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Hitting probabilities for systems of non-linear stochastic heat equations in spatial dimension $k \geq 1$

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Abstract We consider a system of d non-linear stochastic heat equations in spatial dimension $k \geq 1$, whose solution is an $\mathbb{R}^d$-valued random field $u = \{u(t\,, x),\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^k\}$. The d-dimensional driving noise is white in time and with a spatially homogeneous covariance defined as a Riesz kernel with exponent β, where $0 < \beta < (2 \wedge k)$. The non-linearities appear both as additive drift terms and as multipliers of the noise. Using techniques of Malliavin calculus, we establish an upper bound on the two-point density, with respect to Lebesgue measure, of the $\mathbb{R}^{2d}$-valued random vector $(u(s, y), u(t, x))$, that, in particular, quantifies how this density degenerates as $(s, y) \to (t, x)$. From this result, we deduce a lower bound on hitting probabilities of the process u, in terms of Newtonian capacity. We also establish an upper bound on hitting probabilities of the process in terms of Hausdorff measure. These estimates make it possible to show that points are polar when $d > \frac{4+2k}{2-\beta}$ and are not polar when

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$d < \frac{4+2k}{2-\beta}$. In the first case, we also show that the Hausdorff dimension of the range of the process is $\frac{4+2k}{2-\beta}$ a.s.

Keywords Hitting probabilities · Systems of non-linear stochastic heat equations · Spatially homogeneous Gaussian noise · Malliavin calculus

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1 Introduction and main results

Consider the following system of stochastic partial differential equations:

$$\begin{cases} \frac{\partial}{\partial t} u_i(t,x) = \frac{1}{2}\Delta_x u_i(t,x) + \sum_{j=1}^{d} \sigma_{i,j}(u(t,x))\, \dot{F}^j(t,x) + b_i(u(t,x)), \\ u_i(0,x) = 0, \qquad\qquad\qquad\qquad i \in \{1,\dots,d\}, \end{cases} \tag{1.1}$$

where $t \geq 0$, $x \in \mathbb{R}^k$, $k \geq 1$, $\sigma_{i,j}, b_i : \mathbb{R}^d \to \mathbb{R}$ are globally Lipschitz functions, i, $j \in \{1,\dots,d\}$, and the Δ_x denotes the Laplacian in the spatial variable x.

The noise $\dot{F} = (\dot{F}^1,\dots,\dot{F}^d)$ is a spatially homogeneous centered Gaussian generalized random field with covariance of the form

$$\mathrm{E}\,[\dot{F}^i(t,x)\dot{F}^j(s,y)] = \delta(t-s)\|x-y\|^{-\beta}\delta_{ij}, \quad 0 < \beta < (2 \wedge k). \tag{1.2}$$

Here, $\delta(\cdot)$ denotes the Dirac delta function, δ_{ij} the Kronecker symbol and $\|\cdot\|$ is the Euclidean norm. In particular, the d-dimensional driving noise $\dot{F}$ is white in time and with a spatially homogeneous covariance given by the Riesz kernel $f(x) = \|x\|^{-\beta}$.

The solution u of (1.1) is known to be a d-dimensional random field (see Sect. 2, where precise definitions and references are given), and the aim of this paper is to develop potential theory for u. In particular, given a set $A \subset \mathbb{R}^d$, we want to determine whether or not the process u hits A with positive probability. For systems of linear and/or nonlinear stochastic heat equations in spatial dimension 1 driven by a d-dimensional space-time white noise, this type of question was studied in Dalang, Khoshnevisan, and Nualart [5] and [6]. For systems of linear and/or nonlinear stochastic wave equations, this was studied first in Dalang and Nualart [7] for the reduced wave equation in spatial dimension 1, and in higher spatial dimensions in Dalang and Sanz-Solé [9,10]. The approach of this last paper is used for some of our estimates (see Proposition 5.7).

We note that for the Gaussian random fields, and, in particular, for (1.1) when $b \equiv 0$ and $\sigma = I_d$, the $d \times d$-identity matrix, there is a well-developed potential theory [2,27]. The main effort here concerns the case where b and/or σ are not constant, in which case u is *not* Gaussian.

Let us introduce some notation concerning potential theory. For all Borel sets $F \subseteq \mathbb{R}^d$, let $\mathscr{P}(F)$ denote the set of all probability measures with compact support in F. For all $\alpha \in \mathbb{R}$ and $\mu \in \mathscr{P}(\mathbb{R}^k)$, we let $I_\alpha(\mu)$ denote the *α-dimensional energy* of μ, that is,

$$I_\alpha(\mu) := \iint \mathrm{K}_\alpha(\|x-y\|)\,\mu(dx)\,\mu(dy),$$

where

$$\mathrm{K}_\alpha(r) := \begin{cases} r^{-\alpha} & \text{if } \alpha > 0, \\ \log(N_0/r) & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha < 0, \end{cases} \tag{1.3}$$

where N_0 is a constant whose value will be specified later (at the end of the proof of Lemma 2.3).

For all $\alpha \in \mathbb{R}$ and Borel sets $F \subset \mathbb{R}^k$, $\mathrm{Cap}_\alpha(F)$ denotes the *α-dimensional capacity of F*, that is,

$$\mathrm{Cap}_\alpha(F) := \left[\inf_{\mu \in \mathscr{P}(F)} I_\alpha(\mu)\right]^{-1},$$

where, by definition, $1/\infty := 0$.

Given $\alpha \geq 0$, the α-dimensional *Hausdorff measure* of F is defined by

$$\mathscr{H}_\alpha(F) = \lim_{\epsilon \to 0^+} \inf \left\{ \sum_{i=1}^{\infty} (2r_i)^\alpha : F \subseteq \bigcup_{i=1}^{\infty} B(x_i\,, r_i),\ \sup_{i\geq 1} r_i \leq \epsilon \right\}, \tag{1.4}$$

where $B(x\,, r)$ denotes the open (Euclidean) ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$. When $\alpha < 0$, we define $\mathscr{H}_\alpha(F)$ to be infinite.

Consider the following hypotheses on the coefficients of the system of equations (1.1), which are common assumptions when using Malliavin calculus:

P1 The functions $\sigma_{i,j}$ and b_i are C^∞ and have bounded partial derivatives of all positive orders, and the $\sigma_{i,j}$ are bounded, $i, j \in \{1, \ldots, d\}$.
P2 The matrix $\sigma = (\sigma_{i,j})_{1 \leq i,j \leq d}$ is *strongly elliptic*, that is, there is $\rho > 0$ such that $\|\sigma(x) \cdot \xi\|^2 \geq \rho^2 > 0$, for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$ with $\|\xi\| = 1$.

Remark 1.1 Note that because σ is a square matrix,

$$\inf_{x \in \mathbb{R}^d} \inf_{\|\xi\|=1} \|\sigma(x) \cdot \xi\|^2 = \inf_{x \in \mathbb{R}^d} \inf_{\|\xi\|=1} \|\xi^{\mathsf{T}} \cdot \sigma(x)\|^2$$

(for non square matrices, this equality is false in general). Therefore, it follows from **P2** that $\|\xi^{\mathsf{T}} \cdot \sigma(x)\|^2 \geq \rho^2 > 0$, for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$ with $\|\xi\| = 1$.

For $T > 0$ fixed, we say that $I \times J \subset (0, T] \times \mathbb{R}^k$ is a closed non-trivial rectangle if $I \subset (0, T]$ is a closed non-trivial interval and J is of the form $[a_1, b_1] \times \cdots \times [a_k, b_k]$, where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$, $i = 1, \ldots, k$.

The main result of this article is the following.

Theorem 1.2 *Let u denote the solution of (1.1). Assume conditions* **P1** *and* **P2**. *Fix $T > 0$ and let $I \times J \subset (0, T] \times \mathbb{R}^k$ be a closed non-trivial rectangle. Fix $M > 0$ and $\eta > 0$.*

(a) *There exists $C > 0$ such that for all compact sets $A \subseteq [-M, M]^d$,*

$$\mathrm{P}\{u(I \times J) \cap A \neq \varnothing\} \leq C\, \mathscr{H}_{d-(\frac{4+2k}{2-\beta})-\eta}(A).$$

(b) *There exists $c > 0$ such that for all compact sets $A \subseteq [-M, M]^d$,*

$$\mathrm{P}\{u(I \times J) \cap A \neq \varnothing\} \geq c\, \mathrm{Cap}_{d-(\frac{4+2k}{2-\beta})+\eta}(A).$$

As a consequence of Theorem 1.2, we deduce the following result on the polarity of points. Recall that A is a *polar set* for u if $P\{u(I \times J) \cap A \neq \varnothing\} = 0$, for any $I \times J$ as in Theorem 1.2.

Corollary 1.3 *Let u denote the solution of* (1.1). *Assume* **P1** *and* **P2**. *Then points are not polar for u when $d < \frac{4+2k}{2-\beta}$, and are polar when $d > \frac{4+2k}{2-\beta}$ (if $\frac{4+2k}{2-\beta}$ is an integer, then the case $d = \frac{4+2k}{2-\beta}$ is open).*

Another consequence of Theorem 1.2 is the Hausdorff dimension of the range of the process u.

Corollary 1.4 *Let u denote the solution of* (1.1). *Assume* **P1** *and* **P2**. *If $d > \frac{4+2k}{2-\beta}$, then a.s.,*

$$\dim_{\mathrm{H}}\left(u\left(\mathbb{R}_+ \times \mathbb{R}^k\right)\right) = \frac{4+2k}{2-\beta}.$$

The result of Theorem 1.2 can be compared to the best result available for the Gaussian case, using the result of [27, Theorem 7.6].

Theorem 1.5 *Let v denote the solution of* (1.1) *when $b \equiv 0$ and $\sigma \equiv I_d$. Fix $T, M > 0$ and let $I \times J \subset (0, T] \times \mathbb{R}^k$ be a closed non-trivial rectangle. There exists $c > 0$ such that for all compact sets $A \subseteq [-M, M]^d$,*

$$c^{-1}\, \mathrm{Cap}_{d-\left(\frac{4+2k}{2-\beta}\right)}(A) \leq \mathrm{P}\Big\{v\,(I \times J) \cap A \neq \varnothing\Big\} \leq c\, \mathscr{H}_{d-\left(\frac{4+2k}{2-\beta}\right)}(A).$$

Theorem 1.5 is proved in Sect. 2. Comparing Theorems 1.2 and 1.5, we see that Theorem 1.2 is nearly optimal.

In order to prove Theorem 1.2, we shall use techniques of Malliavin calculus in order to establish first the following result. Let $p_{t,x}(z)$ denote the probability density function of the $\mathbb{R}^d$-valued random vector $u(t, x) = (u_1(t, x), \ldots, u_d(t, x))$ and for $(s, y) \neq (t, x)$, let $p_{s,y;\,t,x}(z_1, z_2)$ denote the joint density function of the $\mathbb{R}^{2d}$-valued random vector

$$(u(s, y), u(t, x)) = (u_1(s, y), \ldots, u_d(s, y), u_1(t, x), \ldots, u_d(t, x))\,.$$

The existence (and smoothness) of $p_{t,x}(\cdot)$ when $d = 1$ follows from [13, Theorem 2.1] and Lemma 6.1 (see also [18, Theorem 6.2]). The extension of this fact to $d \geq 1$ is proved in Proposition 4.2. The existence (and smoothness) of $p_{s,y;\,t,x}(\cdot,\cdot)$ is a consequence of Theorem 5.8 and [16, Theorem 2.1.2 and Corollary 2.1.2].

The main technical effort in this paper is the proof of the following theorem.

Theorem 1.6 *Assume* **P1** *and* **P2**. *Fix $T > 0$ and let $I \times J \subset (0, T] \times \mathbb{R}^k$ be a closed non-trivial rectangle.*

(a) *The density $p_{t,x}(z)$ is a C^∞ function of z and is uniformly bounded over $z \in \mathbb{R}^d$ and $(t, x) \in I \times J$.*
(b) *For all $\eta > 0$ and $\gamma \in (0, 2 - \beta)$, there exists $c > 0$ such that for any $(s, y), (t, x) \in I \times J$, $(s, y) \neq (t, x)$, $z_1, z_2 \in \mathbb{R}^d$, and $p \geq 1$,*

$$p_{s,y;\,t,x}(z_1, z_2) \leq c(|t-s|^{\frac{2-\beta}{2}} + \|x-y\|^{2-\beta})^{-(d+\eta)/2} \times \left[\frac{|t-s|^{\gamma/2} + \|x-y\|^{\gamma}}{\|z_1 - z_2\|^2} \wedge 1\right]^{p/(2d)}. \tag{1.5}$$

Statement (a) of this theorem is proved at the end of Sect. 4, and statement (b) is proved in Sect. 5.3.

Remark 1.7 (a) Theorem 1.6(a) remains valid under a slightly weaker version of **P1**, in which the $\sigma_{i,j}$ need not be bounded (but their derivatives of all positive orders are bounded).

(b) The last factor on the right-hand side of (1.5) is similar to the one obtained in [10, Remark 3.1], while in the papers [5,6], which concern spatial dimension 1, it was replaced by

$$\exp\left(-\frac{\|z_1 - z_2\|^2}{c\left(|t-s|^{\gamma/2} + \|x-y\|^{\gamma}\right)}\right).$$

This exponential factor was obtained by first proving this bound in the case where $b_i \equiv 0$, $i = 1, \ldots, d$, and then using Girsanov's theorem. In the case of higher spatial dimensions that we consider here, we can obtain this same bound when $b_i \equiv 0$, $i = 1, \ldots, d$ (see Lemma 5.12 in Sect. 5.3). Since there is no applicable Girsanov's theorem in higher spatial dimensions and for equations on all of $\mathbb{R}^d$, we establish (1.5) and, following [10], show in Sect. 2.4 that this estimate is sufficient for our purposes.

One further fact about $p_{t,x}(\cdot)$ that we will need is provided by the following recent result of Nualart [19].

Theorem 1.8 *Assume* **P1** *and* **P2**. *Fix $T > 0$ and let $I \times J \subset (0, T] \times \mathbb{R}^k$ be a closed non-trivial rectangle. Then for all $z \in \mathbb{R}^d$ and $(t, x) \in (0, T] \times \mathbb{R}^k$, the density $p_{t,x}(z)$ is strictly positive.*

2 Proof of Theorems 1.2, 1.5 and Corollaries 1.3, 1.4 (assuming Theorem 1.6)

We first define precisely the driving noise that appears in (1.1). Let $\mathscr{D}(\mathbb{R}^k)$ be the space of C^∞ test-functions with compact support. Then $F = \{F(\phi) = (F^1(\phi), \dots, F^d(\phi)),\ \phi \in \mathscr{D}(\mathbb{R}^{k+1})\}$ is an $L^2(\Omega, \mathscr{F}, \mathrm{P})^d$-valued mean zero Gaussian process with covariance

$$\mathrm{E}\left[F^i(\phi)F^j(\psi)\right] = \delta_{ij} \int\limits_{\mathbb{R}_+} dr \int\limits_{\mathbb{R}^k} dy \int\limits_{\mathbb{R}^k} dz\, \phi(r, y) \|y - z\|^{-\beta} \psi(r, z).$$

Using elementary properties of the Fourier transform (see Dalang [3]), this covariance can also be written as

$$\mathrm{E}\left[F^i(\phi)F^j(\psi)\right] = \delta_{ij}\ c_{k,\beta} \int\limits_{\mathbb{R}_+} dr \int\limits_{\mathbb{R}^k} d\xi\ \|\xi\|^{\beta-k} \mathscr{F}\phi(r, \cdot)(\xi)\overline{\mathscr{F}\psi(r, \cdot)(\xi)},$$

where $c_{k,\beta}$ is a constant and $\mathscr{F} f(\cdot)(\xi)$ denotes the Fourier transform of f, that is,

$$\mathscr{F} f(\cdot)(\xi) = \int\limits_{\mathbb{R}^k} e^{-2\pi i \xi\cdot x} f(x)\, dx.$$

Since Eq. (1.1) is formal, we first provide, following Walsh [25, pp. 289–290], a rigorous formulation of (1.1) through the notion of *mild solution* as follows. Let $M = (M^1, \dots, M^d)$, $M^i = \{M^i_t(A), t \geq 0, A \in \mathscr{B}_b(\mathbb{R}^k)\}$ be the d-dimensional worthy martingale measure obtained as an extension of the process $\dot{F}$ as in Dalang and Frangos [4]. Then a *mild solution* of (1.1) is a jointly measurable $\mathbb{R}^d$-valued process $u = \{u(t, x),\ t \geq 0,\ x \in \mathbb{R}^k\}$, adapted to the natural filtration generated by M, such that

$$\begin{aligned} u_i(t, x) = &\int\limits_0^t \int\limits_{\mathbb{R}^k} S(t - s, x - y) \sum_{j=1}^d \sigma_{i,j}(u(s, y)) M^j(ds, dy) \\ &+ \int\limits_0^t ds \int\limits_{\mathbb{R}^k} dy\, S(t - s, x - y)\, b_i(u(s, y)), \quad i \in \{1, \dots, d\}, \end{aligned} \tag{2.1}$$

where $S(t, x)$ is the fundamental solution of the deterministic heat equation in $\mathbb{R}^k$, that is,

$$S(t, x) = (2\pi t)^{-k/2} \exp\left(-\frac{\|x\|^2}{2t}\right),$$

and the stochastic integral is interpreted in the sense of [25]. We note that the covariation measure of M^i is

$$Q\left([0,t]\times A\times B\right)=\langle M^i(A),M^i(B)\rangle_t=t\int_{\mathbb{R}^k}dx\int_{\mathbb{R}^k}dy\,1_A(x)\,\|x-y\|^{-\beta}\,1_B(y),$$

and its dominating measure is $K\equiv Q$. In particular,

$$\begin{aligned}&E\left[\left(\int_0^t\int_{\mathbb{R}^k}S(t-s,x-y)\,M^i(ds,dy)\right)^2\right]\\&=\int_0^t ds\int_{\mathbb{R}^k}dy\int_{\mathbb{R}^k}dz\,S(t-s,x-y)\,\|y-z\|^{-\beta}\,S(t-s,x-z)\\&=c_{k,\beta}\int_0^t ds\int_{\mathbb{R}^k}d\xi\,\|\xi\|^{\beta-k}\,|\mathscr{F}S(t-s,\cdot)(\xi)|^2,\end{aligned}\tag{2.2}$$

where we have used elementary properties of the Fourier transform (see also Dalang [3], Nualart and Quer-Sardanyons [18], and Dalang and Quer-Sardanyons [8] for properties of the stochastic integral). This last formula is convenient since

$$\mathscr{F}S(r,\cdot)(\xi)=\exp\left(-2\pi^2 r\|\xi\|^2\right).\tag{2.3}$$

The existence and uniqueness of the solution of (1.1) is studied in Dalang [3] for general space correlation functions f which are non-negative, non-negative definite and continuous on $\mathbb{R}^k\setminus\{0\}$ (in the case where $k=1$; for these properties, the extension to $k>1$ is straightforward). In particular, it is proved that if the *spectral measure* of $\dot F$, that is, the non-negative tempered measure μ on $\mathbb{R}^k$ such that $\mathscr{F}\mu=f$, satisfies

$$\int_{\mathbb{R}^k}\frac{\mu(d\xi)}{1+\|\xi\|^2}<+\infty,\tag{2.4}$$

then there exists a unique solution of (1.1) such that $(t,x)\mapsto u(t,x)$ is L^2-continuous, and condition (2.4) is also necessary for existence of a mild solution.

