilon e^{-2s\mathrm{Re}\,\Psi(\xi)}\, \mathrm{d}s = \epsilon\, \frac{1 - e^{-2\epsilon\mathrm{Re}\,\Psi(\xi)}}{2\epsilon\mathrm{Re}\,\Psi(\xi)}.$$

Because

(5.27)

$$\frac{1}{2}\frac{1}{1+\theta} \le \frac{1-e^{-\theta}}{\theta} \le \frac{2}{1+\theta} \quad \text{for all } \theta > 0,$$

it follows that $\frac{1}{4}(2\pi)^d \mathscr{E}(\epsilon\,;\varphi) \le T_2 \le (2\pi)^d \mathscr{E}(\epsilon\,;\varphi)$. Since $T_1 \ge 0$ we obtain the first inequality of the proposition.

We can use Lemma 3.5 with $\lambda = 1$ and

(5.28)

$$g(s) := \int_{\mathbf{R}^d} e^{-2s\mathrm{Re}\,\Psi(\xi)} \left|1 - e^{-\epsilon\Psi(\xi)}\right|^2 |\hat{\varphi}(\xi)|^2\, \mathrm{d}\xi$$

to obtain the following bound:

(5.29)

$$T_1 \le \frac{e^{2t}}{2} \int_{\mathbf{R}^d} \left|1 - e^{-\epsilon\Psi(\xi)}\right|^2 \frac{|\hat{\varphi}(\xi)|^2}{1 + \mathrm{Re}\,\Psi(\xi)}\, \mathrm{d}\xi.$$

Because $|1 - e^{-\epsilon\Psi(\xi)}|^2 \le 1 \wedge \epsilon^2 |\Psi(\xi)|^2$, it follows that $T_1 \le (2\pi)^d e^{2t} \mathscr{F}(\epsilon\,;\varphi)$, and hence the proof is completed. $\qquad\square$

5.2. **Estimate in the hyperbolic case.**

Proposition 5.7. *Assume the symmetry condition* (2.12), *and let* W *denote the weak solution to the stochastic wave equation* (2.4). *Then, for all* $t \geq 0$, $\epsilon > 0$, *and* $\varphi \in L^2_{\mathscr{L}}(\mathbf{R}^d)$,

$$(5.30) \qquad \mathrm{E}\left(|W(t+\epsilon, \varphi) - W(t, \varphi)|^2\right) \leq (4t + 2\epsilon)\, \mathscr{E}(\epsilon^2; \varphi).$$

Proof. By density, it suffices to prove the proposition for all functions $\varphi \in \mathscr{S}(\mathbf{R}^d)$ of rapid decrease. Henceforth, we choose and fix such a function φ.

In accordance with (2.21) we write

$$(5.31) \qquad \mathrm{E}\left(|W(t+\epsilon, \varphi) - W(t, \varphi)|^2\right) := T_1 + T_2,$$

where

$$(5.32) \quad T_1 := \frac{1}{(2\pi)^d} \int_0^t \mathrm{d}r \int_{\mathbf{R}^d} \mathrm{d}\xi\, \frac{\left|\sin\left(\sqrt{\Psi(\xi)}\,(r+\epsilon)\right) - \sin\left(\sqrt{\Psi(\xi)}\,r\right)\right|^2}{\Psi(\xi)} |\hat{\varphi}(\xi)|^2$$

and

$$(5.33) \qquad T_2 := \frac{1}{(2\pi)^d} \int_t^{t+\epsilon} \mathrm{d}r \int_{\mathbf{R}^d} \mathrm{d}\xi\, \frac{\sin^2\left(\sqrt{\Psi(\xi)}\,r\right)}{\Psi(\xi)} |\hat{\varphi}(\xi)|^2.$$

We estimate T_2 first: Since $|\sin\theta| \leq \min(|\theta|, 1)$ for all $\theta \in [-\pi, \pi)$,

$$(5.34) \qquad \int_{\mathbf{R}^d} \frac{\sin^2\left(\sqrt{\Psi(\xi)}\,r\right)}{\Psi(\xi)} |\hat{\varphi}(\xi)|^2\, \mathrm{d}\xi \leq \int_{\mathbf{R}^d} \left(r^2 \wedge \frac{1}{\Psi(\xi)}\right) |\hat{\varphi}(\xi)|^2\, \mathrm{d}\xi.$$

Because $\int_\epsilon^{2\epsilon} \left(r^2 \wedge a\right)\, dr \leq \epsilon\left(\epsilon^2 \wedge a\right)$ for all $a, \epsilon > 0$,

$$(5.35) \qquad \begin{aligned} T_2 &\leq \frac{\epsilon}{(2\pi)^d} \int_{\mathbf{R}^d} \left(\epsilon^2 \wedge \frac{1}{\Psi(\xi)}\right) |\hat{\varphi}(\xi)|^2\, \mathrm{d}\xi \\ &\leq 2\epsilon\, \mathscr{E}(\epsilon^2; \varphi). \end{aligned}$$

Also, because $|\sin\alpha - \sin\beta|^2 \leq 2[1 - \cos(\beta - \alpha)]$,

$$(5.36)$$
$$\int_{\mathbf{R}^d} \frac{\left|\sin\left(\sqrt{\Psi(\xi)}\,(r+\epsilon)\right) - \sin\left(\sqrt{\Psi(\xi)}\,r\right)\right|^2}{\Psi(\xi)} |\hat{\varphi}(\xi)|^2\, \mathrm{d}\xi$$
$$\leq 2 \int_{\mathbf{R}^d} \frac{1 - \cos\left(\sqrt{\Psi(\xi)}\,\epsilon\right)}{\Psi(\xi)} |\hat{\varphi}(\xi)|^2\, \mathrm{d}\xi.$$

This and the elementary inequality $1 - \cos x \leq x^2/2$ together yield the bound

$$(5.37) \qquad \frac{1 - \cos\left(\sqrt{\Psi(\xi)}\,\epsilon\right)}{\Psi(\xi)} \leq 2\left(\epsilon^2 \wedge \frac{1}{\Psi(\xi)}\right).$$

Consequently,

$$(5.38) \quad \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{\left|\sin\left(\sqrt{\Psi(\xi)}\,(r+\epsilon)\right) - \sin\left(\sqrt{\Psi(\xi)}\,r\right)\right|^2}{\Psi(\xi)} |\hat{\varphi}(\xi)|^2\, \mathrm{d}\xi \leq 4\, \mathscr{E}(\epsilon^2; \varphi),$$

whence T_1 is at most t times the right-hand side of the preceding. This and (5.35) together yield the proof. $\square$

Proof of Theorem 5.3. We start with the weak solution H to the stochastic heat equation. Throughout, $\varphi \in L^2_{\mathscr{L}}(\mathbf{R}^d)$ is held fixed.