In the case of the noise (1.2), $f(x)=\|x\|^{-\beta}$ and $\mu(d\xi)=c_d\|\xi\|^{\beta-k}\,d\xi$, where c_d is a constant (see Stein [24, Chap.V, Sect. 1, Lemma 2(b)]), and the condition (2.4) is equivalent to

$$0<\beta<(2\wedge k).\tag{2.5}$$

Therefore, by Dalang [3], there exists a unique L^2-continuous solution of (1.1), satisfying

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^k}\mathrm{E}\left[|u_i(t,x)|^p\right]<+\infty,\quad i\in\{1,\ldots,d\},$$

for any $T>0$ and $p\geq 1$.

2.1 Hölder continuity of the solution

Let $T > 0$ be fixed. In Sanz-Solé and Sarrà [23, Theorem 2.1] it is proved that for any $\gamma \in (0, 2-\beta),\ s, t \in [0, T],\ s \leq t,\ x, y \in \mathbb{R}^k,\ p > 1$,

$$\mathrm{E}\left[\|u(t,x) - u(s,y)\|^p\right] \leq C_{\gamma,p,T}\left(|t-s|^{\gamma/2} + \|x-y\|^{\gamma}\right)^{p/2}. \tag{2.6}$$

In particular, the trajectories of u are a.s. $\gamma/4$-Hölder continuous in t and $\gamma/2$-Hölder continuous in x.

The next result shows that the estimate (2.6) is nearly optimal (the only possible improvement would be to include the value $\gamma = 2-\beta$).

Proposition 2.1 *Let v denote the solution of (1.1) with $\sigma \equiv 1$ and $b \equiv 0$. Then for any $0 < t_0 < T,\ p > 1$ and K a compact set, there exists $c_1 = c_1(p, t_0, K) > 0$ such that for any $t_0 \leq s \leq t \leq T,\ x, y \in K,\ i \in \{1, \ldots, d\}$,*

$$\mathrm{E}\left[|v_i(t,x) - v_i(s,y)|^p\right] \geq c_1\left(|t-s|^{\frac{2-\beta}{4}p} + \|x-y\|^{\frac{2-\beta}{2}p}\right). \tag{2.7}$$

Proof Since v is Gaussian, it suffices to check (2.7) for $p = 2$. Setting $t = s + h$ and $x = y + z$, we observe from (2.2) that

$$\mathrm{E}\left[|v_i(s+h, y+z) - v_i(s,y)|^2\right] = c_{k,\beta}\ (I_1 + I_2),$$

where

$$I_1 = \int_s^{s+h} dr \int_{\mathbb{R}^k} d\xi\ \|\xi\|^{\beta-k}\ |\mathscr{F} S(s+h-r,\cdot)(\xi)|^2,$$

$$I_2 = \int_0^{s} dr \int_{\mathbb{R}^k} d\xi\ \|\xi\|^{\beta-k}\ |\mathscr{F} S(s+h-r,\cdot)(\xi)\, e^{-2\pi i\xi\cdot(y+z)} - \mathscr{F} S(s-r,\cdot)(\xi)\, e^{-2\pi i\xi\cdot y}|^2.$$

Case 1. $h \geq \|z\|^2$. In this case, we notice from (2.3) that

$$\begin{aligned} I_1 + I_2 \geq I_1 &= \int_s^{s+h} dr \int_{\mathbb{R}^k} d\xi\ \|\xi\|^{\beta-k} \exp(-4\pi^2(s+h-r)\|\xi\|^2) \\ &= \int_{\mathbb{R}^k} d\xi\ \|\xi\|^{\beta-k}\left(\frac{1-\exp(-4\pi^2 h\|\xi\|^2)}{4\pi^2\|\xi\|^2}\right). \end{aligned}$$

We now use the change of variables $\tilde{\xi} = h^{1/2}\xi$ to see that the last right-hand side is equal to

$$h^{\frac{2-\beta}{2}} \int_{\mathbb{R}^k} d\tilde{\xi}\ \|\tilde{\xi}\|^{\beta-k} \left(\frac{1 - \exp(-4\pi^2 \|\tilde{\xi}\|^2)}{4\pi^2 \|\tilde{\xi}\|^2} \right).$$

Note that the last integral is positive and finite. Therefore, when $h \geq \|z\|^2$,

$$\mathrm{E}\left[|v_i(t+h, y+z) - v_i(s, y)|^2 \right] \geq c \left(\max\left(h, \|z\|^2\right) \right)^{\frac{2-\beta}{2}}.$$

Case 2. $\|z\|^2 \geq h$. In this case, we notice that

$$\begin{aligned} I_1 + I_2 \geq I_2 = \int_0^s dr \int_{\mathbb{R}^k} d\xi\ \|\xi\|^{\beta-k} \exp\left(-4\pi^2 (s-r) \|\xi\|^2\right) \\ \times \left| 1 - \exp\left(-4\pi^2 h \|\xi\|^2\right) \exp\left(-2\pi i\, \xi \cdot z\right) \right|^2. \end{aligned}$$

We use the elementary inequality $|1 - re^{i\theta}| \geq \frac{1}{2}|1 - e^{i\theta}|$, valid for all $r \in [0, 1]$ and $\theta \in \mathbb{R}$, and we calculate the dr-integral, to see that

$$I_2 \geq \int_{\mathbb{R}^k} d\xi\ \|\xi\|^{\beta-k} \left(\frac{1 - \exp(-4\pi^2 s \|\xi\|^2)}{4\pi^2 \|\xi\|^2} \right) |1 - \exp(-2\pi i\, \xi \cdot z)|^2.$$

Because $z \in K - K$ and K is compact, fix $M > 0$ such that $\|z\| \leq M$. When $z \neq 0$, we use the change of variables $\tilde{\xi} = \|z\|\xi$ and write $e = z/\|z\|$ to see that the last right-hand side is equal to

$$\begin{aligned} & c\|z\|^{2-\beta} \int_{\mathbb{R}^k} d\tilde{\xi}\ \|\tilde{\xi}\|^{\beta-k-2} \left(1 - \exp\left(-4\pi^2 t \|\tilde{\xi}\|^2 / \|z\|^2\right)\right) |1 - \exp(-2\pi i\, \tilde{\xi} \cdot e)|^2 \\ & \geq c\|z\|^{2-\beta} \int_{\mathbb{R}^k} d\tilde{\xi}\ \|\tilde{\xi}\|^{\beta-k-2} \left(1 - \exp\left(-4\pi^2 t_0 \|\tilde{\xi}\|^2 / M^2\right)\right) |1 - \exp(-2\pi i\, \tilde{\xi} \cdot e)|^2. \end{aligned}$$

The last integral is a positive constant. Therefore, when $\|z\|^2 \geq h$,

$$\mathrm{E}\left[|v_i(t+h, y+z) - v_i(s, y)|^2 \right] \geq c \left(\max\left(h, \|z\|^2\right) \right)^{\frac{2-\beta}{2}}.$$

Cases 1 and 2 together establish (2.7). □

2.2 Proof of Theorem 1.5

Under the hypotheses on b and σ, the components of $v = (v_1, \dots, v_d)$ are independent, so v is a $(1+k, d)$-Gaussian random field in the sense of [27]. We apply Theorem 7.6 in [27]. For this, we are going to verify Conditions (C1) and (C2) of [27, Sect. 2.4, p. 158] with $N = k+1$, $H_1 = (2-\beta)/4$, $H_j = (2-\beta)/2$, $j = 1, \dots, k$.

In particular, for (C1), we must check that there are positive constants $c_1, \dots, c_4$ such that for all (t, x) and (s, y) in $I \times J$,

$$c_1 \le E\left(v_1(t,x)^2\right) \le c_2, \tag{2.8}$$

and

$$\begin{aligned} c_3\left(|t-s|^{\frac{2-\beta}{2}} + \|x-y\|^{2-\beta}\right) &\le E\left[(v_1(t,x) - v_1(s,y))^2\right] \\ &\le c_4\left(|t-s|^{\frac{2-\beta}{2}} + \|x-y\|^{2-\beta}\right). \end{aligned} \tag{2.9}$$

Condition (2.8) is satisfied because $\mathrm{E}[v_1(t,x)^2] = Ct^{(2-\beta)/2}$ (see (2.2), (2.3) and Lemma 6.1). The lower bound of (2.9) follows from Proposition 2.1. The upper bound is a consequence of [22, Propositions 2.4 and 3.2].

Finally, in order to establish Condition (C2) it suffices to apply the fourth point of Remark 2.2 in [27]. Indeed, it is stated there that Condition (C1) implies condition (C2) when $(t,x) \mapsto E[v_1(t,x)^2] = Ct^{(2-\beta)/2}$ is continuous in $I \times J$ with continuous partial derivatives, and this is clearly the case.

This completes the proof of Theorem 1.5. □

2.3 Proof of Theorem 1.2(a)

Fix $T > 0$ and let $I \times J \subset (0, T] \times \mathbb{R}^k$ be a closed non-trivial rectangle. Let $\gamma \in (0, 2-\beta)$. For all positive integers n, $i \in \{0, \dots, n\}$ and $j = (j_1, \dots, j_k) \in \{0, \dots, n\}^k$, set $t_i^n = i2^{-\frac{4n}{\gamma}}$, $x_j^n = (x_{j_1}^n = j_1 2^{-\frac{2n}{\gamma}}, \dots, x_{j_k}^n = j_k 2^{-\frac{2n}{\gamma}})$, and

$$I_{i,j}^n = \left[t_i^n, t_{i+1}^n\right] \times \left[x_{j_1}^n, x_{j_1+1}^n\right] \times \dots \times \left[x_{j_k}^n, x_{j_k+1}^n\right].$$

The proof of the following lemma uses Theorem 1.6a and (2.6), but follows along the same lines as [5, Theorem 3.3] with $\Delta((t,x);(s,y))$ there replaced by $|t-s|^{\gamma/2} + \|x-y\|^{\gamma}$, β there replaced by $d - \eta$ and ϵ in Condition (3.2) there replaced by 2^{-n}. It is therefore omitted.

Lemma 2.2 *Fix $\eta > 0$ and $M > 0$. Then there exists $c > 0$ such that for all $z \in [-M, M]^d$, n large and $I_{i,j}^n \subset I \times J$,*

$$P\left\{u\left(I_{i,j}^n\right) \cap B\left(z, 2^{-n}\right) \ne \varnothing\right\} \le c2^{-n(d-\eta)}.$$

Proof of the upper bound in Theorem 1.2 Fix $\epsilon \in (0, 1)$ and $n \in \mathbb{N}$ such that $2^{-n-1} < \epsilon \leq 2^{-n}$, and write

$$\mathrm{P}\{u\,(I \times J) \cap B(z\,,\epsilon) \neq \varnothing\} \leq \sum_{(i,j):I_{i,j}^n \cap (I\times J)\neq\varnothing} \mathrm{P}\{u(I_{i,j}^n) \cap B(z\,,\epsilon) \neq \varnothing\}.$$

The number of $(1+k)$-tuples (i, j) involved in the sum is at most $c\,2^{n(\frac{4}{\gamma}+\frac{2}{\gamma}k)}$. Lemma 2.2 implies therefore that for all $z \in A$, $\eta > 0$ and large n,

$$\mathrm{P}\{u\,(I \times J) \cap B(z\,,\epsilon) \neq \varnothing\} \leq \tilde{C}(2^{-n})^{d-\eta}\, 2^{n\frac{4+2k}{\gamma}}.$$

Let $\eta' = \eta + \left(\frac{1}{\gamma} - \frac{1}{2-\beta}\right)(4+2k)$. Then this is equal to

$$2^{-n(d-(\frac{4+2k}{2-\beta}+\eta'))} \leq C\epsilon^{d-\frac{4+2k}{2-\beta}-\eta'},$$

because $2^{-n-1} < \epsilon \leq 2^{-n}$. Note that C does not depend on $(n\,,\epsilon)$, and η' can be made arbitrarily small by choosing γ close to $2-\beta$ and η small enough. In particular, for all $\epsilon \in (0, 1)$,

$$\mathrm{P}\{u\,(I \times J) \cap B(z\,,\epsilon) \neq \varnothing\} \leq C\,\epsilon^{d-\frac{4+2k}{2-\beta}-\eta'}. \tag{2.10}$$

Now we use a *covering argument*: Choose $\tilde{\epsilon} \in (0, 1)$ and let $\{B_i\}_{i=1}^\infty$ be a sequence of open balls in $\mathbb{R}^d$ with respective radii $r_i \in [0, \tilde{\epsilon})$ such that

$$A \subset \cup_{i=1}^\infty B_i \text{ and } \sum_{i=1}^\infty (2r_i)^{d-\frac{4+2k}{2-\beta}-\eta'} \leq \mathscr{H}_{d-\frac{4+2k}{2-\beta}-\eta'}(A) + \tilde{\epsilon}. \tag{2.11}$$

Because $\mathrm{P}\{u\,(I \times J) \cap A \neq \varnothing\}$ is at most $\sum_{i=1}^\infty \mathrm{P}\{u(I \times J) \cap B_i \neq \varnothing\}$, the bounds in (2.10) and (2.11) together imply that

$$\mathrm{P}\{u\,(I \times J) \cap A \neq \varnothing\} \leq C\left(\mathscr{H}_{d-\frac{4+2k}{2-\beta}-\eta'}(A) + \tilde{\epsilon}\right).$$

Let $\tilde{\epsilon} \to 0$ to conclude. □

2.4 Proof of Theorem 1.2(b)

The following preliminary lemmas are the analogues needed here of [5, Lemma 2.2] and [5, Lemma 2.3], respectively.

Lemma 2.3 *Fix $T > 0$ and let $I \times J \subset (0, T] \times \mathbb{R}^k$ be a closed non-trivial rectangle. Then for all $N > 0$, $b > 0$, $\tilde{\gamma} > \gamma > 0$ and $p > \frac{2d}{\gamma}(\tilde{\gamma}b - 2k - 4)$, there exists a finite and positive constant $C = C(I, J, N, b, \gamma, \tilde{\gamma}, p)$ such that for all $a \in [0\,, N]$,*

$$\int_I dt \int_I ds \int_J dx \int_J dy\, (|t-s|^{\tilde{\gamma}/2} + \|x-y\|^{\tilde{\gamma}})^{-b/2}$$
$$\times \left(\frac{|t-s|^{\gamma/4} + \|x-y\|^{\gamma/2}}{a} \wedge 1 \right)^{p/(2d)} \leq C\, \mathrm{K}_{\frac{\tilde{\gamma}}{\gamma} b - \frac{4+2k}{\gamma}}(a). \tag{2.12}$$

Proof Let $|J|$ denote the diameter of the set J. Using the change of variables $\tilde{u} = t-s$ (t fixed), $\tilde{v} = x-y$ (x fixed), we see that the integral in (2.12) is bounded above by

$$|I|\lambda_k(J) \int_0^{|I|} d\tilde{u} \int_{B(0,|J|)} d\tilde{v}\, (\tilde{u}^{\tilde{\gamma}/2} + \|\tilde{v}\|^{\tilde{\gamma}})^{-b/2} \left(\frac{\tilde{u}^{\gamma/4} + \|\tilde{v}\|^{\gamma/2}}{a} \wedge 1 \right)^{p/(2d)},$$

where λ_k denotes Lebesgue measure in $\mathbb{R}^k$. A change of variables [$\tilde{u} = a^{4/\gamma} u^2$, $\tilde{v} = a^{2/\gamma} v$] implies that this is equal to

$$C\, a^{\frac{4+2k}{\gamma} - \frac{\tilde{\gamma}}{\gamma} b} \int_0^{a^{-2/\gamma}(|I|)^{1/2}} u du$$
$$\times \int_{B(0,|J|a^{-2/\gamma})} dv\, (u^{\tilde{\gamma}} + \|v\|^{\tilde{\gamma}})^{-b/2} \left(\left(u^{\gamma/4} + \|v\|^{\gamma/2} \right) \wedge 1 \right)^{p/(2d)}.$$

We pass to polar coordinates in the variable v, to see that this is bounded by

$$C\, a^{\frac{4+2k}{\gamma} - \frac{\tilde{\gamma}}{\gamma} b} \int_0^{a^{-2/\gamma}(|I|)^{1/2}} du \int_0^{|J|a^{-2/\gamma}} dx\, x^{k-1}\, u\, (u^{\tilde{\gamma}} + x^{\tilde{\gamma}})^{-b/2}$$
$$\times \left(\left(u^{\gamma/2} + x^{\gamma/2} \right)^{p/(2d)} \wedge 1 \right).$$

Bounding $x^{k-1}u$ by $(u+x)^k$ and using the fact that all norms in $\mathbb{R}^2$ are equivalent, we bound this above by

$$C\, a^{\frac{4+2k}{\gamma} - \frac{\tilde{\gamma}}{\gamma} b} \int_0^{a^{-2/\gamma}(2|I|)^{1/2}} du \int_0^{2|J|a^{-2/\gamma}} dx\, (u+x)^{k - \frac{\tilde{\gamma} b}{2}} \left((u+x)^{\gamma p/(4d)} \wedge 1 \right).$$

We now pass to polar coordinates of (u, x), to bound this by

$$C\, a^{\frac{4+2k}{\gamma} - \frac{\tilde{\gamma}}{\gamma} b} (I_1 + I_2(a)), \tag{2.13}$$

where

$$I_1 = \int_0^{KN^{-2/\gamma}} d\rho\, \rho^{k+1-\frac{\tilde{\gamma}b}{2}} (\rho^{\gamma p/(4d)} \wedge 1),$$

$$I_2(a) = \int_{KN^{-2/\gamma}}^{Ka^{-2/\gamma}} d\rho\, \rho^{k+1-\frac{\tilde{\gamma}b}{2}},$$

where $K = 2(\sqrt{|I|} \vee |J|)$. Clearly, $I_1 \leq C < \infty$ since $k + 1 - \frac{\tilde{\gamma}}{2}b + \frac{\gamma p}{4d} > -1$ by the hypothesis on p. Moreover, if $k + 2 - \frac{\tilde{\gamma}b}{2} \neq 0$, then

$$I_2(a) = K^{k+2-\frac{\tilde{\gamma}b}{2}} \frac{a^{\frac{\tilde{\gamma}}{\gamma}b-\frac{4+2k}{\gamma}} - N^{\frac{\tilde{\gamma}}{\gamma}b-\frac{4+2k}{\gamma}}}{k + 2 - \frac{\tilde{\gamma}b}{2}}.$$

There are three separate cases to consider. (i) If $k + 2 - \frac{\tilde{\gamma}b}{2} < 0$, then $I_2(a) \leq C$ for all $a \in [0, N]$. (ii) If $k+2-\frac{\tilde{\gamma}b}{2} > 0$, then $I_2(a) \leq c\, a^{\frac{\tilde{\gamma}}{\gamma}b-\frac{4+2k}{\gamma}}$. (iii) If $k+2-\frac{\tilde{\gamma}b}{2} = 0$, then

$$I_2(a) = \frac{2}{\gamma}\left[\ln\frac{1}{a} + \ln N\right].$$

We combine these observations to conclude that the expression in (2.13) is bounded by $C\, \mathrm{K}_{\frac{\tilde{\gamma}}{\gamma}b-\frac{4+2k}{\gamma}}(a)$, provided that N_0 in (1.3) is sufficiently large. This proves the lemma. □

For all $a, \nu, \rho > 0$, define

$$\Psi_{a,\nu}(\rho) := \int_0^a dx\, \frac{x^{k-1}}{\rho + x^\nu}. \tag{2.14}$$

Lemma 2.4 *For all $a, \nu, T > 0$, there exists a finite and positive constant $C = C(a\,, \nu\,, T)$ such that for all $0 < \rho < T$,*

$$\Psi_{a,\nu}(\rho) \leq C\mathrm{K}_{(\nu-k)/\nu}(\rho).$$

Proof If $\nu < k$, then $\lim_{\rho\to 0} \Psi_{a,\nu}(\rho) = \int_0^a x^{k-1-\nu}\, dx < \infty$. In addition, $\rho \mapsto \Psi_{a,\nu}(\rho)$ is nonincreasing, so $\Psi_{a,\nu}$ is bounded on $\mathbb{R}_+$ when $\nu < k$. In this case, $\mathrm{K}_{(\nu-k)/\nu}(\rho) = 1$, so the result follows in the case that $\nu < k$.

For the case $\nu \geq k$, we change variables ($y = x\rho^{-1/\nu}$) to find that

$$\Psi_{a,\nu}(\rho) = \rho^{-(\nu-k)/\nu} \int_0^{a\rho^{-1/\nu}} dy\, \frac{y^{k-1}}{1+y^\nu}.$$

When $\nu > k$, this gives the desired result, with $c = \int_0^{+\infty} dy\, y^{k-1}\,(1+y^\nu)^{-1}$. When $\nu = k$, we simply evaluate the integral in (2.14) explicitly: this gives the result for $0 < \rho < T$, given the choice of $\mathrm{K}_0(r)$ in (1.3). We note that the constraint "$0 < \rho < T$" is needed only in this case. □

Proof of the lower bound of Theorem 1.2 The proof of this result follows along the same lines as the proof of [5, Theorem 2.1(1)], therefore we will only sketch the steps that differ. We need to replace their $\beta - 6$ by our $d - \frac{4+2k}{\tilde{\gamma}} + \eta$.

Note that our Theorem 1.6(a) and Theorem 1.8 prove that

$$\inf_{\|z\|\leq M} \int_I dt \int_J dx\, p_{t,x}(z) \geq C > 0, \tag{2.15}$$

which proves hypothesis **A1'** of [5, Theorem 2.1(1)] (see [5, Remark 2.5(a)]).

Moreover, Theorem 1.6(b) proves a property that is weaker than hypothesis **A2** of [5, Theorem 2.1(1)] with their $\beta = d + \eta$, $\gamma \in (0, 2-\beta)$ and

$$\mathbf{\Delta}((t,x)\,;(s,y)) = |t-s|^{\gamma/2} + \|x-y\|^\gamma,$$

but which will be sufficient for our purposes.

Let us now follow the proof of [5, Theorem 2.1(1)]. Define, for all $z \in \mathbb{R}^d$ and $\epsilon > 0$, $\tilde{B}(z\,,\epsilon) := \{y \in \mathbb{R}^d : \ |y-z| < \epsilon\}$, where $|z| := \max_{1\leq j\leq d} |z_j|$, and

$$J_\epsilon(z) = \frac{1}{(2\epsilon)^d} \int_I dt \int_J dx\, \mathbf{1}_{\tilde{B}(z,\epsilon)}(u(t\,,x)), \tag{2.16}$$

as in [5, (2.28)].

Assume first that $d + \eta < \frac{4+2k}{2-\beta}$. Using Theorem 1.6(b), we find, instead of [5, (2.30)],

$$\mathrm{E}\left[(J_\epsilon(z))^2\right] \leq c \int_I dt \int_I ds \int_J dx \int_J dy\, \left(|t-s|^{\frac{2-\beta}{2}} + \|x-y\|^{2-\beta}\right)^{-(d+\eta)/2}.$$

Use the change of variables $u = t - s$ (t fixed), $v = x - y$ (x fixed) to see that the above integral is bounded above by

$$\tilde{c}\int_0^{|I|} d\tilde{u} \int_{B(0,|J|)} d\tilde{v}\, \left(\tilde{u}^{\frac{2-\beta}{2}} + \|\tilde{v}\|^{2-\beta}\right)^{-(d+\eta)/2}$$

$$= c\int_0^{|I|} du \int_0^{|J|} dx\, x^{k-1} \left(u^{\frac{2-\beta}{2}} + x^{2-\beta}\right)^{-(d+\eta)/2}$$

$$\leq c\int_0^{|I|} du\, \Psi_{|J|,(2-\beta)(d+\eta)/2}\left(u^{(2-\beta)(d+\eta)/4}\right).$$

Hence, Lemma 2.4 implies that for all $\epsilon > 0$,

$$\mathrm{E}\left[(J_\epsilon(z))^2\right] \leq C\int_0^{|I|} du\, \mathrm{K}_{1-\frac{2k}{(2-\beta)(d+\eta)}}\left(u^{(2-\beta)(d+\eta)/4}\right).$$

We now consider three different cases: (i) If $0 < (2-\beta)(d+\eta) < 2k$, then the integral equals $|I|$. (ii) If $2k < (2-\beta)(d+\eta) < 4+2k$, then $\mathrm{K}_{1-\frac{2k}{(2-\beta)(d+\eta)}}(u^{(2-\beta)(d+\eta)/4}) = u^{(k/2)-(2-\beta)(d+\eta)/4}$ and the integral is finite. (iii) If $(2-\beta)(d+\eta) = 2k$, then $\mathrm{K}_0(u^{k/2}) = \log(N_0/u^{k/2})$ and the integral is also finite. The remainder of the proof of the lower bound of Theorem 1.2 when $d+\eta < \frac{4+2k}{2-\beta}$ follows exactly as in [5, Theorem 2.1(1) *Case 1*].

Assume now that $d+\eta > \frac{4+2k}{2-\beta}$. Define, for all $\mu \in \mathscr{P}(A)$ and $\epsilon > 0$,

$$J_\epsilon(\mu) = \frac{1}{(2\epsilon)^d}\int_{\mathbb{R}^d} \mu(dz) \int_I dt \int_J dx\, \mathbf{1}_{\tilde{B}(z,\epsilon)}(u(t\,,x)),$$

as [5, (2.35)].

In order to prove the analogue of [5, (2.41)], we use Theorem 1.6(b) and Lemma 2.3 (instead of [5, Lemma 2.2(1)]), to see that for all $\mu \in \mathscr{P}(A)$, $\epsilon \in (0,1)$ and $\gamma \in (0, 2-\beta)$,

$$\mathrm{E}\left[(J_\epsilon(\mu))^2\right] \leq c\left[\mathrm{Cap}_{\frac{2-\beta}{\gamma}(d+\eta)-\frac{4+2k}{\gamma}}(A)\right]^{-1} = c\left[\mathrm{Cap}_{d+\tilde{\eta}-\frac{4+2k}{2-\beta}}(A)\right]^{-1}.$$

The remainder of the proof of the lower bound of Theorem 1.2 when $d+\eta > \frac{4+2k}{2-\beta}$ follows as in [5, Proof of Theorem 2.1(1) *Case 2*].

The case $d+\eta = \frac{4+2k}{2-\beta}$ is proved exactly along the same lines as the proof of [5, Theorem 2.1(1) *Case 3*], appealing to (2.15), Theorem 1.6(b) and Lemma 2.3. $\square$

2.5 Proof of Corollaries 1.3 and 1.4

Proof of Corollary 1.3 Let $z \in \mathbb{R}^d$. If $d < \frac{4+2k}{2-\beta}$, then there is $\eta > 0$ such that $d - \frac{4+2k}{2-\beta} + \eta < 0$, and thus

$$\mathrm{Cap}_{d-\frac{4+2k}{2-\beta}+\eta}(\{z\}) = 1.$$

Hence, Theorem 1.2(b) implies that $\{z\}$ is not polar. On the other hand, if $d > \frac{4+2k}{2-\beta}$, then there is $\eta > 0$ such that $d - \frac{4+2k}{2-\beta} - \eta > 0$. Therefore,

$$\mathscr{H}_{d-\frac{4+2k}{2-\beta}-\eta}(\{z\}) = 0$$

and Theorem 1.2(a) implies that $\{z\}$ is polar. □

Proof of Corollary 1.4 We first recall, following Khoshnevisan [12, Chap. 11, Sect. 4], the definition of stochastic codimension of a random set E in $\mathbb{R}^d$, denoted $\mathrm{codim}(E)$, if it exists: $\mathrm{codim}(E)$ is the real number $\alpha \in [0, d]$ such that for all compact sets $A \subset \mathbb{R}^d$,

$$P\{E \cap A \neq \varnothing\} \begin{cases} > 0 \text{ whenever } \dim_{\mathrm{H}}(A) > \alpha, \\ = 0 \text{ whenever } \dim_{\mathrm{H}}(A) < \alpha. \end{cases}$$

By Theorem 1.2, $\mathrm{codim}(u(\mathbb{R}_+ \times \mathbb{R}^k)) = (d - \frac{4+2k}{2-\beta})^+$. Moreover, in Khoshnevisan [12, Theorem 4.7.1, Chap.11], it is proved that given a random set E in $\mathbb{R}^d$ whose codimension is strictly between 0 and d,

$$\dim_{\mathrm{H}}(E) + \mathrm{codim}(E) = d, \quad \text{a.s.}$$

This implies the desired statement. □

3 Elements of Malliavin calculus

Let $\mathscr{S}(\mathbb{R}^k)$ be the Schwartz space of $\mathscr{C}^\infty$ functions on $\mathbb{R}^k$ with rapid decrease. Let $\mathscr{H}$ denote the completion of $\mathscr{S}(\mathbb{R}^k)$ endowed with the inner product

$$\begin{aligned}\langle \phi(\cdot), \psi(\cdot)\rangle_{\mathscr{H}} &= \int_{\mathbb{R}^k} dx \int_{\mathbb{R}^k} dy\, \phi(x)\|x-y\|^{-\beta}\psi(y) \\ &= \int_{\mathbb{R}^k} d\xi\, \|\xi\|^{\beta-k}\mathscr{F}\phi(\cdot)(\xi)\overline{\mathscr{F}\psi(\cdot)(\xi)},\end{aligned}$$

$\phi, \psi \in \mathscr{S}(\mathbb{R}^k)$. Notice that $\mathscr{H}$ may contain Schwartz distributions (see [3]).

For $h = (h^1, \dots, h^d) \in \mathscr{H}^d$ and $\tilde{h} = (\tilde{h}^1, \dots, \tilde{h}^d) \in \mathscr{H}^d$, we set $\langle h, \tilde{h}\rangle_{\mathscr{H}^d} = \sum_{i=1}^d \langle h^i, \tilde{h}^i\rangle_{\mathscr{H}}$. Let $T > 0$ be fixed. We set $\mathscr{H}_T^d = L^2([0,T]; \mathscr{H}^d)$ and for $0 \le s \le t \le T$, we will write $\mathscr{H}_{s,t}^d = L^2([s,t]; \mathscr{H}^d)$.

The centered Gaussian noise F can be used to construct an isonormal Gaussian process $\{W(h),\ h \in \mathscr{H}_T^d\}$ (that is, $\mathrm{E}[W(h)W(\tilde{h})] = \langle h, \tilde{h}\rangle_{\mathscr{H}_T^d}$) as follows. Let $\{e_j,\ j \ge 0\} \subset \mathscr{S}(\mathbb{R}^k)$ be a complete orthonormal system of the Hilbert space $\mathscr{H}$. Then for any $t \in [0,T]$, $i \in \{1, \dots, d\}$ and $j \ge 0$, set

$$W_j^i(t) = \int_0^t \int_{\mathbb{R}^k} e_j(x) \cdot F^i(ds, dx),$$

so that $(W_j^i,\ j \ge 1)$ is a sequence of independent standard real-valued Brownian motions such that for any $\phi \in \mathscr{D}([0,T] \times \mathbb{R}^k)$,

$$F^i(\phi) = \sum_{j=0}^{\infty} \int_0^T \left\langle \phi(s,\cdot), e_j(\cdot)\right\rangle_{\mathscr{H}} dW_j^i(s),$$

where the series converges in $L^2(\Omega, \mathscr{F}, \mathrm{P})$. For $h^i \in \mathscr{H}_T$, we set

$$W^i(h^i) = \sum_{j=0}^{\infty} \int_0^T \left\langle h^i(s,\cdot), e_j(\cdot)\right\rangle_{\mathscr{H}} dW_j^i(s),$$

where, again, this series converges in $L^2(\Omega, F, P)$. In particular, for $\phi \in \mathscr{D}([0,T] \times \mathbb{R}^k)$, $F^i(\phi) = W^i(\phi)$. Finally, for $h = (h^1, \dots, h^d) \in \mathscr{H}_T^d$, we set

$$W(h) = \sum_{i=1}^d W^i(h^i).$$

With this isonormal Gaussian process, we can use the framework of Malliavin calculus. Let $\mathscr{S}$ denote the class of smooth random variables of the form $G = g(W(h_1), \dots, W(h_n))$, where $n \ge 1$, $g \in \mathscr{C}_P^\infty(\mathbb{R}^n)$, the set of real-valued functions g such that g and all its partial derivatives have at most polynomial growth, $h_i \in \mathscr{H}_T^d$. Given $G \in \mathscr{S}$, its derivative $(D_r G = (D_r^{(1)} G, \dots, D_r^{(d)} G),\ r \in [0,T])$, is an $\mathscr{H}_T^d$-valued random vector defined by

$$D_r G = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(W(h_1), \dots, W(h_n)) h_i(r).$$

For $\phi \in \mathscr{H}^d$ and $r \in [0,T]$, we write $D_{r,\phi} G = \langle D_r G, \phi(\cdot)\rangle_{\mathscr{H}^d}$. More generally, the derivative $D^m G = (D^m_{(r_1,\dots,r_m)} G,\ (r_1, \dots, r_m) \in [0,T]^m)$ of order $m \ge 1$ of G is the

$(\mathscr{H}_T^d)^{\otimes j}$-valued random vector defined by

$$D^m_{(r_1,\dots,r_m)}G = \sum_{i_1,\dots,i_m=1}^{n} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_m}} g(W(h_1),\dots,W(h_n)) h_{i_1}(r_1)\otimes\cdots\otimes h_{i_m}(r_m).$$

For $p, m \geq 1$, the space $\mathbb{D}^{m,p}$ is the closure of $\mathscr{S}$ with respect to the seminorm $\|\cdot\|_{m,p}$ defined by

$$\|G\|^p_{m,p} = \mathrm{E}\left[|G|^p\right] + \sum_{j=1}^{m} \mathrm{E}\left[\|D^j G\|^p_{(\mathscr{H}_T^d)^{\otimes j}}\right].$$

We set $\mathbb{D}^\infty = \cap_{p\geq 1}\cap_{m\geq 1}\mathbb{D}^{m,p}$.

The derivative operator D on $L^2(\Omega)$ has an adjoint, termed the Skorohod integral and denoted by δ, which is an unbounded operator on $L^2(\Omega, \mathscr{H}_T^d)$. Its domain, denoted by Dom δ, is the set of elements $u \in L^2(\Omega, \mathscr{H}_T^d)$ for which there exists a constant c such that $|\mathrm{E}[\langle DF, u\rangle_{\mathscr{H}_T^d}]| \leq c\|F\|_{0,2}$, for any $F \in \mathbb{D}^{1,2}$. If $u \in \text{Dom}\,\delta$, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the following duality relation:

$$\mathrm{E}\left[F\delta(u)\right] = \mathrm{E}\left[\langle DF, u\rangle_{\mathscr{H}_T^d}\right], \quad \text{for all } F \in \mathbb{D}^{1,2}.$$

An important application of Malliavin calculus is the following global criterion for existence and smoothness of densities of probability laws.

Theorem 3.1 [16, Thm.2.1.2 and Cor.2.1.2] or [21, Thm.5.2] *Let $F = (F^1,\dots,F^d)$ be an $\mathbb{R}^d$-valued random vector satisfying the following two conditions:*

(i) $F \in (\mathbb{D}^\infty)^d$;
(ii) *the Malliavin matrix of F defined by $\gamma_F = (\langle DF^i, DF^j\rangle_{\mathscr{H}_T^d})_{1\leq i,j\leq d}$ is invertible a.s. and* $(\det \gamma_F)^{-1} \in L^p(\Omega)$ *for all $p \geq 1$.*

Then the probability law of F has an infinitely differentiable density function.

A random vector F that satisfies conditions (i) and (ii) of Theorem 3.1 is said to be nondegenerate. The next result gives a criterion for uniform boundedness of the density of a nondegenerate random vector.

Proposition 3.2 [6, Proposition 3.4] *For all $p > 1$ and $\ell \geq 1$, let $c_1 = c_1(p) > 0$ and $c_2 = c_2(\ell, p) \geq 0$ be fixed. Let $F \in (\mathbb{D}^\infty)^d$ be a nondegenerate random vector such that*

(a) $\mathrm{E}[(\det\gamma_F)^{-p}] \leq c_1$;
(b) $\mathrm{E}[\|D^l(F^i)\|^p_{(\mathscr{H}_T^d)^{\otimes\ell}}] \leq c_2, \quad i = 1,\dots,d.$

Then the density of F is C^∞ and uniformly bounded, and the bound does not depend on F but only on the constants $c_1(p)$ and $c_2(\ell, p)$.

In [13], the Malliavin differentiability and the smoothness of the density of $u(t,x)$ was established when $d=1$, and the extension to $d>1$ can easily be done by working coordinate by coordinate. These results were extended in [18, Proposition 5.1]. In particular, letting $\cdot$ denote the spatial variable, for $r\in[0,t]$ and $i,l\in\{1,\dots,d\}$, the derivative of $u_i(t,x)$ satisfies the system of equations

$$\begin{aligned} D_r^{(l)}(u_i(t,x)) &= \sigma_{il}(u(r,\cdot))\, S(t-r,x-\cdot) \\ &\quad + \int_r^t \int_{\mathbb{R}^k} S(t-\theta,x-\eta) \sum_{j=1}^d D_r^{(l)}\left(\sigma_{i,j}(u(\theta,\eta))\right) M^j(d\theta,d\eta) \\ &\quad + \int_r^t d\theta \int_{\mathbb{R}^k} d\eta\, S(t-\theta,x-\eta) D_r^{(l)}\left(b_i(u(\theta,\eta))\right), \end{aligned} \tag{3.1}$$

and $D_r^{(l)}(u_i(t,x)) = 0$ if $r>t$. Moreover, by [18, Proposition 6.1], for any $p>1$, $m\ge 1$ and $i\in\{1,\dots,d\}$, the order m derivative satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^k} \mathrm{E}\left[\left\| D^m(u_i(t,x))\right\|^p_{(\mathscr{H}_T^d)^{\otimes m}}\right] < +\infty, \tag{3.2}$$

and D^m also satisfies the system of stochastic partial differential equations given in [18, (6.29)] and obtained by iterating the calculation that leads to (3.1). In particular, $u(t,x)\in(\mathbb{D}^\infty)^d$, for all $(t,x)\in[0,T]\times\mathbb{R}^k$.