According to Proposition 5.6, for all $T > 0$,

$$(5.39) \qquad \sup_{\substack{0 < \epsilon < 1 \\ 0 \le t \le t+\epsilon \le T}} \frac{\mathrm{E}\left(|H(t+\epsilon, \varphi) - H(t, \varphi)|^2\right)}{\mathscr{E}(\epsilon; \varphi) + \mathscr{F}(\epsilon; \varphi)} \le 2e^{2T} < \infty.$$

Since $\epsilon \mapsto \mathscr{E}(\epsilon; \varphi)$ is nondecreasing, a direct application of Gaussian process theory implies that $\{H(t, \varphi)\}_{t \in [0,T]}$ has a continuous modification provided that

$$(5.40) \qquad \int_{0+} \frac{\sqrt{\mathscr{E}(\epsilon; \varphi) + \mathscr{F}(\epsilon; \varphi)}}{\epsilon\sqrt{\log(1/\epsilon)}} \, \mathrm{d}\epsilon < \infty.$$

See Lemma 6.4.6 of Marcus and Rosen [40, p. 275]. A similar argument works for the weak solution W to the stochastic wave equation (2.4), but we appeal to Proposition 5.7 in place of Proposition 5.6. Thus, the first portion of our proof will be completed, once we prove that condition (5.3) implies (5.40).

Suppose the integrability condition (5.3) holds, and let us write

$$(5.41) \qquad \int_{0+} \frac{\sqrt{\mathscr{E}(\epsilon; \varphi) + \mathscr{F}(\epsilon; \varphi)}}{\epsilon\sqrt{\log(1/\epsilon)}} \, \mathrm{d}\epsilon \le \int_{0+} \frac{\sqrt{\mathscr{E}(\epsilon; \varphi)}}{\epsilon\sqrt{\log(1/\epsilon)}} \, \mathrm{d}\epsilon + \int_{0+} \frac{\sqrt{\mathscr{F}(\epsilon; \varphi)}}{\epsilon\sqrt{\log(1/\epsilon)}} \, \mathrm{d}\epsilon$$
$$:= I_1 + I_2.$$

We wish to prove that I_1 and I_2 are finite.

Let us first consider I_1. We multiply and divide the integrand of I_1 by the square root of $g(1/\epsilon)$, and then apply the Cauchy–Schwarz inequality to obtain the following:

$$(5.42) \qquad I_1 \le \left(\int_{0+} \frac{\mathscr{E}(\epsilon; \varphi)g(1/\epsilon)}{\epsilon} \, \mathrm{d}\epsilon\right)^{1/2} \left(\int_{0+} \frac{1}{\epsilon \log(1/\epsilon)g(1/\epsilon)} \, \mathrm{d}\epsilon\right)^{1/2}$$
$$= \mathrm{const} \cdot \left(\int_{0+} \frac{\mathscr{E}(\epsilon; \varphi)g(1/\epsilon)}{\epsilon} \, \mathrm{d}\epsilon\right)^{1/2}.$$

Let c be a positive constant to be chosen later. Note that

$$\int_0^{1/c} \frac{\mathscr{E}(\epsilon; \varphi)g(1/\epsilon)}{\epsilon} \, \mathrm{d}\epsilon = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{\varphi}(\xi)|^2 \, \mathrm{d}\xi \left(\int_0^{1/c} \frac{g(1/\epsilon)}{1 + \epsilon \mathrm{Re}\,\Psi(\xi)} \, \mathrm{d}\epsilon\right)$$

$$(5.43) \qquad = \frac{1}{(2\pi)^d} \int_{\mathrm{Re}\,\Psi \ge c} |\hat{\varphi}(\xi)|^2 \, \mathrm{d}\xi(\cdots)$$
$$+ \frac{1}{(2\pi)^d} \int_{\mathrm{Re}\,\Psi < c} |\hat{\varphi}(\xi)|^2 \, \mathrm{d}\xi(\cdots)$$
$$:= \frac{I_3 + I_4}{(2\pi)^d},$$

with the notation being clear enough. We will look at I_3 and I_4 separately. We begin with I_4 first. We have

$$
\begin{aligned}
I_4 &\le \int_{\mathrm{Re}\,\Psi < c} \int_0^{1/c} \frac{|\hat{\varphi}(\xi)|^2 g(1/\epsilon)}{1 + \epsilon \mathrm{Re}\,\Psi(\xi)}\, \mathrm{d}\epsilon\, \mathrm{d}\xi \\
&\le \int_{\mathrm{Re}\,\Psi < c} |\hat{\varphi}(\xi)|^2\, \mathrm{d}\xi \cdot \int_0^{1/c} g(1/\epsilon)\, \mathrm{d}\epsilon \\
&\le \mathrm{const} \cdot \mathscr{E}(1\,;\varphi),
\end{aligned}
$$

(5.44)

where we have used Proposition 5.2 to obtain the last inequality. Thus, we have $I_1 < \infty$ as soon as we establish that $I_3 < \infty$.

In order to bound I_3 we let $N > c$ and consider the following:

$$
\begin{aligned}
J(N) &:= \int_0^{1/c} \frac{g(1/\epsilon)}{1 + \epsilon N}\, \mathrm{d}\epsilon \\
&= \int_0^{1/N} \frac{g(1/\epsilon)}{1 + \epsilon N}\, \mathrm{d}\epsilon + \int_{1/N}^{1/c} \frac{g(1/\epsilon)}{1 + \epsilon N}\, \mathrm{d}\epsilon \\
&= I_5 + I_6.
\end{aligned}
$$

(5.45)

On one hand, some calculus shows that

$$
I_6 = O\left(\frac{\log N \cdot g(N)}{N} \right) \quad \text{as } N \to \infty.
$$

(5.46)

On the other hand, we can change variables and appeal to Proposition 5.2 and deduce that

$$
\begin{aligned}
I_5 &\le \int_N^\infty \frac{g(u)}{u^2}\, \mathrm{d}u \\
&\sim \frac{g(N)}{N} \quad \text{as } N \to \infty.
\end{aligned}
$$

(5.47)

Consequently $J(N) = O(\log N \cdot g(N)/N)$ as $N \to \infty$. Since

$$
I_3 = \frac{1}{(2\pi)^d} \int_{\mathrm{Re}\,\Psi \ge c} |\hat{\varphi}(\xi)|^2\, J(\mathrm{Re}\,\Psi(\xi))\, \mathrm{d}\xi,
$$

(5.48)

the inequalities (5.44)–(5.46) together yield the following:

(5.49) $\qquad\qquad$ (5.3) $\quad\Longrightarrow\quad I_1 < \infty,$

upon taking the constant c large enough.

We now look at I_2. Let us recall the definition of $\mathscr{F}(\epsilon\,;\varphi)$ and write

(5.50)

$$
\begin{aligned}
\mathscr{F}(\epsilon\,;\varphi) &= \frac{1}{(2\pi)^d} \left[\int_{|\Psi| \le 1/c} (\cdots)\, \mathrm{d}\xi + \int_{1/c \le |\Psi| \le 1/\epsilon} (\cdots)\, \mathrm{d}\xi + \int_{|\Psi| > 1/\epsilon} (\cdots)\, \mathrm{d}\xi \right] \\
&= \frac{1}{(2\pi)^d} [I_7 + I_8 + I_9].
\end{aligned}
$$

Because $I_7 \le (2\pi)^d c^{-2} \epsilon^2 \mathscr{E}(1\,;\varphi)$,

$$
\int_0^{1/c} \frac{\sqrt{I_7}}{\epsilon \sqrt{\log(1/\epsilon)}}\, \mathrm{d}\epsilon \le \mathrm{const} \cdot \sqrt{\mathscr{E}(1\,;\varphi)},
$$

(5.51)

which is finite.

Since we have assumed that

$$(5.52) \qquad A := \int_0^{1/c} \frac{\mathrm{d}\epsilon}{\epsilon \log(1/\epsilon) g(1/\epsilon)} < \infty,$$

it follows that

$$(5.53) \qquad \begin{aligned} \int_0^{1/c} \frac{\sqrt{I_8}}{\epsilon \sqrt{\log(1/\epsilon)}} \, \mathrm{d}\epsilon &= \int_0^{1/c} \frac{1}{\sqrt{\epsilon \log(1/\epsilon) g(1/\epsilon)}} \cdot \sqrt{\frac{I_8 \cdot g(1/\epsilon)}{\epsilon}} \, \mathrm{d}\epsilon \\ &\le A^{1/2} \left(\int_0^{1/c} \frac{I_8 \cdot g(1/\epsilon)}{\epsilon} \, \mathrm{d}\epsilon \right)^{1/2}. \end{aligned}$$

But

$$(5.54) \qquad \begin{aligned} \int_0^{1/c} \frac{I_8 \cdot g(1/\epsilon)}{\epsilon} \, \mathrm{d}\epsilon &= \int_0^{1/c} \epsilon g(1/\epsilon) \, \mathrm{d}\epsilon \int_{1/c \le |\Psi| \le 1/\epsilon} \mathrm{d}\xi \, \frac{|\Psi(\xi)|^2 \cdot |\hat{\varphi}(\xi)|^2}{1 + \mathrm{Re}\, \Psi(\xi)} \\ &\le \mathrm{const} \cdot \int_{1/c \le |\Psi|} \frac{|\hat{\varphi}(\xi)|^2 g(1 + \Psi(\xi))}{1 + \mathrm{Re}\, \Psi(\xi)} \, \mathrm{d}\xi, \end{aligned}$$

since $\int_{\epsilon \le N} \epsilon g(1/\epsilon) \, \mathrm{d}\epsilon = O(N^2 g(1/N))$ as $N \to \infty$ [this follows from Proposition 5.2, after a change of variables]. As a result of this development, and thanks to condition (5.3), we have

$$(5.55) \qquad \int_0^{1/c} \frac{\sqrt{I_8}}{\epsilon \sqrt{\log(1/\epsilon)}} \, \mathrm{d}\epsilon < \infty.$$

Similarly, we multiply and divide the integrand below by the square root of $g(1/\epsilon)$ and apply the Cauchy–Schwarz inequality in order to obtain

(5.56)

$$\begin{aligned} \int_0^{1/c} \frac{\sqrt{I_9}}{\epsilon \sqrt{\log(1/\epsilon)}} \, \mathrm{d}\epsilon &\le A^{1/2} \left(\int_0^{1/c} \mathrm{d}\epsilon \int_{|\Psi| > 1/\epsilon} \mathrm{d}\xi \, \frac{|\hat{\varphi}(\xi)|^2 g(1/\epsilon)}{\epsilon(1 + \mathrm{Re}\, \Psi(\xi))} \right)^{1/2} \\ &\le \mathrm{const} \cdot \left(\int_{\mathbf{R}^d} \frac{\log(1 + |\Psi(\xi)|) g(1 + |\Psi(\xi)|)}{(1 + \mathrm{Re}\, \Psi(\xi))} |\hat{\varphi}(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2}, \end{aligned}$$

where A is defined in (5.52). In the last inequality we have changed the order of integration. After taking into account inequalites (5.46)–(5.56), we obtain

$$(5.57) \qquad\qquad (5.3) \implies I_2 < \infty.$$

Assertions (5.49) and (5.57), together with (5.41), imply that (5.40) holds provided that (5.3) does. This proves the first assertion of the theorem.

Suppose $\gamma > 0$, where

$$(5.58) \qquad\qquad \gamma := \underline{\mathrm{ind}}\, \mathscr{E}(\bullet\,; \varphi),$$

for brevity. Then by definition, $\mathscr{E}(\epsilon\,; \varphi) \le \epsilon^{\gamma + o(1)}$ as $\epsilon \downarrow 0$. Another standard result from Gaussian analysis, used in conjunction with Proposition 5.6, proves that H has a Hölder-continuous modification with Hölder exponent $\le \gamma/2$ [40, Theorem 7.2.1, p. 298]. The proof for W is analogous.

For the remainder of the proof we consider only the weak solution H to the stochastic heat equation (2.3), and we write $H_t := H(t\,, \varphi)$ for typographical ease.

If $\gamma > 0$, then Proposition 5.6 and elementary properties of normal laws together imply that

$$(5.59) \qquad \inf_{t \geq 0} \|H_{t+\epsilon} - H_t\|_{L^2(\mathrm{P})} \geq \epsilon^{\gamma + o(1)} \quad \text{for infinitely-many } \epsilon \downarrow 0.$$

Consequently, for all $\delta \in (0, 1)$, $q > \gamma/2$, $T > S > 0$, and $t \in [S, T]$—all fixed—the following holds for infinitely-many values of $\epsilon \downarrow 0$:

$$(5.60) \qquad \mathrm{P}\left\{\sup_{r \in [S,T]} |H_{r+\epsilon} - H_r| \geq \epsilon^q\right\} \geq \mathrm{P}\left\{|H_{t+\epsilon} - H_t| \geq \delta \|H_{t+\epsilon} - H_t\|_{L^2(\mathrm{P})}\right\}$$
$$= 1 - \sqrt{\frac{2}{\pi}} \int_0^\delta \exp(-x^2/2)\, \mathrm{d}x.$$

The inequality $\exp(-x^2/2) \leq 1$ then implies that

$$(5.61) \qquad \mathrm{P}\left\{\sup_{r \in [S,T]} |H_{r+\epsilon} - H_r| \geq \epsilon^q \text{ for infinitely many } \epsilon \downarrow 0\right\} \geq 1 - \sqrt{\frac{2}{\pi}}\, \delta.$$

Since $q > \gamma/2$ is arbitrary,

$$(5.62) \qquad \mathrm{P}\left\{\limsup_{\epsilon \downarrow 0} \sup_{r \in [S,T]} \frac{|H_{r+\epsilon} - H_r|}{\epsilon^q} = \infty\right\} \geq 1 - \sqrt{\frac{2}{\pi}}\, \delta.$$

Let $\delta \downarrow 0$ to find that

$$(5.63) \qquad \limsup_{\epsilon \downarrow 0} \sup_{r \in [S,T]} \frac{|H_{r+\epsilon} - H_r|}{\epsilon^q} = \infty \qquad \text{a.s.}$$

This proves that any $q > \gamma/2$ is an almost-sure lower bound for the critical Hölder exponent of H. Moreover, in the case that $\gamma = 0$, we find that (5.63) holds a.s. for all $q > 0$. Thus, it follows that with probability one, H has no Hölder-continuous modification in that case. The proof is now complete. $\qquad \square$

6. Spatial and joint continuity: Proofs of Theorems 2.5 and 2.6

Proof of Theorem 2.5. We begin by proving the portion of Theorem 2.5 that relates to the stochastic heat equation and its random-field solution $\{H(t, x); t \geq 0, x \in \mathbf{R}\}$; Theorem 2.1 guarantees the existence of the latter process.