4 Existence, smoothness and uniform boundedness of the density

The aim of this section is to prove Theorem 1.6(a). For this, we will use Proposition 3.2. The following proposition proves condition (a) of Proposition 3.2.

Proposition 4.1 *Fix $T>0$ and assume hypotheses* **P1** *and* **P2**. *Then, for any $p\ge 1$, $\mathrm{E}\left[(\det\gamma_{u(t,x)})^{-p}\right]$ is uniformly bounded over (t,x) in any closed non-trivial rectangle $I\times J\subset(0,T]\times\mathbb{R}^k$.*

Proof Let $(t,x)\in I\times J$ be fixed, where $I\times J$ is a closed non-trivial rectangle of $(0,T]\times\mathbb{R}^k$. We write

$$\det\gamma_{u(t,x)} \ge \left(\inf_{\xi\in\mathbb{R}^d:\|\xi\|=1}(\xi^T\gamma_{u(t,x)}\xi)\right)^d.$$

Let $\xi \in \mathbb{R}^d$ with $\|\xi\| = 1$ and fix $\epsilon \in (0, 1)$. Using (3.1), we see that

$$\begin{aligned}\xi^T \gamma_{u(t,x)} \xi &\geq \sum_{l=1}^{d} \int_{t-\epsilon}^{t} dr \left\| \sum_{i=1}^{d} D_r^{(l)}(u_i(t,x))\xi_i \right\|_{\mathscr{H}}^2 \\ &= \sum_{l=1}^{d} \int_{t-\epsilon}^{t} dr \left\| \sum_{i=1}^{d} \sigma_{i,l}(u(r,\cdot))S(t-r, x-\cdot)\xi_i + \sum_{i=1}^{d} a_i(l,r,t,x)\xi_i \right\|_{\mathscr{H}}^2,\end{aligned}$$

where, for $r < t$,

$$\begin{aligned}a_i(l,r,t,x) &= \int_r^t \int_{\mathbb{R}^k} S(t-\theta, x-\eta) \sum_{j=1}^{d} D_r^{(l)}\left(\sigma_{i,j}(u(\theta,\eta))\right) M^j(d\theta, d\eta) \\ &\quad + \int_r^t d\theta \int_{\mathbb{R}^k} d\eta\, S(t-\theta, x-\eta) D_r^{(l)}\left(b_i(u(\theta,\eta))\right). \end{aligned} \tag{4.1}$$

We use the inequality

$$\|a+b\|_{\mathscr{H}}^2 \geq \frac{2}{3}\|a\|_{\mathscr{H}}^2 - 2\|b\|_{\mathscr{H}}^2, \tag{4.2}$$

to see that

$$\xi^{\mathsf{T}} \gamma_{u(t,x)} \xi \geq \frac{2}{3} \sum_{l=1}^{d} \int_{t-\epsilon}^{t} dr \left\| \left(\xi^{\mathsf{T}} \cdot \sigma(u(r,\cdot))\right)_l S(t-r, x-\cdot) \right\|_{\mathscr{H}}^2 - 2A_3,$$

where

$$A_3 = \int_{t-\epsilon}^{t} dr \sum_{l=1}^{d} \left\| \sum_{i=1}^{d} a_i(l,r,t,x)\,\xi_i \right\|_{\mathscr{H}}^2.$$

The same inequality (4.2) shows that

$$\sum_{l=1}^{d} \int_{t-\epsilon}^{t} dr \left\| \left(\xi^{\mathsf{T}} \cdot \sigma(u(r,\cdot))\right)_l S(t-r, x-\cdot) \right\|_{\mathscr{H}}^2 \geq \frac{2}{3}A_1 - 2A_2,$$

where

$$A_1 = \int_{t-\epsilon}^{t} dr \sum_{l=1}^{d} \left\| \left(\xi^{\mathsf{T}} \cdot \sigma(u(r,x)) \right)_l \, S(t-r, x-\cdot) \right\|_{\mathscr{H}}^2,$$

$$A_2 = \int_{t-\epsilon}^{t} dr \sum_{l=1}^{d} \left\| \left(\xi^{\mathsf{T}} \cdot (\sigma(u(r,\cdot)) - \sigma(u(r,x))) \right)_l \, S(t-r, x-\cdot) \right\|_{\mathscr{H}}^2. \quad (4.3)$$

Note that we have added and subtracted a "localized" term so as to be able to use the ellipticity property of σ (a similar idea is used in [15] in dimension 1).

Hypothesis **P2** (see also Remark 1.1) and Lemma 6.1 together yield $A_1 \geq C\epsilon^{\frac{2-\beta}{2}}$, where C is uniform over $(t,x) \in I \times J$.

Now, using the Lipschitz property of σ and Hölder's inequality with respect to the measure $\|y-z\|^{-\beta} S(t-r, x-y) S(t-r, x-z)\, dr dy dz$, we get that for $q \geq 1$,

$$\begin{aligned}
&\mathrm{E}\left[\sup_{\xi \in \mathbb{R}^d:\, \|\xi\|=1} |A_2|^q \right] \\
&\leq \left(\int_{t-\epsilon}^{t} dr \int_{\mathbb{R}^k} dy \int_{\mathbb{R}^k} dz\, \|y-z\|^{-\beta} S(t-r, x-y) S(t-r, x-z) \right)^{q-1} \\
&\quad \times \left(\int_{t-\epsilon}^{t} dr \int_{\mathbb{R}^k} dy \int_{\mathbb{R}^k} dz\, \|y-z\|^{-\beta} S(t-r, x-y) S(t-r, x-z) \right. \\
&\quad \left. \times \mathrm{E}\left[\|u(r,y) - u(r,x)\|^q \, \|u(r,z) - u(r,x)\|^q \right] \right).
\end{aligned}$$

Using Lemma 6.1 and (2.6) we get that for any $q \geq 1$ and $\gamma \in (0, 2-\beta)$,

$$\mathrm{E}\left[|A_2|^q \right] \leq C\epsilon^{(q-1)\frac{2-\beta}{2}} \times \Psi,$$

where

$$\Psi = \int_{0}^{\epsilon} dr \int_{\mathbb{R}^k} dy \int_{\mathbb{R}^k} dz\, \|y-z\|^{-\beta} S(r, x-y) S(r, x-z) \|y-x\|^{\frac{\gamma q}{2}} \|z-x\|^{\frac{\gamma q}{2}}.$$

Changing variables $[\tilde{y} = \frac{x-y}{\sqrt{r}}, \tilde{z} = \frac{x-z}{\sqrt{r}}]$, this becomes

$$
\begin{aligned}
\Psi &= \int_0^{\epsilon} dr\, r^{-\frac{\beta}{2}+\frac{\gamma q}{2}} \int_{\mathbb{R}^k} d\tilde{y} \int_{\mathbb{R}^k} d\tilde{z}\, S(1,\tilde{y}) S(1,\tilde{z}) \|\tilde{y}-\tilde{z}\|^{-\beta} \|\tilde{y}\|^{\frac{\gamma q}{2}} \|\tilde{z}\|^{\frac{\gamma q}{2}} \\
&= C\epsilon^{\frac{2-\beta}{2}+\frac{\gamma q}{2}}.
\end{aligned}
$$

Therefore, we have proved that for any $q \geq 1$ and $\gamma \in (0, 2-\beta)$,

$$
\mathrm{E}\left[\sup_{\xi \in \mathbb{R}^d : \|\xi\|=1} |A_2|^q\right] \leq C\epsilon^{\frac{2-\beta}{2}q+\frac{\gamma}{2}q}, \tag{4.4}
$$

where C is uniform over $(t, x) \in I \times J$.

On the other hand, applying Lemma 6.2 with $s = t$, we find that for any $q \geq 1$,

$$
\mathrm{E}\left[\sup_{\xi \in \mathbb{R}^d : \|\xi\|=1} |A_3|^q\right] \leq C\epsilon^{(2-\beta)q},
$$

where C is uniform over $(t, x) \in I \times J$.

Finally, we apply [6, Proposition 3.5] with $Z := \inf_{\|\xi\|=1}(\xi^{\mathsf{T}}\gamma_{u(t,x)}\xi)$, $Y_{1,\epsilon} = Y_{2,\epsilon} = \sup_{\|\xi\|=1}(|A_2| + |A_3|)$, $\epsilon_0 = 1$, $\alpha_1 = \alpha_2 = \frac{2-\beta}{2}$, and $\beta_1 = \beta_2 = \frac{2-\beta}{2} + \frac{\gamma}{2}$, for any $\gamma \in (0, 2-\beta)$, to conclude that for any $p \geq 1$,

$$
\mathrm{E}\left[\left(\det \gamma_{u(t,x)}\right)^{-p}\right] \leq C(p),
$$

where the constant $C(p) < \infty$ does not depend on $(t, x) \in I \times J$. □

In [13, Theorem 3.2] the existence and smoothness of the density of the solution of equation (1.1) with one single equation ($d = 1$) was proved (see also [18, Theorem 6.2]). The extension of this fact for a system of d equations is given in the next proposition.

Proposition 4.2 *Fix $t > 0$ and $x \in \mathbb{R}^k$. Assume hypotheses* **P1** *and* **P2**. *Then the law of $u(t, x)$, solution of (1.1), is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$. Moreover, its density $p_{t,x}(\cdot)$ is C^∞.*

Proof This is a consequence of Theorem 3.1 and Proposition 4.1. □

Proof of Theorem 1.6(a) This follows directly from Proposition 4.1 and (3.2), using Proposition 3.2. □

5 Gaussian upper bound for the bivariate density

The aim of this section is to prove Theorem 1.6(b).

5.1 Upper bound for the derivative of the increment

Proposition 5.1 *Assume hypothesis* **P1**. *Then for any $T > 0$ and $p \geq 1$, there exists $C := C(T, p) > 0$ such that for any $0 \leq s \leq t \leq T$, $x, y \in \mathbb{R}^k$, $m \geq 1$, $i \in \{1, \ldots, d\}$, and $\gamma \in (0, 2-\beta)$,*

$$\|D^m(u_i(t,x) - u_i(s,y))\|_{L^p(\Omega;(\mathscr{H}_T^d)^{\otimes m})} \leq C\left(|t-s|^{\gamma/2} + \|x-y\|^{\gamma}\right)^{1/2}.$$

Proof Assume $m = 1$ and fix $p \geq 2$, since it suffices to prove the statement in this case. Let

$$g_{t,x;s,y}(r, \cdot) := S(t-r, x-\cdot)1_{\{r\leq t\}} - S(s-r, y-\cdot)1_{\{r\leq s\}}.$$

Using (3.1), we see that

$$\|D\left(u_i(t,x) - u_i(s,y)\right)\|^p_{L^p(\Omega;\mathscr{H}_T^d)} \leq c_p\left(A_1 + A_{2,1} + A_{2,2} + A_{3,1} + A_{3,2}\right),$$

where

$$A_1 = \mathrm{E}\left[\left(\int_0^T dr \sum_{j=1}^d \|g_{t,x;s,y}(r,\cdot)\sigma_{ij}(u(r,\cdot))\|^2_{\mathscr{H}}\right)^{p/2}\right],$$

$$A_{2,1} = \mathrm{E}\left[\left\|\int_0^T\int_{\mathbb{R}^k} g_{t,x;t,y}(\theta,\eta)\sum_{j=1}^d D(\sigma_{i,j}(u(\theta,\eta)))M^j(d\theta,d\eta)\right\|^p_{\mathscr{H}_T^d}\right],$$

$$A_{2,2} = \mathrm{E}\left[\left\|\int_0^T\int_{\mathbb{R}^k} g_{t,y;s,y}(\theta,\eta)\sum_{j=1}^d D(\sigma_{i,j}(u(\theta,\eta)))M^j(d\theta,d\eta)\right\|^p_{\mathscr{H}_T^d}\right],$$

$$A_{3,1} = \mathrm{E}\left[\left\|\int_0^T d\theta\int_{\mathbb{R}^k} d\eta\, g_{t,x;t,y}(\theta,\eta)D(b_i(u(\theta,\eta)))\right\|^p_{\mathscr{H}_T^d}\right],$$

$$A_{3,2} = \mathrm{E}\left[\left\|\int_0^T d\theta\int_{\mathbb{R}^k} d\eta\, g_{t,y;s,y}(\theta,\eta)D(b_i(u(\theta,\eta)))\right\|^p_{\mathscr{H}_T^d}\right].$$

Using Burkholder's inequality, (2.2) and (2.6), we see that for any $\gamma \in (0, 2-\beta)$,

$$A_1 \leq c_p\mathrm{E}\left[\left|\int_0^T\int_{\mathbb{R}^k} g_{t,x;s,y}(\theta,\eta)\sum_{j=1}^d \sigma_{ij}(u(\theta,\eta))M^j(d\theta,d\eta)\right|^p\right]. \tag{5.1}$$

In order to bound the right-hand side of (5.1), one proceeds as in [23], where the so-called "factorization method" is used. In fact, the calculation used in [23] in order to obtain [23, (10)] and [23, (19)] (see in particular the treatment of the terms $I_2(t,h,x)$, $I_3(t,h,x)$, and $J_2(t,x,z)$ in this reference) show that for any $\gamma \in (0, 2-\beta)$,

$$A_1 \le c_p \left(|t-s|^{\frac{\gamma}{2}} + \|x-y\|^{\gamma}\right)^{\frac{p}{2}}.$$

We do not expand on this further since we will be using this method several times below, with details

In order to bound the terms $A_{2,1}$ and $A_{2,2}$, we will also apply the factorisation method used in [23]. That is, using the semigroup property of S, the Beta function and a stochastic Fubini's theorem (whose assumptions can be seen to be satisfied, see e.g. [25, Theorem 2.6]), we see that, for any $\alpha \in (0, \frac{2-\beta}{4})$,

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^k} S(t-\theta, x-\eta) \sum_{j=1}^d D(\sigma_{i,j}(u(\theta,\eta))) M^j(d\theta, d\eta) \\ = \frac{\sin(\pi\alpha)}{\pi} \int_0^t dr \int_{\mathbb{R}^k} dz\, S(t-r, x-z)(t-r)^{\alpha-1} Y_\alpha^i(r,z),\end{aligned} \tag{5.2}$$

where $Y = (Y_\alpha^i(r,z), r \in [0,T], z \in \mathbb{R}^k)$ is the $\mathscr{H}_T^d$-valued process defined by

$$Y_\alpha^i(r,z) = \int_0^r \int_{\mathbb{R}^k} S(r-\theta, z-\eta)(r-\theta)^{-\alpha} \sum_{j=1}^d D(\sigma_{i,j}(u(\theta,\eta))) M^j(d\theta, d\eta).$$

Let us now bound the $L^p(\Omega; \mathscr{H}_T^d)$-norm of the process Y. Using [18, (3.13)] and the boundedness of the derivatives of the coefficients of σ, we see that for any $p \ge 2$,

$$\mathrm{E}\left[\|Y_\alpha^i(r,z)\|_{\mathscr{H}_T^d}^p\right] \le c_p \sum_{i=1}^d \sup_{(t,x)\in[0,T]\times\mathbb{R}^k} \mathrm{E}\left[\|D(u_i(t,x))\|_{\mathscr{H}_T^d}^p\right] (\nu_{r,z})^{p/2},$$

where

$$\nu_{r,z} := \|S(r-*, z-\cdot)(r-*)^{-\alpha}\|_{\mathscr{H}_r^d}. \tag{5.3}$$

We have that

$$\nu_{r,z} = \int_0^r ds \int_{\mathbb{R}^k} d\xi\, \|\xi\|^{\beta-k} (r-s)^{-2\alpha} \exp\left(-2\pi^2 (r-s)\|\xi\|^2\right)$$

$$
\begin{aligned}
&= \int_0^r ds\, (r-s)^{-2\alpha-\frac{\beta}{2}} \int_{\mathbb{R}^k} d\tilde{\xi}\, \|\tilde{\xi}\|^{\beta-k} \exp\left(-2\pi^2 \|\tilde{\xi}\|^2\right) \\
&= r^{\frac{2-\beta}{2}-2\alpha}.
\end{aligned}
\tag{5.4}
$$

Hence, we conclude from (3.2) that

$$
\sup_{(r,z)\in[0,T]\times\mathbb{R}^k} \mathrm{E}\left[\|Y^i_\alpha(r,z)\|^p_{\mathscr{H}^d_T}\right] < +\infty. \tag{5.5}
$$

Now, in order to bound $A_{2,1}$, first note that by (5.2) we can write

$$
A_{2,1} \le \mathrm{E}\left[\left\| \int_0^t dr \int_{\mathbb{R}^k} dz\, (\psi_\alpha(t-r,x-z) - \psi_\alpha(t-r,y-z)) Y^i_\alpha(r,z) \right\|^p_{\mathscr{H}^d_T}\right],
$$

where $\psi_\alpha(t,x) = S(t,x)t^{\alpha-1}$. Then, appealing to Minkowski's inequality, (5.5) and Lemma 5.2(a) below, we find that, for any $\gamma \in (0, 4\alpha)$,

$$
\begin{aligned}
A_{2,1} &\le c_p \left(\int_0^t dr \int_{\mathbb{R}^k} dz\, |\psi_\alpha(t-r,x-z) - \psi_\alpha(t-r,y-z)| \right)^p \\
&\quad \times \sup_{(r,z)\in[0,T]\times\mathbb{R}^k} \mathrm{E}\left[\|Y^i_\alpha(r,z)\|^p_{\mathscr{H}^d_T}\right] \\
&\le c_p \|x-y\|^{\frac{\gamma}{2}p}.
\end{aligned}
$$

We next treat $A_{2,2}$. Using (5.2), we have that $A_{2,2} \le c_{p,\alpha}(A_{2,2,1} + A_{2,2,2})$, where

$$
\begin{aligned}
A_{2,2,1} &= \mathrm{E}\left[\left\| \int_0^s dr \int_{\mathbb{R}^k} dz\, \left(\psi_\alpha(t-r,x-z) - \psi_\alpha(s-r,x-z)\right) Y^i_\alpha(r,z) \right\|^p_{\mathscr{H}^d_T}\right], \\
A_{2,2,2} &= \mathrm{E}\left[\left\| \int_s^t dr \int_{\mathbb{R}^k} dz\, \psi_\alpha(t-r,x-z) Y^i_\alpha(r,z) \right\|^p_{\mathscr{H}^d_T}\right].
\end{aligned}
$$

Now, by Minkowski's inequality, (5.5) and Lemma 5.2(b) below, we find that, for any $\gamma \in (0, 4\alpha)$,

$$\begin{aligned}
A_{2,2,1} &\leq c_p \left(\int_0^s dr \int_{\mathbb{R}^k} dz \, |\psi_\alpha(t-r, x-z) - \psi_\alpha(s-r, x-z)| \right)^p \\
&\quad \times \sup_{(r,z)\in[0,T]\times\mathbb{R}^k} \mathrm{E}\left[\|Y_\alpha^i(r,z)\|^p_{\mathscr{H}_T^d} \right] \\
&\leq c_p |t-s|^{\frac{\gamma}{4}p}.
\end{aligned}$$

In the same way, using Minkowski's inequality, (5.5) and Lemma 5.2(c) below, for any $\gamma \in (0, 4\alpha)$, we have that

$$\begin{aligned}
A_{2,2,2} &\leq c_p \left(\int_s^t dr \int_{\mathbb{R}^k} dz \, \psi_\alpha(t-r, x-z) \right)^p \sup_{(r,z)\in[0,T]\times\mathbb{R}^k} \mathrm{E}\left[\|Y_\alpha^i(r,z)\|^p_{\mathscr{H}_T^d} \right] \\
&\leq c_p |t-s|^{\frac{\gamma}{4}p}.
\end{aligned}$$

Finally, we bound $A_{3,1}$ and $A_{3,2}$, which can be written

$$\begin{aligned}
A_{3,1} &= \mathrm{E}\left[\left\| \int_0^t d\theta \int_{\mathbb{R}^k} d\eta \, \left(S(t-\theta, x-\eta) - S(t-\theta, y-\eta)\right) D\left(b_i(u(\theta,\eta))\right) \right\|^p_{\mathscr{H}_T^d} \right], \\
A_{3,2} &= \mathrm{E}\left[\left\| \int_0^t d\theta \int_{\mathbb{R}^k} d\eta \, S(t-\theta, y-\eta) D\left(b_i(u(\theta,\eta))\right) \right.\right. \\
&\quad \left.\left. - \int_0^s d\theta \int_{\mathbb{R}^k} d\eta \, S(s-\theta, y-\eta) D\left(b_i(u(\theta,\eta))\right) \right\|^p_{\mathscr{H}_T^d} \right].
\end{aligned}$$

The factorisation method used above is also needed in this case, that is, using the semigroup property of S, the Beta function and Fubini's theorem, we see that for any $\alpha \in (0, 1)$,

$$\begin{aligned}
&\int_0^t d\theta \int_{\mathbb{R}^k} d\eta \, S(t-\theta, x-\eta) D(b_i(u(\theta,\eta))) \\
&\qquad = \frac{\sin(\pi\alpha)}{\pi} \int_0^t dr \int_{\mathbb{R}^k} dz S(t-r, x-z)(t-r)^{\alpha-1} Z_\alpha^i(r,z),
\end{aligned}$$

where $Z = (Z^i_\alpha(r,z), r \in [0,T], z \in \mathbb{R}^k)$ is the $\mathscr{H}_T^d$-valued process defined as

$$Z^i_\alpha(r,z) = \int_0^r d\theta \int_{\mathbb{R}^k} d\eta\, S(r-\theta, z-\eta)(r-\theta)^{-\alpha} D(b_i(u(\theta,\eta))).$$

Hence, we can write

$$A_{3,1} \le \mathrm{E}\Bigg[\Bigg\| \int_0^t dr \int_{\mathbb{R}^k} dz\, (\psi_\alpha(t-r, x-z) - \psi_\alpha(t-r, y-z)) Z^i_\alpha(r,z) \Bigg\|^p_{\mathscr{H}_T^d}\Bigg],$$

and $A_{3,2} \le c_{p,\alpha}(A_{3,2,1} + A_{3,2,2})$, where

$$A_{3,2,1} = \mathrm{E}\Bigg[\Bigg\| \int_0^s dr \int_{\mathbb{R}^k} dz\, (\psi_\alpha(t-r, y-z) - \psi_\alpha(s-r, y-z)) Z^i_\alpha(r,z) \Bigg\|^p_{\mathscr{H}_T^d}\Bigg],$$

$$A_{3,2,2} = \mathrm{E}\Bigg[\Bigg\| \int_s^t dr \int_{\mathbb{R}^k} dz\, \psi_\alpha(t-r, y-z) Z^i_\alpha(r,z) \Bigg\|^p_{\mathscr{H}_T^d}\Bigg].$$

We next compute the $L^p(\Omega; \mathscr{H}_T^d)$-norm for the process Z. Using Minkowski's inequality and the boundedness of the derivatives of the coefficients of b, we get that

$$E[\|Z^i_\alpha(r,z)\|^p_{\mathscr{H}_T^d}] \le c_p \sum_{i=1}^d \sup_{(t,x)\in[0,T]\times\mathbb{R}^k} \mathrm{E}\Big[\|D(u_i(t,x))\|^p_{\mathscr{H}_T^d}\Big] (\gamma_{r,z})^{p/2},$$

where

$$\gamma_{r,z} = \int_0^r d\theta \int_{\mathbb{R}^k} d\eta\, S(r-\theta, z-\eta)(r-\theta)^{-\alpha} = r^{1-\alpha}.$$

Hence, using (3.2), we conclude that

$$\sup_{(r,z)\in[0,T]\times\mathbb{R}^k} \mathrm{E}\Big[\|Z^i_\alpha(r,z)\|^p_{\mathscr{H}_T^d}\Big] < +\infty. \tag{5.6}$$

Then, proceeding as above, using Minkowski's inequality, (5.6) and Lemma 5.2, we conclude that for any $\gamma \in (0, 4\alpha)$,

$$A_{3,1} + A_{3,2} \le c_p \left(\|x-y\|^{\frac{\gamma}{2}p} + \|t-s\|^{\frac{\gamma}{4}p}\right).$$

This concludes the proof of the proposition for $m = 1$.

The case $m > 1$ follows along the same lines by induction using (3.2) and the stochastic partial differential equations satisfied by the iterated derivatives (cf. [18, Proposition 6.1]). □

The following lemma was used in the proof of Proposition 5.1.

Lemma 5.2 *For $\alpha > 0$, set $\psi_\alpha(t,x) = S(t,x)t^{\alpha-1}$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^k$.*

(a) *For $\alpha \in (0, \frac{2-\beta}{4})$, $\gamma \in (0, 4\alpha)$, there is $c > 0$ such that for all $t \in [0,T]$, $x, y \in \mathbb{R}^k$, and $\epsilon \in [0,t]$,*

$$\int_{t-\epsilon}^{t} dr \int_{\mathbb{R}^k} dz \, |\psi_\alpha(t-r, x-z) - \psi_\alpha(t-r, y-z)| \leq c\epsilon^{\alpha-\frac{\gamma}{2}} \|x-y\|^{\gamma/2}.$$

(b) *For $\alpha \in (0, \frac{2-\beta}{4})$, $\gamma \in (0, 4\alpha)$, there is $c > 0$ such that for all $s \leq t \in [0,T]$, $x, y \in \mathbb{R}^k$, and $\epsilon \in [0,s]$,*

$$\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}^k} dz \, |\psi_\alpha(t-r, x-z) - \psi_\alpha(s-r, x-z)| \leq c\epsilon^{\alpha-\frac{\gamma}{4}} |t-s|^{\gamma/4}.$$

(c) *For $\alpha \in (0, \frac{2-\beta}{4})$, $\gamma \in (0, 4\alpha)$, there is $c > 0$ such that for all $s \leq t \in [0,T]$, $x, y \in \mathbb{R}^k$,*

$$\int_{s}^{t} dr \int_{\mathbb{R}^k} dz \, \psi_\alpha(t-r, x-z) \leq c\,|t-s|^{\gamma/4}.$$

Proof (a) This is similar to the proof of [23, (21)].
(b) This is similar to the proof of [23, (14)].
(c) This is a consequence of [23, (15)]. □

5.2 Study of the malliavin matrix

Let $T > 0$ be fixed. For $s, t \in [0,T]$, $s \leq t$, and $x, y \in \mathbb{R}^k$ consider the $2d$-dimensional random vector

$$Z := (u(s,y), u(t,x) - u(s,y)) . \tag{5.7}$$

Let γ_Z be the Malliavin matrix of Z. Note that $\gamma_Z = ((\gamma_Z)_{m,l})_{m,l=1,\ldots,2d}$ is a symmetric $2d \times 2d$ random matrix with four $d \times d$ blocks of the form

$$\gamma_Z = \begin{pmatrix} \gamma_Z^{(1)} & \vdots & \gamma_Z^{(2)} \\ \cdots & \vdots & \cdots \\ \gamma_Z^{(3)} & \vdots & \gamma_Z^{(4)} \end{pmatrix},$$

where

$$\gamma_Z^{(1)} = \left(\left\langle D(u_i(s,y)), D(u_j(s,y))\right\rangle_{\mathscr{H}_T^d}\right)_{i,j=1,\ldots,d},$$
$$\gamma_Z^{(2)} = \left(\left\langle D(u_i(s,y)), D(u_j(t,x)-u_j(s,y))\right\rangle_{\mathscr{H}_T^d}\right)_{i,j=1,\ldots,d},$$
$$\gamma_Z^{(3)} = \left(\left\langle D(u_i(t,x)-u_i(s,y)), D(u_j(s,y))\right\rangle_{\mathscr{H}_T^d}\right)_{i,j=1,\ldots,d},$$
$$\gamma_Z^{(4)} = (\langle D(u_i(t,x)-u_i(s,y)), D(u_j(t,x)-u_j(s,y))\rangle_{\mathscr{H}_T^d})_{i,j=1,\ldots,d}.$$

We let **(1)** denote the set of couples $\{1,\ldots,d\}\times\{1,\ldots,d\}$, **(2)** the set $\{1,\ldots,d\}\times\{d+1,\ldots,2d\}$, **(3)** the set $\{d+1,\ldots,2d\}\times\{1,\ldots,d\}$ and **(4)** the set $\{d+1,\ldots,2d\}\times\{d+1,\ldots,2d\}$.

The following two results follow exactly along the same lines as [6, Propositions 6.5 and 6.7] using (3.2) and Proposition 5.1, so their proofs are omitted.

Proposition 5.3 *Fix $T>0$ and let $I\times J\subset(0,T]\times\mathbb{R}^k$ be a closed non-trivial rectangle. Let A_Z denote the cofactor matrix of γ_Z. Assuming* **P1**, *for any $p>1$ and $\gamma\in(0,2-\beta)$, there is a constant $c_{\gamma,p,T}$ such that for any $(s,y),(t,x)\in I\times J$ with $(s,y)\neq(t,x)$,*

$$\mathrm{E}\left[|(A_Z)_{m,l}|^p\right]^{1/p} \le \begin{cases} c_{\gamma,p,T}\left(|t-s|^{\gamma/2}+\|x-y\|^{\gamma}\right)^{d} & if\ (m,l)\in\textbf{(1)},\\ c_{\gamma,p,T}\left(|t-s|^{\gamma/2}+\|x-y\|^{\gamma}\right)^{d-\frac{1}{2}} & if\ (m,l)\in\textbf{(2)}\ or\ \textbf{(3)},\\ c_{\gamma,p,T}\left(|t-s|^{\gamma/2}+\|x-y\|^{\gamma}\right)^{d-1} & if\ (m,l)\in\textbf{(4)}.\end{cases}$$

Proposition 5.4 *Fix $T>0$ and let $I\times J\subset(0,T]\times\mathbb{R}^k$ be a closed non-trivial rectangle. Assuming* **P1**, *for any $p>1$, $k\ge 1$, and $\gamma\in(0,2-\beta)$, there is a constant $c_{\gamma,k,p,T}$ such that for any $(s,y),(t,x)\in I\times J$ with $(s,y)\neq(t,x)$,*

$$\mathrm{E}\left[\|D^k(\gamma_Z)_{m,l}\|^p_{(\mathscr{H}_T^d)^{\otimes k}}\right]^{1/p} \le \begin{cases} c_{\gamma,k,p,T} & if\ (m,l)\in\textbf{(1)},\\ c_{\gamma,k,p,T}\left(|t-s|^{\gamma/2}+\|x-y\|^{\gamma}\right)^{1/2} & if\ (m,l)\in\textbf{(2)}\ or\ \textbf{(3)},\\ c_{\gamma,k,p,T}\left(|t-s|^{\gamma/2}+\|x-y\|^{\gamma}\right) & if\ (m,l)\in\textbf{(4)}.\end{cases}$$

The main technical effort in this section is the proof of the following proposition.

Proposition 5.5 *Fix $\eta,T>0$. Assume* **P1** *and* **P2**. *Let $I\times J\subset(0,T]\times\mathbb{R}^k$ be a closed non-trivial rectangle. There exists C depending on T and η such that for any $(s,y),(t,x)\in I\times J$, $(s,y)\neq(t,x)$, and $p>1$,*

$$\mathrm{E}\left[\left(\det\gamma_Z\right)^{-p}\right]^{1/p} \le C\left(|t-s|^{\frac{2-\beta}{2}}+\|x-y\|^{2-\beta}\right)^{-d(1+\eta)}. \tag{5.8}$$

Proof The proof has the same general structure as that of [6, Proposition 6.6]. We write

$$\det \gamma_Z = \prod_{i=1}^{2d} \left(\xi^i\right)^{\mathsf{T}} \gamma_Z \xi^i, \tag{5.9}$$

where $\xi = \{\xi^1, \ldots, \xi^{2d}\}$ is an orthonormal basis of $\mathbb{R}^{2d}$ consisting of eigenvectors of γ_Z.

We now carry out the perturbation argument of [6, Proposition 6.6]. Let $\mathbf{0} \in \mathbb{R}^d$ and consider the spaces $E_1 = \{(\lambda, \mathbf{0}) : \lambda \in \mathbb{R}^d\}$ and $E_2 = \{(\mathbf{0}, \mu) : \mu \in \mathbb{R}^d\}$. Each ξ^i can be written

$$\xi^i = \left(\lambda^i, \mu^i\right) = \alpha_i \left(\tilde{\lambda}^i, \mathbf{0}\right) + \sqrt{1-\alpha_i^2}\left(\mathbf{0}, \tilde{\mu}^i\right), \tag{5.10}$$

where $\lambda^i, \mu^i \in \mathbb{R}^d$, $(\tilde{\lambda}^i, \mathbf{0}) \in E_1$, $(\mathbf{0}, \tilde{\mu}^i) \in E_2$, with $\|\tilde{\lambda}^i\| = \|\tilde{\mu}^i\| = 1$ and $0 \le \alpha_i \le 1$. In particular, $\|\xi^i\|^2 = \|\lambda^i\|^2 + \|\mu^i\|^2 = 1$.

The result of [6, Lemma 6.8] gives us at least d eigenvectors $\xi^1, \ldots, \xi^d$ that have a "large projection on E_1", and we will show that these will contribute a factor of order 1 to the product in (5.9). Recall that for a fixed small $\alpha_0 > 0$, ξ^i has a "large projection on E_1" if $\alpha_i \ge \alpha_0$. The at most d other eigenvectors with a "small projection on E_1" will each contribute a factor of order $(|t-s|^{\frac{2-\beta}{2}} + \|x-y\|^{2-\beta})^{-1-\eta}$, as we will make precise below.

Hence, by [6, Lemma 6.8] and Cauchy–Schwarz inequality, one can write

$$\begin{aligned} \mathrm{E}\left[\left(\det \gamma_Z\right)^{-p}\right]^{1/p} \le \sum_{K \subset \{1,\ldots,2d\},\, |K|=d} & \left(\mathrm{E}\left[\mathbf{1}_{A_K}\left(\prod_{i\in K}(\xi^i)^{\mathsf{T}}\gamma_Z\xi^i\right)^{-2p}\right]\right)^{1/(2p)} \\ & \times\left(\mathrm{E}\left[\left(\inf_{\substack{\xi=(\lambda,\mu)\in\mathbb{R}^{2d}:\\ \|\lambda\|^2+\|\mu\|^2=1}} \xi^{\mathsf{T}}\gamma_Z\xi\right)^{-2dp}\right]\right)^{1/(2p)}, \end{aligned} \tag{5.11}$$

where $A_K = \cap_{i\in K}\{\alpha_i \ge \alpha_0\}$.

With this, Propositions 5.6 and 5.7 below will conclude the proof of Proposition 5.5. □

Proposition 5.6 *Fix $\eta, T > 0$. Assume* **P1** *and* **P2**. *There exists C depending on η and T such that for all $s, t \in I$, $0 \le t-s < 1$, $x, y \in J$, $(s, y) \ne (t, x)$, and $p > 1$,*

$$\mathrm{E}\left[\left(\inf_{\substack{\xi=(\lambda,\mu)\in\mathbb{R}^{2d}:\\ \|\lambda\|^2+\|\mu\|^2=1}} \xi^{\mathsf{T}}\gamma_Z\xi\right)^{-2dp}\right] \le C\left(|t-s|^{\frac{2-\beta}{2}} + \|x-y\|^{2-\beta}\right)^{-2dp(1+\eta)}. \tag{5.12}$$

Proposition 5.7 *Assume* **P1** *and* **P2**. *Fix* $T > 0$ *and* $p > 1$. *Then there exists* $C = C(p, T)$ *such that for all* $s, t \in I$ *with* $0 \le t - s < \frac{1}{2}$, $x, y \in J$, $(s, y) \neq (t, x)$,

$$\mathrm{E}\left[\mathbf{1}_{A_K}\left(\prod_{i \in K}(\xi^i)^{\mathsf{T}}\gamma_Z \xi^i\right)^{-p}\right] \le C, \tag{5.13}$$

where A_K *is defined just below (5.11).*

Proof of Proposition 5.6 Fix $\gamma \in (0, 2-\beta)$. It suffices to prove this for η sufficiently small, in particular, we take $\eta < \gamma/2$. The proof of this lemma follows lines similar to those of [6, Proposition 6.9], with significantly different estimates needed to handle the spatially homogeneous noise.

For $\epsilon \in (0, t-s)$,

$$\xi^{\mathsf{T}}\gamma_Z \xi \ge J_1 + J_2,$$

where

$$J_1 \coloneqq \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \left\| \sum_{i=1}^{d} (\lambda_i - \mu_i)\left(S(s-r, y-\cdot)\sigma_{i,l}(u(r,\cdot)) + a_i(l, r, s, y)\right) + W \right\|_{\mathscr{H}}^2,$$

$$J_2 := \int_{t-\epsilon}^{t} dr \sum_{l=1}^{d} \|W\|_{\mathscr{H}}^2, \tag{5.14}$$

where

$$W := \sum_{i=1}^{d} \mu_i S(t-r, x-\cdot)\sigma_{i,l}(u(r,\cdot)) + \mu_i a_i(l, r, t, x), \tag{5.15}$$

and $a_i(l, r, t, x)$ is defined in (4.1).

We now consider two different cases.