Throughout we can—and will—assume without loss of generality that $x \mapsto H(t, x)$ is continuous, and hence so is the local time of $\bar{X}$ in its spatial variable (Theorem 2.4). As we saw, during the course of the proof of Theorem 2.4, this automatically implies condition (4.6), which we are free to assume henceforth.

According to (4.2),

$$(6.1) \qquad \tfrac{2}{3}(1 - e^{-2t}) h(|x - y|) \leq \mathrm{E}\left(|H(t, x) - H(t, y)|^2\right) \leq 8e^{2t} h(|x - y|).$$

This implies that the critical Hölder exponent of $z \mapsto H(t, z)$ is almost surely equal to one-half of the following quantity:

$$(6.2) \qquad \underline{\mathrm{ind}}\, h := \liminf_{\epsilon \downarrow 0} \frac{\log h(\epsilon)}{\log(1/\epsilon)}.$$

We will not prove this here, since it is very similar to the proof of temporal Hölder continuity (Theorem 5.3). Consequently,

(6.3) $z \mapsto H(t, z)$ has a Hölder-continuous modification iff $\underline{\text{ind}}\, h > 0$.

Among other things, this implies the equivalence of parts (1) and (2) of the theorem.

Let $Z(t, x)$ denote the local time of $\bar{X}$ at spatial value x at time $t \geq 0$. We prove that (1), (2), and (3) are equivalent by proving that (6.3) continues to hold when H is replaced by Z. Fortunately, this can be read from the work of Barlow [2]. We explain the details briefly. Because (4.6) holds, Theorem 5.3 of Barlow [2] implies that there exists a finite constant $c > 0$ such that for all $t \geq 0$ and finite intervals $I \subset \mathbf{R}$,

$$(6.4) \qquad \lim_{\delta \downarrow 0} \sup_{\substack{a,b \in I \\ |a-b| < \delta}} \frac{|Z(t, a) - Z(t, b)|}{\sqrt{h(|a-b|)\log(1/|b-a|)}} \geq c \left(\sup_{x \in I} Z(s, x) \right)^{1/2} \qquad \text{a.s.}$$

Moreover, we can choose the null set to be independent of all intervals $I \subset \mathbf{R}$ with rational endpoints. In fact, we can replace I by $\mathbf{R}$, since $a \mapsto Z(t, a)$ is supported by the closure of the range of the process $\bar{X}$ up to time t, and the latter range is a.s. bounded since $\bar{X}$ is cadlag.

By their very definition local times satisfy $\int_{-\infty}^{\infty} Z(s, x)\, \mathrm{d}x = s$ a.s. Thus, $\sup_{x \in \mathbf{R}} Z(t, x) > 0$ a.s., whence it follows that for all $q > \frac{1}{2}\underline{\text{ind}}\, h$,

$$(6.5) \qquad \lim_{\delta \downarrow 0} \sup_{|a-b| < \delta} \frac{|Z(t, a) - Z(t, b)|}{|a-b|^q} = \infty \qquad \text{a.s.}$$

That is, there is no Hölder-continuous modification of $a \mapsto Z(t, a)$ of order $> \frac{1}{2}\underline{\text{ind}}\, h$. In particular, if $\underline{\text{ind}}\, h = 0$, then $a \mapsto Z(t, a)$ does not have a Hölder-continuous modification.

Define $d(a, b) := \sqrt{h(|a-b|)}$; it is easy to see that d is a pseudo-metric on $\mathbf{R}$; that is, d satisfies all but the following property of a metric: $d(a, b) = 0$ need not imply $a = b$.

According to Bass and Khoshnevisan [4],

$$(6.6) \qquad \limsup_{\delta \downarrow 0} \sup_{d(a,b) < \delta} \frac{|Z(t, a) - Z(t, b)|}{\int_0^{d(a,b)} (\log N(u))^{1/2}\, \mathrm{d}u} \leq 2 \left(\sup_x Z(t, x) \right)^{1/2} \qquad \text{a.s.,}$$

where $N(u)$ denotes the smallest number of d-balls of radius $\leq u$ needed to cover $[-1, 1]$. This sharpened an earlier result of Barlow [3, Theorem 1.1]. Furthermore, $\sup_x Z(t, x) < \infty$ a.s. [4, Theorem 3.1]. Consequently,

$$(6.7) \qquad \sup_{d(a,b) < \delta} |Z(t, a) - Z(t, b)| = O \left(\int_0^\delta (\log N(u))^{1/2}\, \mathrm{d}u \right) \qquad \text{as } \delta \downarrow 0.$$

According to equations (6.128) and (6.130) of Marcus and Rosen [40, Lemma 6.4.1, p. 271], there exists a finite constant $c > 0$ such that for all $u > 0$ small,

$$(6.8) \qquad \begin{aligned} N(u) &\leq \frac{c}{\ell_2 \{(x, y) \in [-1, 1]^2 : \ h(|x-y|) < u/4\}} \\ &\leq \frac{\text{const}}{h^{-1}(u/4)}, \end{aligned}$$

where $\ell_2(A)$ denotes the Lebesgue measure of $A \subset \mathbf{R}^2$. [Specifically, we apply Lemma 6.4.1 of that reference with their $K := [-1, 1]$ and their $\mu_4 := c$.]

If $\gamma := \underline{\mathrm{ind}}\, h > 0$, then $h(u) = O(u^{\gamma+o(1)})$ as $u \to 0^+$. This and (6.8) together imply that $\log N(u) = O(\log(1/u))$ as $u \to 0^+$, whence

$$(6.9) \qquad \sup_{d(a,b)<\delta} |Z(t,a) - Z(t,b)| = O\left(\delta^{1+o(1)}\right) \qquad \text{as } \delta \downarrow 0.$$

Since h is increasing, the following holds: As $\eta \to 0^+$,

$$(6.10) \qquad \sup_{|a-b|<\eta} |Z(t,a) - Z(t,b)| = O\left(|h(\eta)|^{(1/2)+o(1)}\right) = O\left(\eta^{(\gamma/2)+o(1)}\right).$$

Therefore, $a \mapsto Z(t,a)$ is Hölder continuous of any order $< \gamma/2$. Among other things, this implies (6.3) with Z replacing H, whence it follows that (1)–(3) of the theorem are equivalent.