Case 1. Assume $t - s > 0$ and $\|x - y\|^2 \le t - s$. Fix $\epsilon \in \left(0, (t-s) \wedge (\frac{1}{4})^{2/\eta}\right)$. We write

$$\inf_{\|\xi\|=1} \xi^{\mathsf{T}}\gamma_Z \xi \ge \min\left(\inf_{\|\xi\|=1\,,\|\mu\|\ge\epsilon^{\eta/2}} J_2\,,\ \inf_{\|\xi\|=1\,,\|\mu\|\le\epsilon^{\eta/2}} J_1\right). \tag{5.16}$$

We will now bound the two terms in the above minimum. We start by bounding the term containing J_2. Using (4.2) and adding and subtracting a "local" term as in (4.3), we find that $J_2 \ge \frac{2}{3} J_2^{(1)} - 4(J_2^{(2)} + J_2^{(3)})$, where

$$J_2^{(1)} = \sum_{l=1}^{d} \int_{t-\epsilon}^{t} dr \int_{\mathbb{R}^k} dv \int_{\mathbb{R}^k} dz$$
$$\times \|v - z\|^{-\beta} S(t-r, x-v) S(t-r, x-z)(\mu^{\mathsf{T}} \cdot \sigma(u(r,x)))_l^2,$$

$$J_2^{(2)} = \sum_{l=1}^{d} \int_{t-\epsilon}^{t} dr \int_{\mathbb{R}^k} dv \int_{\mathbb{R}^k} dz \, \|v - z\|^{-\beta} S(t-r, x-v) S(t-r, x-z)$$
$$\times \Big(\mu^{\mathsf{T}} \cdot [\sigma(u(r,v)) - \sigma(u(r,x))]\Big)_l \Big(\mu^{\mathsf{T}} \cdot [\sigma(u(r,z)) - \cdot\sigma(u(r,x))]\Big)_l,$$
$$J_2^{(3)} = \int_{t-\epsilon}^{t} dr \sum_{l=1}^{d} \Big\| \sum_{i=1}^{d} a_i(l,r,t,x)\, \mu_i \Big\|_{\mathscr{H}}^2,$$

Now, hypothesis **P2** (see also Remark 1.1) and Lemma 6.1 together imply that $J_2^{(1)} \geq c\,\|\mu\|^2 \epsilon^{\frac{2-\beta}{2}}$. Therefore,

$$\inf_{\|\xi\|=1, \|\mu\| \geq \epsilon^{\eta/2}} J_2 \geq c\epsilon^{\frac{2-\beta}{2}+\eta} - \sup_{\|\xi\|=1, \|\mu\| \geq \epsilon^{\eta/2}} 2\big(|J_2^{(2)}| + J_2^{(3)}\big). \tag{5.17}$$

Moreover, (4.4) and Lemma 6.2 imply that for any $q \geq 1$,

$$\mathrm{E}\left[\sup_{\|\xi\|=1, \|\mu\| \geq \epsilon^{\eta/2}} (|J_2^{(2)}| + J_2^{(3)})^q\right] \leq c\epsilon^{\frac{2-\beta}{2}q + \frac{\gamma}{2}q}. \tag{5.18}$$

This bounds the first term in (5.16) and gives an analogue of the first inequality in [6, (6.12)].

In order to bound the second infimum in (5.16), we use again (4.2) and we add and subtract a "local" term as in (4.3) to see that

$$J_1 \geq \frac{2}{3} J_1^{(1)} - 8\big(J_1^{(2)} + J_1^{(3)} + J_1^{(4)} + J_1^{(5)}\big),$$

where

$$J_1^{(1)} = \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \, ((\lambda - \mu)^{\mathsf{T}} \cdot \sigma(u(r,y)))_l^2 \int_{\mathbb{R}^k} d\xi \;\; \|\xi\|^{\beta-k} |\mathscr{F} S(s-r, y-\cdot)(\xi)|^2,$$
$$J_1^{(2)} = \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}^k} dv \int_{\mathbb{R}^k} dz \, \|v-z\|^{-\beta} S(s-r, y-v) S(s-r, y-z)$$
$$\times \Big((\lambda - \mu)^{\mathsf{T}} \cdot \big[\sigma(u(r,v)) - \sigma(u(r,y))\big]\Big)_l$$
$$\times \Big((\lambda - \mu)^{\mathsf{T}} \cdot [\sigma(u(r,z)) - \sigma(u(r,y))]\Big)_l,$$
$$J_1^{(3)} := \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \Big\| \sum_{i=1}^{d} \mu_i S(t-r, x-\cdot) \sigma_{i,l}(u(r,\cdot)) \Big\|_{\mathscr{H}}^2,$$

$$J_1^{(4)} := \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \left\| \sum_{i=1}^{d} (\lambda_i - \mu_i) a_i(l, r, s, y) \right\|_{\mathscr{H}}^2,$$

$$J_1^{(5)} := \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \left\| \sum_{i=1}^{d} \mu_i a_i(l, r, t, x) \right\|_{\mathscr{H}}^2.$$

Hypothesis **P2** (see also Remark 1.1) and Lemma 6.1 together imply that $J_1^{(1)} \geq c \left\|\lambda - \mu\right\|^2 \epsilon^{\frac{2-\beta}{2}}$. Therefore,

$$\inf_{\|\xi\|=1, \|\mu\| \leq \epsilon^{\eta/2}} J_1 \geq \tilde{c} \epsilon^{\frac{2-\beta}{2}} - \sup_{\|\xi\|=1, \|\mu\| \leq \epsilon^{\eta/2}} 8 \left(|J_1^{(2)}| + J_1^{(3)} + J_1^{(4)} + J_1^{(5)} \right). \tag{5.19}$$

Now, (4.4) implies that for any $q \geq 1$,

$$\mathrm{E}\left[\sup_{\|\xi\|=1, \|\mu\| \leq \epsilon^{\eta/2}} |J_1^{(2)}|^q \right] \leq c \epsilon^{\frac{2-\beta}{2} q + \frac{\gamma}{2} q}.$$

Moreover, hypothesis **P1** (in particular, the fact that σ is bounded), the Cauchy-Schwarz inequality and Lemma 6.1 imply that for any $q \geq 1$,

$$\mathrm{E}\left[\sup_{\|\xi\|=1, \|\mu\| \leq \epsilon^{\eta/2}} |J_1^{(3)}|^q \right] \leq c \epsilon^{\frac{2-\beta}{2} q + \eta q}.$$

Applying Lemma 6.2 with $t = s$, we get that for any $q \geq 1$,

$$\mathrm{E}\left[\sup_{\|\xi\|=1, \|\mu\| \leq \epsilon^{\eta/2}} |J_1^{(4)}|^q \right] \leq c \epsilon^{\frac{2-\beta}{2} q + \frac{2-\beta}{2} q}.$$

Again Lemma 6.2 gives, for any $q \geq 1$,

$$\mathrm{E}\left[\sup_{\|\xi\|=1, \|\mu\| \leq \epsilon^{\eta/2}} |J_1^{(5)}|^q \right] \leq c \epsilon^{\frac{2-\beta}{2} q + \eta q}.$$

Since we have assumed that $\eta < \frac{\gamma}{4}$, the above bounds in conjunction prove that for any $q \geq 1$,

$$\mathrm{E}\left[\sup_{\|\xi\|=1, \|\mu\| \leq \epsilon^{\eta/2}} \left(|J_1^{(2)}| + J_1^{(3)} + J_1^{(4)} + J_1^{(5)} \right)^q \right] \leq c \epsilon^{\frac{2-\beta}{2} q + \eta q}. \tag{5.20}$$

We finally use (5.16)–(5.20) together with [6, Proposition 3.5] with $\alpha_1 = \frac{2-\beta}{2} + \eta$, $\beta_1 = \frac{2-\beta}{2} + \frac{\gamma}{4}$, $\alpha_2 = \frac{2-\beta}{2}$ and $\beta_2 = \frac{2-\beta}{2} + \eta$ to conclude that

$$\begin{aligned} \mathrm{E}\left[\left(\inf_{\|\xi\|=1} \xi^{\mathsf{T}} \gamma_Z \xi\right)^{-2pd}\right] &\leq c\left[(t-s) \wedge \left(\frac{1}{4}\right)^{2/\eta}\right]^{-2pd(\frac{2-\beta}{2}+\eta)} \\ &\leq c'(t-s)^{-2pd\left(\frac{2-\beta}{2}+\eta\right)} \\ &\leq \tilde{c}\left[(t-s)^{\frac{2-\beta}{2}} + \|x-y\|^{2-\beta}\right]^{-2pd(1+\eta')}, \end{aligned}$$

(for the second inequality, we have used the fact that $t-s < 1$, and for the third, that $\|x-y\|^2 \leq t-s$), whence follows the proposition in the case that $\|x-y\|^2 \leq t-s$.

Case 2. Assume that $\|x-y\| > 0$ and $\|x-y\|^2 \geq t-s \geq 0$. Then

$$\xi^{\mathsf{T}} \gamma_Z \xi \geq J_1 + \tilde{J}_2,$$

where J_1 is defined in (5.14),

$$\tilde{J}_2 := \int_{(t-\epsilon)\vee s}^{t} dr \sum_{l=1}^{d} \|W\|_{\mathscr{H}}^2,$$

and W is defined in (5.15). Let $\epsilon > 0$ be such that $(1+\alpha)\epsilon^{1/2} < \frac{1}{2}\|x-y\|$, where $\alpha > 0$ is large but fixed; its specific value will be decided on later. From here on, Case 2 is divided into two further sub-cases.

Sub-Case A. Suppose that $\epsilon \geq t-s$. Apply inequality (4.2) and add and subtract a "local" term as in (4.3), to find that

$$\begin{aligned} J_1 &\geq \frac{2}{3} A_1 - 8(A_2 + A_3 + A_4 + A_5), \\ \tilde{J}_2 &\geq \frac{2}{3} B_1 - 4(B_2 + B_3), \end{aligned}$$

where

$$\begin{aligned} A_1 &:= \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \, \Big\| S(s-r, y-\cdot) \left((\lambda-\mu)^{\mathsf{T}} \cdot \sigma(u(r,y))\right)_l \\ &\quad + S(t-r, x-\cdot)\left(\mu^{\mathsf{T}} \cdot \sigma(u(r,x))\right)_l \Big\|_{\mathscr{H}}^2, \\ A_2 &:= \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \, \left\| S(s-r, y-\cdot)\left((\lambda-\mu)^{\mathsf{T}} \cdot \left[\sigma(u(r,\cdot)) - \sigma(u(r,y))\right]\right)_l \right\|_{\mathscr{H}}^2 \\ A_3 &:= \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \, \left\| S(t-r, x-\cdot)\left(\mu^{\mathsf{T}} \cdot \left[\sigma(u(r,\cdot)) - \sigma(u(r,x))\right]\right)_l \right\|_{\mathscr{H}}^2 \end{aligned}$$

$$A_4 := \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \left\| \sum_{i=1}^{d} (\lambda_i - \mu_i) a_i(l, r, s, y) \right\|_{\mathscr{H}}^2,$$

$$A_5 := \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \left\| \sum_{i=1}^{d} \mu_i a_i(l, r, t, x) \right\|_{\mathscr{H}}^2,$$

$$B_1 := \sum_{l=1}^{d} \int_{s}^{t} dr \left\| S(t-r, x-\cdot)(\mu^{\mathsf{T}} \cdot \sigma(u(r,x)))_l \right\|_{\mathscr{H}}^2,$$

$$B_2 := \sum_{l=1}^{d} \int_{s}^{t} dr \left\| S(t-r, x-\cdot) \left(\mu^{\mathsf{T}} \cdot \left[\sigma(u(r,\cdot)) - \sigma(u(r,x)) \right] \right)_l \right\|_{\mathscr{H}}^2,$$

$$B_3 := \sum_{l=1}^{d} \int_{s}^{t} dr \left\| \sum_{i=1}^{d} \mu_i a_i(l, r, t, x) \right\|_{\mathscr{H}}^2.$$

Using the inequality $(a+b)^2 \geq a^2 + b^2 - 2|ab|$, we see that $A_1 \geq \tilde{A}_1 + \tilde{A}_2 - 2\tilde{B}_4$, where

$$\tilde{A}_1 = \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \left\| S(s-r, y-\cdot)((\lambda-\mu)^{\mathsf{T}} \cdot \sigma(u(r,y)))_l \right\|_{\mathscr{H}}^2,$$

$$\tilde{A}_2 = \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \left\| S(t-r, x-\cdot)(\mu^{\mathsf{T}} \cdot \sigma(u(r,x)))_l \right\|_{\mathscr{H}}^2,$$

$$\tilde{B}_4 = \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \left\langle S(s-r, y-\cdot)((\lambda-\mu)^{\mathsf{T}} \cdot \sigma(u(r,y)))_l, \right.$$
$$\left. S(t-r, x-\cdot)(\mu^{\mathsf{T}} \cdot \sigma(u(r,x)))_l \right\rangle_{\mathscr{H}}.$$

By hypothesis **P2** (see also Remark 1.1) and Lemma 6.1, we see that

$$\begin{aligned}
\tilde{A}_2 + B_1 &= \sum_{l=1}^{d} \int_{s-\epsilon}^{t} dr \left\| S(t-r, x-\cdot)(\mu^{\mathsf{T}} \cdot \sigma(u(r,x)))_l \right\|_{\mathscr{H}}^2 \\
&\geq \sum_{l=1}^{d} \int_{t-\epsilon}^{t} dr \left\| S(t-r, x-\cdot)(\mu^{\mathsf{T}} \cdot \sigma(u(r,x)))_l \right\|_{\mathscr{H}}^2 \\
&\geq \|\mu\|^2 \epsilon^{\frac{2-\beta}{2}}.
\end{aligned}$$

Similarly, $\tilde{A_1} \geq \|\lambda - \mu\|^2 \epsilon^{\frac{2-\beta}{2}}$, and so

$$\tilde{A_1} + \tilde{A_2} + B_1 \geq \left(\|\lambda - \mu\|^2 + \|\mu\|^2\right)\epsilon^{\frac{2-\beta}{2}} \geq c\epsilon^{\frac{2-\beta}{2}}. \tag{5.21}$$

Turning to the terms that are to be bounded above, we see as in (4.4) that

$$\mathrm{E}\big[|A_2|^q\big] \leq c\epsilon^{\frac{2-\beta}{2}q+\frac{\gamma}{2}q}, \quad \text{and} \quad \mathrm{E}\big[|B_2|^q\big] \leq c\epsilon^{\frac{2-\beta}{2}q+\frac{\gamma}{2}q}.$$

Using Lemma 6.2 and the fact that $t - s \leq \epsilon$, we see that

$$\mathrm{E}\big[|B_3|^q\big] \leq c\epsilon^{(2-\beta)q}, \quad \mathrm{E}\big[|A_4|^q\big] \leq c\epsilon^{(2-\beta)q}, \quad \text{and} \quad \mathrm{E}\big[|A_5|^q\big] \leq c\epsilon^{(2-\beta)q}.$$

In order to bound the q-th moment of A_3, we proceed as we did for the random variable A_2 in (4.3). It suffices to bound the q-th moment of

$$\begin{aligned}&\sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}^k} dv \int_{\mathbb{R}^k} dz \|v - z\|^{-\beta} S(t-r, x-v) S(t-r, x-z)\\ &\quad\times \left(\mu^{\mathsf{T}} \cdot \left[\sigma(u(r,v)) - \sigma(u(r,x))\right]\right)_l \left(\mu^{\mathsf{T}} \cdot \left[\sigma(u(r,z)) - \sigma(u(r,x))\right]\right)_l.\end{aligned}$$

Using Hölder's inequality, the Lipschitz property of σ and (2.6), this q-th moment is bounded by

$$\begin{aligned}&\left(\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}^k} dv \int_{\mathbb{R}^k} dz \|v - z\|^{-\beta} S(t-r, x-v) S(t-r, x-z)\right)^{q-1}\\ &\quad\times \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}^k} dv \int_{\mathbb{R}^k} dz \|v-z\|^{-\beta} S(t-r, x-v) S(t-r, x-z) \|v-x\|^{\frac{\gamma q}{2}} \|z-x\|^{\frac{\gamma q}{2}}\\ &=: a_1 \times a_2.\end{aligned}$$

By Lemma 6.1, $a_1 \leq \epsilon^{\frac{2-\beta}{2}(q-1)}$. For a_2, we use the change of variables $\tilde{v} = \frac{x-v}{\sqrt{t-r}}$, $\tilde{z} = \frac{x-z}{\sqrt{t-r}}$, to see that

$$\begin{aligned}a_2 &= \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}^k} d\tilde{v} \int_{\mathbb{R}^k} d\tilde{z}\, \|\tilde{v} - \tilde{z}\|^{-\beta} (t-r)^{-\beta/2} S(1, \tilde{v}) S(1, \tilde{z})\, \|\tilde{v}\|^{\frac{\gamma q}{2}} \|\tilde{z}\|^{\frac{\gamma q}{2}} (t-r)^{\frac{\gamma q}{2}}\\ &= \int_{s-\epsilon}^{s} dr\, (t-r)^{\frac{\gamma q}{2}-\frac{\beta}{2}} \int_{\mathbb{R}^k} d\tilde{v} \int_{\mathbb{R}^k} d\tilde{z}\, S(1, \tilde{v}) S(1, \tilde{z}) \|\tilde{v} - \tilde{z}\|^{-\beta} \|\tilde{v}\|^{\frac{\gamma q}{2}} \|\tilde{z}\|^{\frac{\gamma q}{2}}\end{aligned}$$

$$
\begin{aligned}
&= c\left((t-s+\epsilon)^{\frac{2-\beta}{2}+\frac{\gamma q}{2}} - (t-s)^{\frac{2-\beta}{2}+\frac{\gamma q}{2}}\right)\\
&\leq c\,\epsilon^{\frac{2-\beta}{2}+\frac{\gamma q}{2}},
\end{aligned}
$$

since $t-s<\epsilon$. Putting together these bounds for a_1 and a_2 yields $\mathrm{E}[|A_3|^q] \leq c\epsilon^{\frac{2-\beta}{2}+\frac{\gamma q}{2}}$.

We now study the term $\tilde{B}_4$, with the objective of showing that $\tilde{B}_4 \leq \Phi(\alpha)\epsilon^{\frac{2-\beta}{2}}$, with $\lim_{\alpha\to+\infty}\Phi(\alpha)=0$. We note that by hypothesis **P1**,

$$
\begin{aligned}
\tilde{B}_4 &\leq c\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}^k} dv \int_{\mathbb{R}^k} dz\, \|v-z\|^{-\beta} S(s-r,y-v)S(t-r,x-z)\\
&= c\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}^k} dv\, \|v\|^{-\beta} (S(s-r,y-\cdot) * S(t-r,\cdot-x))(v)\\
&= c\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}^k} dv\, \|v\|^{-\beta} S(t+s-2r,y-x+v),
\end{aligned}
$$

where we have used the semigroup property of $S(t,v)$. Using the change of variables $\bar{r}=s-r$, it follows that

$$
\begin{aligned}
\tilde{B}_4 &\leq c\int_{0}^{\epsilon} d\bar{r} \int_{\mathbb{R}^k} dv\, \|v\|^{-\beta}(t-s+2\bar{r})^{-k/2}\exp\left(-\frac{\|y-x+v\|^2}{2(t-s+2\bar{r})}\right)\\
&=: c(I_1+I_2),
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int_0^{\epsilon} dr \int_{\|v\|<\sqrt{r}(1+\alpha)} dv\, \|v\|^{-\beta}(t-s+2r)^{-k/2}\exp\left(-\frac{\|y-x+v\|^2}{2(t-s+2r)}\right),\\
I_2 &= \int_0^{\epsilon} dr \int_{\|v\|\geq\sqrt{r}(1+\alpha)} dv\, \|v\|^{-\beta}(t-s+2r)^{-k/2}\exp\left(-\frac{\|y-x+v\|^2}{2(t-s+2r)}\right).
\end{aligned}
$$

Concerning I_1, observe that when $\|v\|<\sqrt{r}(1+\alpha)$, then

$$
\|y-x+v\| \geq \|y-x\| - \|v\| \geq \|y-x\| - \sqrt{\epsilon}(1+\alpha) \geq \frac{1}{2}\|y-x\| \geq \alpha\sqrt{\epsilon},
$$

since we have assumed that $(1+\alpha)\sqrt{\epsilon} < \frac{1}{2}\|y-x\|$. Therefore,

$$I_1 \le \int_0^{\epsilon} dr\,(t-s+2r)^{-k/2}\exp\left(-\frac{\alpha^2\epsilon}{2(t-s+2r)}\right)\int_{\|v\|<\sqrt{r}(1+\alpha)} dv\,\|v\|^{-\beta},$$

and the dv-integral is equal to $(1+\alpha)^{k-\beta} r^{\frac{k-\beta}{2}}$, so

$$\begin{aligned} I_1 &\le (1+\alpha)^{k-\beta}\int_0^{\epsilon} dr\,(t-s+2r)^{-k/2} r^{\frac{k-\beta}{2}}\exp\left(-\frac{\alpha^2\epsilon}{2(t-s+2r)}\right) \\ &\le (1+\alpha)^{k-\beta}\int_0^{\epsilon} dr\,(t-s+2r)^{-\beta/2}\exp\left(-\frac{\alpha^2\epsilon}{2(t-s+2r)}\right), \end{aligned}$$

where the second inequality uses the fact that $k-\beta>0$. Use the change of variables $\rho = \frac{t-s+2r}{\alpha^2\epsilon}$ and the inequality $t-s\le\epsilon$ to see that

$$\begin{aligned} I_1 &\le (1+\alpha)^{k-\beta}\int_{\frac{t-s}{\alpha^2\epsilon}}^{\frac{t-s+2\epsilon}{\alpha^2\epsilon}} d\rho\,\alpha^2\epsilon\,(\alpha^2\epsilon\rho)^{-\beta/2}\exp\left(-\frac{1}{2\rho}\right) \\ &\le \epsilon^{\frac{2-\beta}{2}}(1+\alpha)^{k-\beta}\alpha^{2-\beta}\int_0^{3/\alpha^2} d\rho\,\rho^{-\beta/2}\exp\left(-\frac{1}{2\rho}\right) =: \epsilon^{\frac{2-\beta}{2}}\Phi_1(\alpha). \end{aligned}$$

We note that $\lim_{\alpha\to+\infty}\Phi_1(\alpha)=0$.

Concerning I_2, note that

$$\begin{aligned} I_2 &\le \int_0^{\epsilon} dr \int_{\|v\|>\sqrt{r}(1+\alpha)} dv\, r^{-\beta/2}(1+\alpha)^{-\beta}(t-s+2r)^{-k/2}\exp\left(-\frac{\|y-x+v\|^2}{2(t-s+2r)}\right) \\ &\le (1+\alpha)^{-\beta}\int_0^{\epsilon} dr\, r^{-\beta/2}\int_{\mathbb{R}^k} dv\,(t-s+2r)^{-k/2}\exp\left(-\frac{\|y-x+v\|^2}{2(t-s+2r)}\right) \\ &= c(1+\alpha)^{-\beta}\epsilon^{\frac{2-\beta}{2}}. \end{aligned}$$

We note that $\lim_{\alpha\to+\infty}(1+\alpha)^{-\beta}=0$, and so we have shown that $\tilde{B}_4 \le \Phi(\alpha)\epsilon^{\frac{2-\beta}{2}}$, with $\lim_{\alpha\to+\infty}\Phi(\alpha)=0$.

Using (5.21), we have shown that

$$\begin{aligned}\inf_{\|\xi\|=1} \xi^{\mathsf{T}} \gamma_Z \xi &\geq \frac{2}{3} A_1 - 8\big(A_2 + A_3 + A_4 + A_5\big) + \frac{2}{3} B_1 - 4(B_2 + B_3) \\ &\geq \frac{2}{3}\big(\tilde{A}_1 + \tilde{A}_2 + B_1\big) - \frac{4}{3} \tilde{B}_4 - 8(A_2 + A_3 + A_4 + A_5) - 4(B_2 + B_3) \\ &\geq \frac{2}{3} c\, \epsilon^{\frac{2-\beta}{2}} - 4\Phi(\alpha)\epsilon^{\frac{2-\beta}{2}} - Z_{1,\epsilon},\end{aligned}$$

where $\mathrm{E}[|Z_{1,\epsilon}|^q] \leq \epsilon^{\frac{2-\beta}{2}q + \frac{\gamma}{2}q}$. We choose α large enough so that $\Phi(\alpha) < \frac{1}{12}c$, to get

$$\inf_{\|\xi\|=1} \xi^{\mathsf{T}} \gamma_Z \xi \geq \frac{1}{3} c\epsilon^{\frac{2-\beta}{2}} - Z_{1,\epsilon}.$$

Sub-Case B. Suppose that $\epsilon \leq t - s \leq |x - y|^2$. As in (5.16), we have

$$\inf_{\|\xi\|=1} \xi^{\mathsf{T}} \gamma_Z \xi \geq \min\left(c\epsilon^{\frac{2-\beta}{2}+\eta} - Y_{1,\epsilon},\, c\epsilon^{\frac{2-\beta}{2}} - Y_{2,\epsilon}\right),$$

where $\mathrm{E}[|Y_{1,\epsilon}|^q] \leq c\,\epsilon^{\frac{2-\beta}{2}q + \frac{\gamma}{2}q}$ and $\mathrm{E}[|Y_{2,\epsilon}|^q] \leq c\,\epsilon^{\frac{2-\beta}{2}q + \eta q}$. This suffices for *Sub-Case B*.

Now, we combine *Sub-Cases A* and *B* to see that for $0 < \epsilon < \frac{1}{4}(1+\alpha)^{-2}\|x - y\|^2$,

$$\inf_{\|\xi\|=1} \xi^{\mathsf{T}} \gamma_Z \xi \geq \min\left(c\epsilon^{\frac{2-\beta}{2}+\eta} - Y_{1,\epsilon}, \quad c\epsilon^{\frac{2-\beta}{2}} - Y_{2,\epsilon} 1_{\{\epsilon \leq t-s\}} - Z_{1,\epsilon} 1_{\{t-s<\epsilon\}}\right).$$

By [6, Proposition 3.5], we see that

$$\begin{aligned}\mathrm{E}\left[\left(\inf_{\|\xi\|=1} \xi^{\mathsf{T}} \gamma_Z \xi\right)^{-2dp}\right] &\leq c\|x - y\|^{2(-2dp)\left(\frac{2-\beta}{2}+\eta\right)} \\ &\leq c\big(|t - s| + \|x - y\|^2\big)^{-2dp\left(\frac{2-\beta}{2}+\eta\right)} \\ &\leq c\big(|t - s|^{\frac{2-\beta}{2}} + \|x - y\|^{2-\beta}\big)^{-2dp(1+\tilde{\eta})}\end{aligned}$$

(in the second inequality, we have used the fact that $\|x - y\|^2 \geq t - s$). This concludes the proof of Proposition 5.6. □

Proof of Proposition 5.7 Let $0 < \epsilon < s \leq t$. Fix $i_0 \in \{1, \ldots, 2d\}$ and write $\tilde{\lambda}^{i_0} = (\tilde{\lambda}_1^{i_0}, \ldots, \tilde{\lambda}_d^{i_0})$ and $\tilde{\mu}^{i_0} = (\tilde{\mu}_1^{i_0}, \ldots, \tilde{\mu}_d^{i_0})$. We look at $(\xi^{i_0})^{\mathsf{T}} \gamma_Z \xi^{i_0}$ on the event $\{\alpha_{i_0} \geq \alpha_0\}$. As in the proof of Proposition 5.6 and using the notation from (5.10), this is bounded below by

$$\int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \Bigg\| \sum_{i=1}^{d} \Bigg[\Bigg(\alpha_{i_0} \tilde{\lambda}_i^{i_0} S(s-r, y-\cdot)$$
$$+\tilde{\mu}_i^{i_0} \sqrt{1-\alpha_{i_0}^2}\, (S(t-r, x-\cdot) - S(s-r, y-\cdot)) \Bigg) \sigma_{i,l}(u(r,\cdot))$$
$$+\alpha_{i_0} \tilde{\lambda}_i^{i_0} a_i(l, r, s, y)$$
$$+\tilde{\mu}_i^{i_0} \sqrt{1-\alpha_{i_0}^2}\, (a_i(l, r, t, x) - a_i(l, r, s, y)) \Bigg] \Bigg\|_{\mathscr{H}}^2$$
$$+ \int_{s\vee(t-\epsilon)}^{t} dr \sum_{l=1}^{d} \Bigg\| \sum_{i=1}^{d} \Bigg[\tilde{\mu}_i^{i_0} \sqrt{1-\alpha_{i_0}^2} S(t-r, x-\cdot) \sigma_{i,l}(u(r,\cdot))$$
$$+\tilde{\mu}_i^{i_0} \sqrt{1-\alpha_{i_0}^2}\, a_i(l, r, t, x) \Bigg] \Bigg\|_{\mathscr{H}}^2. \tag{5.22}$$

We seek lower bounds for this expression for $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0 \in (0, \frac{1}{2})$. In the remainder of this proof, we will use the generic notation α, $\tilde{\lambda}$ and $\tilde{\mu}$ for the realizations $\alpha_{i_0}(\omega)$, $\tilde{\lambda}^{i_0}(\omega)$, and $\tilde{\mu}^{i_0}(\omega)$. Our proof follows the structure of [10, Theorem 3.4], rather than [6, Proposition 6.13].

Case 1. $t - s > \epsilon$. Fix $\gamma \in (0, 2-\beta)$ and let η be such that $\eta < \gamma/2$. We note that

$$\inf_{1 \geq \alpha \geq \alpha_0} \left(\xi^{i_0}\right)^{\mathsf{T}} \gamma_Z \xi^{i_0} := \min(E_{1,\epsilon}, E_{2,\epsilon}),$$

where

$$E_{1,\epsilon} := \inf_{\alpha_0 \leq \alpha \leq \sqrt{1-\epsilon^\eta}} \left(\xi^{i_0}\right)^{\mathsf{T}} \gamma_Z \xi^{i_0}, \qquad E_{2,\epsilon} := \inf_{\sqrt{1-\epsilon^\eta} \leq \alpha \leq 1} \left(\xi^{i_0}\right)^{\mathsf{T}} \gamma_Z \xi^{i_0}.$$

Using (4.2) and (5.22), we see that

$$E_{1,\epsilon} \geq \inf_{\alpha_0 \leq \alpha \leq \sqrt{1-\epsilon^\eta}} \left(\frac{2}{3} G_{1,\epsilon} - 2\bar{G}_{1,\epsilon} \right),$$

where

$$G_{1,\epsilon} := (1-\alpha^2) \int_{s\vee(t-\epsilon)}^{t} dr \sum_{l=1}^{d} \|(\tilde{\mu}^{\mathsf{T}} \cdot \sigma(u(r,\cdot)))_l\, S(t-r, x-\cdot)\|_{\mathscr{H}}^2,$$
$$\bar{G}_{1,\epsilon} := \int_{t-\epsilon}^{t} dr \sum_{l=1}^{d} \Bigg\| \sum_{i=1}^{d} \tilde{\mu}_i \sqrt{1-\alpha^2}\, a_i(l, r, t, x) \Bigg\|_{\mathscr{H}}^2.$$

Using the same "localisation argument" as in the proof of Proposition 4.1 (see (4.4)), we have that there exists a random variable W_ϵ such that

$$G_{1,\epsilon} \geq \rho^2 c(1-\alpha^2)((t-s)\wedge\epsilon)^{\frac{2-\beta}{2}} - 2W_\epsilon, \tag{5.23}$$

where, for any $q \geq 1$,

$$\mathrm{E}\big[|W_\epsilon|^q\big] \leq c_q \epsilon^{\frac{2-\beta}{2}q + \frac{\gamma}{2}q}.$$

Hence, using the fact that $1-\alpha^2 \geq \epsilon^\eta$ and $t-s > \epsilon$, we deduce that

$$E_{1,\epsilon} \geq c\epsilon^{\frac{2-\beta}{2}+\eta} - 2W_\epsilon - 2\bar{G}_{1,\epsilon},$$

where, from Lemma 6.2, $\mathrm{E}\,[|\bar{G}_{1,\epsilon}|^q] \leq c_q \epsilon^{(2-\beta)q}$, for any $q \geq 1$,

We now estimate $E_{2,\epsilon}$. Using (4.2) and (5.22), we see that

$$E_{2,\epsilon} \geq \frac{2}{3} G_{2,\epsilon} - 8\big(\bar{G}_{2,1,\epsilon} + \bar{G}_{2,2,\epsilon} + \bar{G}_{2,3,\epsilon} + \bar{G}_{2,4,\epsilon}\big),$$

where

$$\begin{aligned}
G_{2,\epsilon} &:= \alpha^2 \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \|\big(\tilde{\lambda}^{\mathsf{T}} \cdot \sigma(u(r,\cdot))\big)_l\, S(s-r, y-\cdot)\|^2_{\mathscr{H}},\\
\bar{G}_{2,1,\epsilon} &:= (1-\alpha^2) \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \|\big(\tilde{\mu}^{\mathsf{T}} \cdot \sigma(u(r,\cdot))\big)_l\, S(t-r, x-\cdot)\|^2_{\mathscr{H}},\\
\bar{G}_{2,2,\epsilon} &:= (1-\alpha^2) \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \|\big(\tilde{\mu}^{\mathsf{T}} \cdot \sigma(u(r,\cdot))\big)_l\, S(s-r, y-\cdot)\|^2_{\mathscr{H}},\\
\bar{G}_{2,3,\epsilon} &:= \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \left\| \sum_{i=1}^{d} \left(\alpha\tilde{\lambda}_i - \tilde{\mu}_i\sqrt{1-\alpha^2}\right) a_i(l,r,s,y) \right\|^2_{\mathscr{H}},\\
\bar{G}_{2,4,\epsilon} &:= (1-\alpha^2) \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \left\| \sum_{i=1}^{d} \tilde{\mu}_i a_i(l,r,t,x) \right\|^2_{\mathscr{H}}.
\end{aligned}$$

As for the term $G_{1,\epsilon}$ in (5.23) and using the fact that $\alpha^2 \geq 1-\epsilon^\eta$, we get that

$$G_{2,\epsilon} \geq c\epsilon^{\frac{2-\beta}{2}} - 2W_\epsilon,$$

where, for any $q \geq 1$, $\mathrm{E}[|W_\epsilon|^q] \leq c_q \epsilon^{\frac{2-\beta}{2}q+\frac{\gamma}{2}q}$. On the other hand, since $1-\alpha^2 \leq \epsilon^\eta$, we can use hypothesis **P1** and Lemma 6.1 to see that

$$\mathrm{E}\left[|\bar{G}_{2,1,\epsilon}|^q\right] \leq c_q \epsilon^{\left(\frac{2-\beta}{2}+\eta\right)q},$$

and similarly, using Lemma 6.1,

$$\mathrm{E}\left[|\bar{G}_{2,2,\epsilon}|^q\right] \leq c_q \epsilon^{\left(\frac{2-\beta}{2}+\eta\right)q}.$$

Finally, using Lemma 6.2, we have that

$$\begin{aligned}
&\mathrm{E}\left[|\bar{G}_{2,3,\epsilon}|^q\right] \leq c_q \epsilon^{(2-\beta)q}, \quad \text{and} \\
&\mathrm{E}\left[|\bar{G}_{2,4,\epsilon}|^q\right] \leq c_q \epsilon^{\eta q}(t-s+\epsilon)^{\frac{2-\beta}{2}q} \epsilon^{\frac{2-\beta}{2}q} \leq c_q \epsilon^{\left(\frac{2-\beta}{2}+\eta\right)q}.
\end{aligned}$$

We conclude that $E_{2,\epsilon} \geq c\epsilon^{\frac{2-\beta}{2}} - J_\epsilon$, where $\mathrm{E}[|J_\epsilon|^q] \leq c_q \epsilon^{(\frac{2-\beta}{2}+\eta)q}$. Therefore, when $t-s > \epsilon$,

$$1_{\{\alpha_{i_0} \geq \alpha_0\}} \left(\xi^{i_0}\right)^{\mathsf{T}} \gamma_Z \xi^{i_0} \geq 1_{\{\alpha_{i_0} \geq \alpha_0\}} \min\left(c\epsilon^{\frac{2-\beta}{2}+\eta} - V_\epsilon\ ,\ c\epsilon^{\frac{2-\beta}{2}} - J_\epsilon\right),$$

where $\mathrm{E}[|V_\epsilon|^q] \leq c_q \epsilon^{\frac{2-\beta}{2}q+\frac{\gamma}{2}q}$.