The hyperbolic portion of the theorem is proved similarly, but we use Proposition 3.8 in place of Proposition 3.3 everywhere. $\qquad\square$

Proof of Theorem 2.6. Since β'' is positive, $\operatorname{Re}\Psi(\xi) \to \infty$ as $|\xi| \to \infty$. Therefore, for all $\epsilon, \vartheta > 0$ and $x \in \mathbf{R}$,

$$(6.11) \qquad \begin{aligned} \mathscr{E}(\epsilon;\delta_x) &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{\mathrm{d}\xi}{(1/\epsilon) + \operatorname{Re}\Psi(\xi)} \\ &\le \text{const} \cdot \epsilon + \int_{\{\operatorname{Re}\Psi > \vartheta\}} \frac{\mathrm{d}\xi}{(1/\epsilon) + \operatorname{Re}\Psi(\xi)}. \end{aligned}$$

If ξ is sufficiently large, then for all $\gamma \in (d, \beta'')$ we can find a constant $C_\gamma \in (0, \infty)$ such that $\operatorname{Re}\Psi(\xi) \ge C_\gamma |\xi|^\gamma$ for all $\xi \in \{\operatorname{Re}\Psi > \vartheta\}$. Consequently,

$$(6.12) \qquad \begin{aligned} \mathscr{E}(\epsilon;\delta_x) &= O\left(\epsilon + \int_{\{\operatorname{Re}\Psi > \vartheta\}} \frac{\mathrm{d}\xi}{(1/\epsilon) + |\xi|^\gamma}\right) \\ &= O\left(\epsilon^{1-(d/\gamma)}\right), \end{aligned}$$

as $\epsilon \downarrow 0$. Thus, $\underline{\mathrm{ind}}\, \mathscr{E}(\bullet;\delta_x) \le 1 - (d/\beta'')$, where this index was introduced in (5.4). A similar calculation shows that $\underline{\mathrm{ind}}\, h \le \beta'' - d$; confer with (6.2) for the definition of this quantity. Thus, for all fixed $T > 0$, Proposition 5.6 and (6.1) together prove the following: For all $\tau < (\beta'' - d)/\beta''$, $\zeta < \beta'' - d$, $x, y \in \mathbf{R}$, and $s, t \in [0, T]$,

$$(6.13) \qquad \mathrm{E}\left(|H(t,x) - H(s,y)|^2\right) \le \text{const} \cdot \left(|s-t|^\tau + |x-y|^\zeta\right).$$

A two-dimensional version of Kolmogorov's continuity theorem finishes the proof; see the proof of Theorem (2.1) of Revuz and Yor [53, p. 25].

The proof, in the case of the stochastic wave equation, is similar, but we use Propositions 3.8 and 5.7 instead of Propositions 3.3 and 5.6, respectively. $\qquad\square$

7. Heat equation via generators of Markov processes

We now briefly consider the stochastic heat equation, where the spatial movement is governed by the generator $\mathscr{L}$ of a Markov process $X := \{X_t\}_{t\ge0}$ that takes values in a locally compact separable metric space F. We assume further that X admits a symmetrizing measure m that is Radon and positive. Let us emphasize that m satisfies $(P_t f, g) = (f, P_t g)$ for all $t \ge 0$ and $f, g \in L^2(m)$, where $\{P_t\}_{t\ge0}$ denotes the transition operators of X and where $(\varphi_1, \varphi_2) := \int \varphi_1\varphi_2 \, dm$ for all $\varphi_1, \varphi_2 \in L^2(m)$.

7.1. **The general problem.** Consider the stochastic heat equation

(7.1)
$$\left| \begin{array}{l} \partial_t u(t\,,x) = (\mathscr{L}u)(t\,,x) + \dot{w}(t\,,x), \\ u(0\,,x) = 0, \end{array} \right.$$

valid for all $t \geq 0$ and $x \in F$. Here, the underlying noise w in (7.1) is a Gaussian martingale measure on $\mathbf{R}_+ \times F$ in the sense of Walsh [55]: w is defined on the filtered probability space $(\Omega\,,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0}, \mathrm{P})$ and $w_t(\varphi) := \int_0^t \int_F \varphi(s\,,x)\,w(\mathrm{d}x\,\mathrm{d}s)$ defines an $\{\mathscr{F}_t\}_{t\geq 0}$-martingale for $\varphi \in L^2(ds \times m)$, and w can be characterized by the covariance functional for the corresponding Wiener integrals:

(7.2) $$\mathrm{E}\left(\int f\,\mathrm{d}w \cdot \int g\,\mathrm{d}w \right) = \int_0^\infty \mathrm{d}s \int_F m(\mathrm{d}y) \int_F m(\mathrm{d}x)\; f(s\,,x)g(s\,,y)$$

for all $f, g \in L^2(ds \times m)$.

We can follow the description of Walsh [55] and write the weak form of equation (7.1) as follows: For all $\varphi \in L^2(m)$ and $t \geq 0$,

(7.3) $$u(t\,,\varphi) = \int_0^t \int_F (P_{t-s}\varphi)(x)\,w(\mathrm{d}x\,\mathrm{d}s).$$

Lemma 7.1. *The integral defined by (7.3) is well defined for all $\varphi \in L^2(m)$.*

Proof. We follow closely the proof of Proposition 2.8 and apply the fact that the semigroup $\{P_s\}_{s\geq 0}$ is a contraction on $L^2(m)$.   □

Let Z denote the occupation measure of X; consult (1.3). The following is the key result of this section. It identifies an abstract Hilbertian quasi-isometry between the occupation-measure L^2-norm of X and a similar norm for the solution to the stochastic heat equation (7.1) for $\mathscr{L}$.

Theorem 7.2. *If u denotes the weak solution to (7.1), then for all $\varphi \in L^2(m)$ and $t \geq 0$,*

(7.4) $$\tfrac{1}{8}\,t\mathrm{E}\left(|u(t\,,\varphi)|^2 \right) \leq \mathrm{E}_m\left(|Z(t\,,\varphi)|^2 \right) \leq 4t\mathrm{E}\left(|u(t\,,\varphi)|^2 \right).$$

As usual, E_m refers to the expectation operator for the process X, started according to the measure m.

The preceding theorem follows from the next formula.

Proposition 7.3. *If u denotes the weak solution to (7.1), then for all $\varphi \in L^2(m)$ and $t \geq 0$,*

(7.5) $$\mathrm{E}_m\left(|Z(t\,,\varphi)|^2 \right) = 4 \int_0^t \mathrm{E}\left(|u(s/2\,,\varphi)|^2 \right)\,\mathrm{d}s.$$

Proof of Proposition 7.3. Since m is a symmetrizing measure for X, the P_m-law of X_u is m for all $u \geq 0$. By the Markov property and Tonelli's theorem,

(7.6) $$\mathrm{E}_m\left(|Z(t\,,\varphi)|^2 \right) = 2 \int_0^t \mathrm{d}u \int_u^t \mathrm{d}v\; (P_{v-u}\varphi\,,\varphi).$$

We compute the Laplace transform of both sides, viz.,

$$\int_0^\infty e^{-\lambda t} \mathrm{E}_m\left(|Z(t\,,\varphi)|^2\right)\,\mathrm{d}t = 2\int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}u \int_u^t \mathrm{d}v\; e^{-\lambda t}\left(P_{v-u}\varphi\,,\varphi\right)$$

(7.7)
$$= 2\int_0^\infty \mathrm{d}u \int_u^\infty \mathrm{d}v \left(\int_v^\infty e^{-\lambda t}\,\mathrm{d}t\right)\left(P_{v-u}\varphi\,,\varphi\right)$$

$$= \frac{2}{\lambda^2}\int_0^\infty e^{-\lambda s}\left(P_s\varphi\,,\varphi\right)\,\mathrm{d}s.$$

The exchange of the integrals is justified because $(P_r\varphi\,,\varphi) = \|P_{r/2}\varphi\|_{L^2(m)}^2$ is positive and finite. Because $\varphi \in L^2(m)$, Fubini's theorem implies that for all $\lambda > 0$,

(7.8)
$$\int_0^\infty e^{-\lambda t} \mathrm{E}_m\left(|Z(t\,,\varphi)|^2\right)\,\mathrm{d}t = \frac{2}{\lambda^2}\left(R_\lambda\varphi\,,\varphi\right),$$

where $R_\lambda := \int_0^\infty \exp(-\lambda s)P_s\,\mathrm{d}s$ defines the resolvent of $\{P_t\}_{t\geq 0}$.

Let T_λ denote an independent mean-$(1/\lambda)$ exponential holding time. The preceding display can be rewritten as follows:

(7.9)
$$\mathrm{E}_m\left(|Z(T_\lambda\,,\varphi)|^2\right) = \frac{2}{\lambda}\left(R_\lambda\varphi\,,\varphi\right) \qquad \text{for all } \lambda > 0.$$

Next we consider the weak solution u to the stochastic heat equation (7.1) by first observing that $\mathrm{E}(|u(t\,,\varphi)|^2) = \int_0^t \|P_s\varphi\|_{L^2(m)}^2\,\mathrm{d}s$. It follows from this, and successive applications of Tonelli's theorem, that for all $\beta > 0$,

(7.10)
$$\mathrm{E}\left(|u(T_\beta\,,\varphi)|^2\right) = \int_0^\infty \beta e^{-\beta t}\mathrm{d}t \int_0^t \|P_s\varphi\|_{L^2(m)}^2\,\mathrm{d}s$$

$$= \int_0^\infty e^{-\beta s}\|P_s\varphi\|_{L^2(m)}^2\,\mathrm{d}s.$$

Because $\|P_s\varphi\|_{L^2(m)}^2 = (P_{2s}\varphi\,,\varphi)$, we may apply Fubini's theorem once more, and select $\beta := 2\lambda$, to find that

(7.11)
$$\mathrm{E}\left(|u(T_{2\lambda}\,,\varphi)|^2\right) = \int_0^\infty e^{-2\lambda s}\left(P_{2s}\varphi\,,\varphi\right)\,\mathrm{d}s$$

$$= \left(\varphi\,,\int_0^\infty e^{-2\lambda s}P_{2s}\varphi\,\mathrm{d}s\right)$$

$$= \frac{1}{2}\left(R_\lambda\varphi\,,\varphi\right).$$

The condition of square integrability for φ justifies the appeal to Fubini's theorem. We can compare (7.9) and (7.11) to find that

(7.12)
$$\mathrm{E}_m\left(|Z(T_\lambda\,,\varphi)|^2\right) = \frac{4}{\lambda}\mathrm{E}\left(|u(T_{2\lambda}\,,\varphi)|^2\right) \qquad \text{for all } \lambda > 0.$$

Define $q(t) := \mathrm{E}_m(|Z(t\,,\varphi)|^2)$, $\rho(t) := \mathrm{E}(|u(t/2\,,\varphi)|^2)$ and $\mathbf{1}(t) := 1$ for all $t \geq 0$. The preceding shows that the Laplace transform of q is equal to 4 times the product of the respective Laplace transforms of ρ and $\mathbf{1}$. Thus, we can invert to find that $q = 4\rho * \mathbf{1}$, which is another way to state the theorem. $\square$

Proof of Theorem 7.2. Let us choose and fix a measurable function $\varphi : F \to \mathbf{R}$ such that $|\varphi| \in L^2(m)$. The defining isometry for Wiener integrals yields the following identity, where both sides are convergent: $\mathrm{E}(|u(t\,,\varphi)|^2) = \int_0^t \|P_s\varphi\|_{L^2(m)}^2\,\mathrm{d}s$.

Proposition 7.3 implies that

$$(7.13) \qquad 2t\mathrm{E}\left(|u(t/4\,,\varphi)|^2\right) \le \mathrm{E}_m\left(|Z(t\,,\varphi)|^2\right) \le 4t\mathrm{E}\left(|u(t/2\,,\varphi)|^2\right).$$

[For the lower bound, we use the bound

$$\int_0^t \mathrm{E}(|u(s/2\,,\varphi)|^2)\,\mathrm{d}s \ge \int_{t/2}^t \mathrm{E}(|u(s/2\,,\varphi)|^2)\,\mathrm{d}s.]$$

By monotonicity, $\mathrm{E}(|u(t/2\,,\varphi)|^2) \le \mathrm{E}(|u(t\,,\varphi)|^2)$, whence follows the announced upper bound for $\mathrm{E}_m(|Z(t\,,\varphi)|^2)$.

In order to prove the other bound we first write

$$(7.14) \qquad Z(t\,,\varphi) = Z(t/2\,,\varphi) + Z(t/2\,,\varphi) \circ \theta_{t/2},$$

where $\{\theta_s\}_{s\ge 0}$ denotes the collection of all shifts on the paths of X. We care only about distributional properties. Therefore, by working on an appropriate probability space, we can always insure that these shifts can be constructed; see Blumenthal and Getoor [7].

We apply the Markov property at time $t/2$. Since $\mathrm{P}_m \circ X_{t/2}^{-1} = m$, it follows that

$$(7.15) \quad \mathrm{E}_m\left(\left|Z(t/2\,,\varphi) \circ \theta_{t/2}\right|^2\right) = \mathrm{E}_m\left(|Z(t/2\,,\varphi)|^2 \circ \theta_{t/2}\right) = \mathrm{E}_m\left(|Z(t/2\,,\varphi)|^2\right),$$

and hence $\mathrm{E}_m(|Z(t\,,\varphi)|^2) \le 4\mathrm{E}_m(|Z(t/2\,,\varphi)|^2)$, thanks to (7.14). Consequently,

$$(7.16) \qquad \mathrm{E}_m\left(|Z(t\,,\varphi)|^2\right) \le 16\mathrm{E}_m\left(|Z(t/4\,,\varphi)|^2\right) \qquad \text{for all } t \ge 0.$$

This and the first inequality of (7.13) together imply the remaining bound in the statement of the theorem. $\qquad\square$

7.2. **The stochastic heat equation in dimension** $2 - \epsilon$**.** We now specialize the setup of the preceding subsection to produce an interesting family of examples: We suppose that F is a locally compact subset of $\mathbf{R}^d$ for some integer $d \ge 1$, and m is a positive Radon measure on F, as before. Let $\{R_\lambda\}_{\lambda>0}$ denote the resolvent of X, and suppose that X has jointly continuous and uniformly bounded resolvent densities $\{r_\lambda\}_{\lambda>0}$. In particular, $r_\lambda(x\,,y) \ge 0$ for all $x, y \in F$ and

$$(7.17) \qquad (R_\lambda f)(x) = \int_F r_\lambda(x\,,y)f(y)\,m(\mathrm{d}y),$$

for all measurable functions $f : F \to \mathbf{R}_+$. Recall that X has local times $\{Z(t\,,x)\}_{t\ge 0, x\in F}$ if and only if for all measurable functions $f : F \to \mathbf{R}_+$, and every $t \ge 0$,

$$(7.18) \qquad Z(t\,,f) = \int_F Z(t\,,z)f(z)\,m(\mathrm{d}z),$$

valid P_x-a.s. for all $x \in F$. Choose and fix some point $a \in F$, and define

$$(7.19) \qquad f_\epsilon^a(z) := \frac{\mathbf{1}_{B(a,\epsilon)}(z)}{m(B(a\,,\epsilon))} \qquad \text{for all } z \in F \text{ and } \epsilon > 0.$$

Of course, $B(a\,,\epsilon)$ denotes the ball of radius ϵ about a, measured in the natural metric of F. Because r_λ is jointly continuous, $\lim_{\epsilon\downarrow 0}(R_\lambda f_\epsilon^a)(x) = r_\lambda(x\,,a)$, uniformly for x ranging over a given compact set.

Define $\varphi_{\epsilon,\delta} := f_\epsilon^a - f_\delta^a$, and observe that $\varphi_{\epsilon,\delta}$ is an element of $L^2(m)$. Furthermore,

$$(7.20) \quad \lim_{\epsilon,\delta\downarrow 0} (R_\lambda \varphi_{\epsilon,\delta}, \varphi_{\epsilon,\delta}) = \lim_{\epsilon,\delta\downarrow 0} \left\{ (R_\lambda f_\epsilon^a, f_\epsilon^a) - 2(R_\lambda f_\epsilon^a, f_\delta^a) + (R_\lambda f_\delta^a, f_\delta^a) \right\} = 0.$$

If $h \in L^2(m)$, then the weak solution h to (7.1)—where $\mathscr{L}$ denotes the L^2-generator of X—satisfies

$$
\begin{aligned}
\mathrm{E}\left(|u(t,h)|^2\right) &= \int_0^t \|P_s h\|_{L^2(m)}^2 \, ds \\
&\leq e^{2\lambda t} \int_0^\infty e^{-2\lambda s} \|P_s h\|_{L^2(m)}^2 \, ds \\
&= \frac{e^{2\lambda t}}{2}(R_\lambda h, h).
\end{aligned}
$$
(7.21)

Therefore, Theorem 7.2 and (7.20) together imply that $\{u(t, f_\epsilon^a)\}_{\epsilon>0}$ is a Cauchy sequence in $L^2(\mathrm{P})$ for all $t \geq 0$. In other words, we have shown that the stochastic heat equation (7.1) has a "random-field solution".

Example 7.4. It is now easy to check, using the heat-kernel estimates of Barlow [1, Theorems 8.1.5 and 8.1.6], that for all $d \in (0, 2)$ there exists a compact "fractal" $F \subset \mathbf{R}^2$ of Hausdorff dimension d such that Brownian motion on F satisfies the bounded/continuous resolvent-density properties here. Such fractals include, but are not limited to, finitely-ramified ones such as the Sierpinski triangle [the "gasket"].

The preceding proves that if we replace $\mathscr{L}$ by the Laplacian on F, then the stochastic heat equation (7.1) has a "random-field solution." Specifically, the latter means that for all $t \geq 0$ and $a \in F$, $u(t, f_\epsilon^a)$ converges in $L^2(\mathrm{P})$ as $\epsilon \downarrow 0$, where f_ϵ^a is defined in (7.19). This example comes about because the fractional diffusions of Barlow [1] have local times when the dimension d of the fractal on which they live satisfies $d < 2$. See Barlow [1, Theorem 3.32], for instance. $\square$

8. A SEMILINEAR PARABOLIC PROBLEM

We consider the semilinear problems that correspond to the stochastic heat equation (2.3). At this point in the development of SPDEs, we can make general sense of nonlinear stochastic PDEs only when the linearized SPDE is sensible. Thus, we assume henceforth that $d = 1$.

We investigate the semilinear stochastic heat equation. Let $b : \mathbf{R} \to \mathbf{R}$ be a measurable function, and consider the solution H_b to the following SPDE:

$$(8.1) \quad \left| \begin{aligned} &\partial_t H_b(t, x) = (\mathscr{L} H_b)(t, x) + b(H_b(t, x)) + \dot{w}(t, x), \\ &H_b(0, x) = 0, \end{aligned} \right.$$

where $\mathscr{L}$ denotes the generator of the Lévy process X, as before.

Equation (8.1) has a chance of making sense only if the linearized problem (2.3) has a random-field solution H. In that case we follow Walsh [55] and write the solution H_b as the solution to the following:

$$(8.2) \quad H_b(t, x) = H(t, x) + \int_0^t \int_{-\infty}^\infty b(H_b(s, x - y)) \, P_{t-s}(dy) \, ds,$$

where the measures $\{P_t\}_{t\geq 0}$ are determined from the semigroup of X by $P_t(E) := (P_t \mathbf{1}_E)(0)$ for all Borel sets $E \subset \mathbf{R}$. [This is standard notation.] Apparently, (8.2)

is a deterministic integral equation that is driven by the noise H. We will soon see that this integral equation has a "good solution" H_b under more or less standard conditions on the function b. But first, let us make an observation.

Lemma 8.1. *If (2.3) has a random-field solution, then the process X has a jointly measurable transition density $\{p_t(x)\}_{t>0,x\in\mathbf{R}}$ that satisfies the following: For all $\eta > 0$ there exists a constant $C := C_\eta \in (0,\infty)$ such that for all $t > 0$,*

$$(8.3) \qquad \int_0^t \|p_s\|_{L^2(\mathbf{R})}^2 \, ds \le C \exp(\eta t).$$

Finally, $(t,x) \mapsto p_t(x)$ is uniformly continuous on $[\epsilon,T] \times \mathbf{R}$ for all fixed $\epsilon, T > 0$.

Proof. We can inspect the function $y = x\exp(-x)$ to find that $\exp(-x) \le (1+x)^{-1}$ for all $x \ge 0$. Consequently,

$$(8.4) \qquad \int_{-\infty}^\infty e^{-t\operatorname{Re}\Psi(\xi)} \, d\xi \le \int_{-\infty}^\infty \frac{d\xi}{1 + t\operatorname{Re}\Psi(\xi)}.$$

The finiteness of the second integral, however, has been shown to be equivalent to the existence of random-field solutions to (2.3); see (3.17). It follows that the first integral in (8.4) is convergent. We apply the inversion theorem to deduce from this that the transition densities of X are given by

$$p_t(x) = (2\pi)^{-1} \int_{-\infty}^\infty \exp\{-ix\xi - t\Psi(\xi)\} d\xi,$$

where the integral is absolutely convergent for all $t > 0$ and $x \in \mathbf{R}$. Among other things, this formula implies the uniform continuity of $p_t(x)$ away from $t = 0$. In addition, Plancherel's theorem tells us that for all $s > 0$,

$$(8.5) \qquad \|p_s\|_{L^2(\mathbf{R})}^2 = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-2s\operatorname{Re}\Psi(\xi)} \, d\xi.$$

Therefore, Lemma 3.5 and Tonelli's theorem together imply that for all $\lambda > 0$,

$$(8.6) \qquad \int_0^t \|p_s\|_{L^2(\mathbf{R})}^2 \, ds \le \frac{e^{2t/\lambda}}{4\pi} \int_{-\infty}^\infty \frac{d\xi}{(1/\lambda) + \operatorname{Re}\Psi(\xi)} < \infty.$$

This completes the proof. $\qquad\square$

Because of (8.2) and the preceding lemma, let us define a solution to (8.1) as the random field H_b that solves the following:

$$(8.7) \qquad H_b(t,x) = H(t,x) + \int_0^t ds \int_{-\infty}^\infty dy \, b(H_b(s,x-y)) \, p_{t-s}(y),$$

provided that the preceding indeed has a solution.

For the following we assume that the underlying probability space $(\Omega, \mathscr{F}, \mathrm{P})$ is complete.

Theorem 8.2. *Suppose b is bounded and globally Lipschitz, and the stochastic heat equation (2.3) has a random-field solution H; thus, in particular, $d = 1$. Then, there exists a modification of H, denoted still by H, and a process H_b with the following properties:*

(1) *$H_b \in L_{loc}^p(\mathbf{R}_+ \times \mathbf{R})$ almost surely, for all $p \in [1,\infty)$.*
(2) *With probability one, (8.7) holds simultaneously for all $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$.*

(3) *For all $T > 0$, J is a.s. bounded and continuous on $[0\,,T] \times \mathbf{R}$, where*

$$J(t\,,x) := \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \; b(H_b(s\,,x-y))p_{t-s}(y).$$

Remark 8.3. Before we proceed with the proof, we make a few remarks:

(1) Aspects of this theorem are well known and exist in the literature. Notably, one can apply a very general theorem of Dalang [15, Theorem 13] to conclude that if (1.1) has a random-field solution, then H_b exists, satisfies (1), and solves (8.7) almost surely for every fixed pair $(t\,,x) \in \mathbf{R}_+ \times \mathbf{R}$.

(2) It is possible to adapt the argument of Nualart and Pardoux [45, Proposition 1.6] to deduce that the laws of H and H_b are mutually absolutely continuous; see also Dalang and Nualart [21, Corollary 5.3] and Dalang, Khoshnevisan, and Nualart [22, eqs. (5.2) and (5.3)]. A consequence of this mutual absolute continuity is that H_b is [Hölder] continuous if and only if H is as well.

(3) Theorem 8.2 implies facts that cannot be described by the change-of-measure methods alluded to in the preceding sections. For instance, Theorem 8.2 has the striking consequence that with probability one, H_b *and* H *blow up in exactly the same points*! [This is simply so, because $J = H_b - H$ is locally bounded.] For example, we mention that the operators considered in Example 5.5, when the parameter α there is ≤ 2, lead to discontinuous solutions H that blow up (a.s.) in every open subset of $\mathbf{R}_+ \times \mathbf{R}$ [40, Corollary 5.3.6, p. 208]. In those cases, H_b inherits this property as well.

We prove Theorem 8.2 next. As we mentioned earlier, portions of the theorem—including existence and uniqueness—follow from a much more general result of Dalang [15, Theorem 13], which we use during the course of the proof.

Proof. We will need the following assertion:

$$(8.8) \qquad\qquad\qquad H \text{ is continuous in probability.}$$

In fact, Dalang [15, Theorem 13] has proved the stronger proposition that H is continuous in $L^2(\mathrm{P})$. Among other things, (8.8) and Theorem 13 of Dalang [15] together prove that our random-field solution is also a D-random-field solution.

Since b is globally Lipschitz, Theorem 13 of Dalang [15] implies the existence of a random field u_∞ such that

$$(8.9) \qquad \sup_{t\in[0,T]} \sup_{x\in\mathbf{R}} \mathrm{E}\left(|u_\infty(t\,,x)|^p\right) < \infty \qquad \text{for all } T > 0 \text{ and } p \in [1\,,\infty),$$

and outside a single set of P-measure zero,

$$(8.10) \qquad u_\infty(t\,,x) = H(t\,,x) + \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \; b(u_\infty(s\,,x-y))\,p_{t-s}(y),$$

simultaneously for almost all $(t\,,x) \in \mathbf{R}_+ \times \mathbf{R}$. [Observe the order of the quantifiers.] By the Fubini–Tonelli theorem, $u_\infty \in L^p_{loc}(\mathbf{R}_+ \times \mathbf{R})$ almost surely for all $p \ge 1$. As a particular case, we have $u_\infty \in L^1_{loc}(\mathbf{R}_+ \times \mathbf{R})$ almost surely.

Next we choose and fix a constant $T > 0$, and consider the Borel probability measure Υ_T on $\mathbf{R}_+ \times \mathbf{R}$ defined by

$$(8.11) \qquad\qquad \int f \, \mathrm{d}\Upsilon_T := \frac{1}{2T} \int_0^T \mathrm{d}t \int_{-\infty}^\infty e^{-|x|} \, \mathrm{d}x \; f(t\,,x),$$

valid for all bounded and measurable functions $f : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$. Owing to (8.9) and the Fubini–Tonelli theorem, $u_\infty \in L^1(\Upsilon_T)$ almost surely.

We recall a standard fact from classical analysis: *If $f \in L^1(\Upsilon_T)$, then f is continuous in the measure Υ_T.* This means that for all $\delta > 0$,

$$(8.12) \qquad \lim_{\epsilon, \eta \to 0} \Upsilon_T \left\{ (t, x) : \ \left| f(t + \eta, x + \epsilon) - f(t, x) \right| > \delta \right\} = 0,$$

and it follows immediately from standard approximation arguments [56, §2.201, p. 17]. Consequently, Lemma 8.1 and the fact that $u_\infty \in L^1(\Upsilon_T)$ a.s. together imply that u_∞ is continuous in Υ_T-measure almost surely. Because T is an arbitrary positive number, and since b is bounded and Lipschitz continuous, the integrability/continuity properties of $p_t(x)$—as explained in Lemma 8.1—together imply that the function

$$(8.13) \quad (t, x) \mapsto \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \ b(u_\infty(s, x - y)) \, p_{t-s}(y)$$

$$= \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}z \ b(u_\infty(s, z)) \, p_{t-s}(x - z)$$

is almost surely continuous; it is also a.s.-bounded on $[0, T] \times \mathbf{R}$ for every nonrandom $T > 0$. Now let us *define*

$$(8.14) \qquad H_b(t, x) := H(t, x) + \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \ b(u_\infty(s, x - y)) \, p_{t-s}(y).$$

Thanks to (8.10),

$$(8.15) \qquad \mathrm{P} \left\{ H_b(t, x) = u_\infty(t, x) \right\} = 1 \qquad \text{for all } (t, x) \in \mathbf{R}_+ \times \mathbf{R}.$$

Thus, H_b is a modification of u_∞. In addition, the Tonelli theorem applies to tell us that H_b inherits the almost-sure local integrability property of u_∞. Thus, (8.9) holds also with u_∞ replaced by H_b, whence we have $H_b \in L_{loc}^p(\mathbf{R}_+ \times \mathbf{R})$ a.s., thanks to the Fubini–Tonelli theorem. Moreover, outside a single null set we have
(8.16)

$$J(t, x) = \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \ b(u_\infty(s, x - y)) \, p_{t-s}(y) \qquad \text{for all } (t, x) \in \mathbf{R}_+ \times \mathbf{R}.$$

This proves the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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