Case 2. $t-s \leq \epsilon$, $\frac{|x-y|^2}{\delta_0} \leq \epsilon$. The constant δ_0 will be chosen sufficiently large (see (5.29)). Fix $\theta \in (0, \frac{1}{2})$ and $\gamma \in (0, 2-\beta)$. From (4.2) and (5.22), we have that

$$1_{\{\alpha_{i_0} \geq \alpha_0\}} \left(\xi^{i_0}\right)^{\mathsf{T}} \gamma_Z \xi^{i_0} \geq \frac{2}{3} G_{3,\epsilon^\theta} - 8\big(\bar{G}_{3,1,\epsilon^\theta} - \bar{G}_{3,2,\epsilon^\theta} - \bar{G}_{3,3,\epsilon^\theta} - \bar{G}_{3,4,\epsilon^\theta}\big),$$

where

$$\begin{aligned}
G_{3,\epsilon^\theta} &:= \alpha^2 \int_{s-\epsilon^\theta}^{s} dr \sum_{l=1}^{d} \|(\tilde{\lambda}^{\mathsf{T}} \cdot \sigma(u(r,y)))_l\, S(s-r, y-\cdot)\|^2_{\mathscr{H}}, \\
\bar{G}_{3,1,\epsilon^\theta} &:= \alpha^2 \int_{s-\epsilon^\theta}^{s} dr \sum_{l=1}^{d} \|(\tilde{\lambda}^{\mathsf{T}} \cdot (\sigma(u(r,\cdot)) - \sigma(u(r,y))))_l\, S(s-r, y-\cdot)\|^2_{\mathscr{H}}, \\
\bar{G}_{3,2,\epsilon^\theta} &:= \int_{s-\epsilon^\theta}^{s} dr \sum_{l=1}^{d} \left\| \sum_{i=1}^{d} \left(\alpha\tilde{\lambda}_i - \tilde{\mu}_i\sqrt{1-\alpha^2}\right) a_i(l,r,s,y) \right\|^2_{\mathscr{H}},
\end{aligned}$$

$$\bar{G}_{3,3,\epsilon^\theta} := (1-\alpha^2) \int_{s-\epsilon^\theta}^{s} dr \times \sum_{l=1}^{d} \left\| (\tilde{\mu}^{\mathsf{T}} \cdot \sigma(u(r,\cdot)))_l \left(S(t-r, x-\cdot) - S(s-r, y-\cdot)\right) \right\|_{\mathscr{H}}^2,$$

$$\bar{G}_{3,4,\epsilon^\theta} := (1-\alpha^2) \int_{s-\epsilon^\theta}^{s} dr \sum_{l=1}^{d} \left\| \sum_{i=1}^{d} \tilde{\mu}_i \, a_i(l, r, t, x) \right\|_{\mathscr{H}}^2.$$

By hypothesis **P2** (see also Remark 1.1) and Lemma 6.1, since $t - s \le \epsilon$ and $\alpha \ge \alpha_0$, we have that

$$G_{3,\epsilon^\theta} \ge \alpha_0^2 c \epsilon^{\theta \frac{2-\beta}{2}}.$$

As in the proof of Proposition 4.1 (see in particular (4.3) to (4.4)), we get that for any $q \ge 1$,

$$\mathrm{E}\left[|\bar{G}_{3,1,\epsilon^\theta}|^q\right] \le C \epsilon^{\theta\left(\frac{2-\beta}{2} + \frac{\gamma}{2}\right)q}.$$

Appealing to Lemma 6.2 and using the fact that $t - s \le \epsilon$, we see that

$$\mathrm{E}\left[|\bar{G}_{3,2,\epsilon^\theta}|^q\right] \le c_q \epsilon^{\theta(2-\beta)q}$$

and

$$\mathrm{E}\left[|\bar{G}_{3,4,\epsilon^\theta}|^q\right] \le c_q (t - s + \epsilon^\theta)^{\frac{2-\beta}{2}q} \epsilon^{\theta \frac{2-\beta}{2} q} \le c_q \epsilon^{\theta(2-\beta)q}.$$

It remains to find an upper bound for $\bar{G}_{3,3,\epsilon^\theta}$. From Burkholder's inequality, for any $q \ge 1$,

$$\mathrm{E}\left[|\bar{G}_{3,3,\epsilon^\theta}|^q\right] \le c_q \left(W_{1,\epsilon^\theta} + W_{2,\epsilon^\theta}\right), \tag{5.24}$$

where

$$W_{1,\epsilon^\theta} = \mathrm{E}\left[\left| \int_{s-\epsilon^\theta}^{s} \int_{\mathbb{R}^k} \left(S(t-r, x-z) - S(s-r, x-z)\right) \sum_{l=1}^{d} \left(\tilde{\mu}^{\mathsf{T}} \cdot \sigma(u(r,z))\right)_l M^l(dr, dz) \right|^{2q}\right],$$

$$W_{2,\epsilon^\theta} = \mathrm{E}\left[\left| \int_{s-\epsilon^\theta}^{s} \int_{\mathbb{R}^k} \left(S(s-r, x-z) - S(s-r, y-z)\right) \sum_{l=1}^{d} \left(\tilde{\mu}^{\mathsf{T}} \cdot \sigma(u(r,z))\right)_l M^l(dr, dz) \right|^{2q}\right].$$

As in the proof of Proposition 5.1, using the semigroup property of S, the Beta function and a stochastic Fubini's theorem (whose assumptions can be seen to be satisfied, see e.g. [25, Theorem 2.6]), we see that for any $\alpha \in (0, \frac{2-\beta}{4})$,

$$\int_{s-\epsilon^\theta}^{s}\int_{\mathbb{R}^k} S(s-v, y-\eta) \sum_{l=1}^{d} \left(\tilde{\mu}^{\mathsf{T}} \cdot \sigma(u(v,\eta))\right)_l M^l(dv, d\eta)$$

$$= \frac{\sin(\pi\alpha)}{\pi} \int_{s-\epsilon^\theta}^{s} dr \int_{\mathbb{R}^k} dz\, S(s-r, y-z)(s-r)^{\alpha-1} Y_\alpha(r,z) \tag{5.25}$$

where $Y = (Y_\alpha(r,z), r \in [0,T], z \in \mathbb{R}^k)$ is the real valued process defined as

$$Y_\alpha(r,z) = \int_{s-\epsilon^\theta}^{r}\int_{\mathbb{R}^k} S(r-v, z-\eta)(r-v)^{-\alpha} \sum_{l=1}^{d} \left(\tilde{\mu}^{\mathsf{T}} \cdot \sigma(u(v,\eta))\right)_l M^l(dv, d\eta).$$

We next estimate the $L^p(\Omega)$-norm of the process Y. Using Burkholder's inequality, the boundedness of the coefficients of σ, and the change variables $\tilde{\xi} = \sqrt{r-v}\,\xi$, we see that

$$\begin{aligned}
\mathrm{E}\left[|Y_\alpha(r,z)|^p\right] &\le c_p \left(\int_{s-\epsilon^\theta}^{r} dv \int_{\mathbb{R}^k} d\xi\, \|\xi\|^{\beta-k} |\mathscr{F} S(r-v, z-\cdot)(r-v)^{-\alpha}(\xi)|^2 \right)^{\frac{p}{2}} \\
&= c_p \left(\int_{s-\epsilon^\theta}^{r} dv \int_{\mathbb{R}^k} d\xi\, \|\xi\|^{\beta-k} (r-v)^{-2\alpha} e^{-4\pi^2(r-v)\|\xi\|^2} \right)^{\frac{p}{2}} \\
&= c_p \left(\int_{s-\epsilon^\theta}^{r} dv\, (r-v)^{-2\alpha-\frac{\beta}{2}} \int_{\mathbb{R}^k} d\tilde{\xi}\, \|\tilde{\xi}\|^{\beta-k} e^{-4\pi^2\|\tilde{\xi}\|^2} \right)^{\frac{p}{2}} \\
&\le c_p \left(r - s + \epsilon^\theta\right)^{\left(\frac{2-\beta}{4}-\alpha\right)p}.
\end{aligned}$$

Hence, we conclude that

$$\sup_{(r,z)\in[s-\epsilon^\theta, s]\times\mathbb{R}^k} \mathrm{E}\left[|Y_\alpha(r,z)|^p\right] \le c_p \epsilon^{\theta\left(\frac{2-\beta}{4}-\alpha\right)p}. \tag{5.26}$$

Let us now bound W_{1,ϵ^θ}. Using (5.25) and Minskowski's inequality, we have that

$$\begin{aligned}
W_{1,\epsilon^\theta} \le &\left(\int_{s-\epsilon^\theta}^{s} dr \int_{\mathbb{R}^k} dz \left(\psi_\alpha(t-r, x-z) - \psi_\alpha(s-r, x-z)\right) \right)^{2q} \\
&\times \sup_{(r,z)\in[s-\epsilon^\theta, s]\times\mathbb{R}^k} \mathrm{E}\left[|Y_\alpha(r,z)|^{2q}\right],
\end{aligned}$$

where $\psi_\alpha(t,x) = S(t,x)t^{-\alpha}$. Then by (5.26) and Lemma 5.2(b), we obtain that for any $\gamma < 4\alpha$,

$$W_{1,\epsilon^\theta} \le c_q \epsilon^{\theta q \left(2\alpha - \frac{\gamma}{2}\right)} |t-s|^{\frac{\gamma}{2}q} \epsilon^{\theta\left(\frac{2-\beta}{2} - 2\alpha\right)q} = c_q \epsilon^{\theta\left(\frac{2-\beta}{2} - \frac{\gamma}{2}\right)q} |t-s|^{\frac{\gamma}{2}q}.$$

Thus, using the fact that $t - s \le \epsilon$, we conclude that

$$W_{1,\epsilon^\theta} \le c_q \epsilon^{\theta\left(\frac{2-\beta}{2} - \frac{\gamma}{2}\right)q} \epsilon^{\frac{\gamma}{2}q} = c_q \epsilon^{\theta\frac{2-\beta}{2}q} \epsilon^{\frac{\gamma}{2}(1-\theta)q}. \tag{5.27}$$

We finally treat W_{2,ϵ^θ}. Using (5.25) and Minskowski's inequality, we have that

$$W_{2,\epsilon^\theta} \le \left(\int\limits_{s-\epsilon^\theta}^{s} dr \int\limits_{\mathbb{R}^k} dz \big(\psi_\alpha(s-r, x-z) - \psi_\alpha(s-r, y-z)\big) \right)^{2q} \times \sup_{(r,z)\in\left[s-\epsilon^\theta, s\right]\times\mathbb{R}^k} \mathrm{E}\big[|Y_\alpha(r,z)|^{2q}\big].$$

Then by (5.26) and Lemma 5.2(a), we obtain that for any $\gamma < 4\alpha$,

$$W_{2,\epsilon^\theta} \le c_q\, \epsilon^{\theta q(2\alpha-\gamma)} |x-y|^{\gamma q} \epsilon^{\theta\left(\frac{2-\beta}{2} - 2\alpha\right)q} = c_q \epsilon^{\theta\left(\frac{2-\beta}{2} - \gamma\right)q} |x-y|^{\gamma q}.$$

Thus, using the fact that $|x-y| \le \sqrt{\delta_0 \epsilon}$, we conclude that

$$W_{2,\epsilon^\theta} \le c_q \epsilon^{\theta\left(\frac{2-\beta}{2} - \gamma\right)q} \delta_0^{\frac{\gamma}{2}q} \epsilon^{\frac{\gamma}{2}q} = c_q \delta_0^{\frac{\gamma}{2}q} \epsilon^{\theta\frac{2-\beta}{2}q} \epsilon^{\frac{\gamma}{2}(1-2\theta)q}. \tag{5.28}$$

Finally, substituting (5.27) and (5.28) into (5.24) we conclude that for any $q \ge 1$,

$$\mathrm{E}\left[|\bar{G}_{3,3,\epsilon^\theta}|^q\right] \le c_q \epsilon^{\theta\frac{2-\beta}{2}q} \epsilon^{\frac{\gamma}{2}(1-2\theta)q}.$$

Therefore, we have proved that in the Case 2,

$$1_{\{\alpha_{i_0} \ge \alpha_0\}} \left(\xi^{i_0}\right)^{\mathsf{T}} \gamma_Z \xi^{i_0} \ge 1_{\{\alpha_{i_0} \ge \alpha_0\}} \big(c\epsilon^{\theta\frac{2-\beta}{2}} - W_\epsilon\big),$$

where $\mathrm{E}[|W_\epsilon|^q] \le c_q \epsilon^{\theta\frac{2-\beta}{2}q + \frac{\gamma}{2}q \min(\theta, 1-2\theta)}$.

Case 3. $t - s \le \epsilon,\ 0 < \epsilon < \frac{|x-y|^2}{\delta_0}$. From (4.2) and (5.22), we have that

$$1_{\{\alpha_{i_0} \ge \alpha_0\}} \left(\xi^{i_0}\right)^{\mathsf{T}} \gamma_Z \xi^{i_0} \ge \frac{2}{3} G_{4,\epsilon} - 8(\bar{G}_{4,1,\epsilon} - \bar{G}_{4,2,\epsilon} - \bar{G}_{4,3,\epsilon} - \bar{G}_{4,4,\epsilon}),$$

where

$$G_{4,\epsilon} := \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \Bigg\| \sum_{i=1}^{d} \Big\{ \Big(\alpha\tilde{\lambda}_i - \tilde{\mu}_i\sqrt{1-\alpha^2}\Big)\, \sigma_{i,l}(u(r,y))\, S(s-r, y-\cdot) + \sqrt{1-\alpha^2}\,\tilde{\mu}_i \sigma_{i,l}(u(r,x)) S(t-r, x-\cdot)\Big\} \Bigg\|_{\mathscr{H}}^2,$$

$$\bar{G}_{4,1,\epsilon} := \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \times \Bigg\| \sum_{i=1}^{d} \Big(\alpha\tilde{\lambda}_i - \tilde{\mu}_i\sqrt{1-\alpha^2}\Big) \big[\sigma_{i,l}(u(r,\cdot)) - \sigma_{i,l}(u(r,y))\big]\, S(s-r, y-\cdot) \Bigg\|_{\mathscr{H}}^2,$$

$$\bar{G}_{4,2,\epsilon} := (1-\alpha^2) \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \Big\| \Big(\tilde{\mu}^{\mathsf{T}} \cdot \big[\sigma(u(r,\cdot)) - \sigma(u(r,x))\big]\Big)_l\, S(t-r, x-\cdot) \Big\|_{\mathscr{H}}^2,$$

$$\bar{G}_{4,3,\epsilon} := \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \Big\| \sum_{i=1}^{d} (\alpha\tilde{\lambda}_i - \tilde{\mu}_i\sqrt{1-\alpha^2})\, a_i(l, r, s, y) \Big\|_{\mathscr{H}}^2,$$

$$\bar{G}_{4,4,\epsilon} := (1-\alpha^2) \int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \Big\| \sum_{i=1}^{d} \tilde{\mu}_i\, a_i(l, r, t, x) \Big\|_{\mathscr{H}}^2.$$

We start with a lower bound for $G_{4,\epsilon}$. Observe that this term is similiar to the term A_1 in the Sub-Case A of the proof of Proposition 5.6. Using the inequality $(a+b)^2 \geq a^2 + b^2 - 2|ab|$, we see that $G_{4,\epsilon} \geq G_{4,1,\epsilon} + G_{4,2,\epsilon} - 2G_{4,3,\epsilon}$, where

$$G_{4,1,\epsilon} = \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \Big\| S(s-r, y-\cdot)\big((\alpha\lambda - \sqrt{1-\alpha^2}\mu)^{\mathsf{T}} \cdot \sigma(u(r,y))\big)_l \Big\|_{\mathscr{H}}^2,$$

$$G_{4,2,\epsilon} = \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \Big\| S(t-r, x-\cdot)(\sqrt{1-\alpha^2}\mu^{\mathsf{T}} \cdot \sigma(u(r,x)))_l \Big\|_{\mathscr{H}}^2,$$

$$G_{4,3,\epsilon} = \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \big\langle S(s-r, y-\cdot)((\alpha\lambda - \sqrt{1-\alpha^2}\mu)^{\mathsf{T}} \cdot \sigma(u(r,y)))_l,\; S(t-r, x-\cdot)(\alpha\mu^{\mathsf{T}} \cdot \sigma(u(r,x)))_l \big\rangle_{\mathscr{H}}.$$

Hypothesis **P2** (see also Remark 1.1), Lemma 6.1, and the fact that $t - s \leq \epsilon$ imply that

$$G_{4,1,\epsilon} + G_{4,2,\epsilon} \geq c\big(\|\alpha\lambda - \sqrt{1-\alpha^2}\mu\|^2 + \|\sqrt{1-\alpha^2}\,\mu\|^2\big)\epsilon^{\frac{2-\beta}{2}} \geq c_0\epsilon^{\frac{2-\beta}{2}}.$$

On the other hand, using the same computation as the one done for the term $\tilde{B}_4$ in the Sub-Case A of the proof of Proposition 5.6, we conclude that $G_{4,3,\epsilon} \leq \Phi(\frac{1}{2}\sqrt{\delta_0} - 1)\epsilon^{\frac{2-\beta}{2}}$, with $\lim_{\alpha\to+\infty}\Phi(\alpha) = 0$. Choose δ_0 sufficiently large so that

$$\Phi\Big(\frac{1}{2}\sqrt{\delta_0} - 1\Big) \leq \frac{c_0}{2}, \tag{5.29}$$

so that $G_{4,\epsilon} \geq \frac{c_0}{2}\epsilon^{\frac{2-\beta}{2}}$.

We next treat the terms $\bar{G}_{4,i,\epsilon}$, $i = 1, \ldots, 4$. Using the same argument as for the term $\bar{G}_{3,1,\epsilon^\theta}$, we see that for any $q \geq 1$,

$$\mathrm{E}\left[|\bar{G}_{4,1,\epsilon}|^q\right] \leq C\epsilon^{\left(\frac{2-\beta}{2}+\frac{\gamma}{2}\right)q}.$$

Appealing to Lemma 6.2 and using the fact that $t - s \leq \epsilon$, we find that

$$\mathrm{E}\left[|\bar{G}_{4,3,\epsilon}|^q\right] \leq c_q\epsilon^{(2-\beta)q}, \quad \text{and} \quad \mathrm{E}\left[|\bar{G}_{4,4,\epsilon}|^q\right] \leq c_q\epsilon^{\theta(2-\beta)q}.$$

Finally, we treat $\bar{G}_{4,2,\epsilon}$. As in the proof of Proposition 4.1, using Hölder's inequality, the Lipschitz property of σ, Lemma 6.1 and (2.6), we get that for any $q \geq 1$,

$$\mathrm{E}\left[|\bar{G}_{4,2,\epsilon}|^q\right] \leq C\epsilon^{(\frac{2-\beta}{2})(q-1)} \times \Psi,$$

where

$$\begin{aligned}\Psi = &\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}^k} dv \int_{\mathbb{R}^k} dz\, \|z - v\|^{-\beta} S(t-r, x-v) S(t-r, x-z) \|x - v\|^{\frac{\gamma}{2}q} \\ &\times \|x - z\|^{\frac{\gamma}{2}q}.\end{aligned}$$

Changing variables $[\tilde{v} = \frac{x-v}{\sqrt{t-r}}, \tilde{z} = \frac{x-z}{\sqrt{t-r}}]$, this becomes

$$\begin{aligned}\Psi &= \int_{s-\epsilon}^{s} dr\, (t-r)^{-\frac{\beta}{2}+\frac{\gamma q}{2}} \int_{\mathbb{R}^k} d\tilde{v} \int_{\mathbb{R}^k} d\tilde{z}\, S(1, \tilde{v}) S(1, \tilde{z}) \|\tilde{v} - \tilde{z}\|^{-\beta} \|\tilde{z}\|^{\gamma q/2} \|\tilde{v}\|^{\gamma q/2} \\ &= C\Big((t - s + \epsilon)^{\frac{2-\beta}{2}+\frac{\gamma q}{2}} - (t-s)^{\frac{2-\beta}{2}+\frac{\gamma q}{2}}\Big) \\ &\leq C\epsilon^{\frac{2-\beta}{2}+\frac{\gamma q}{2}}.\end{aligned}$$

Hence, we obtain that for any $q \geq 1$,

$$\mathrm{E}\left[|\bar{G}_{4,2,\epsilon}|^q\right] \leq C\epsilon^{\left(\frac{2-\beta}{2}+\frac{\gamma}{2}\right)q}.$$

Therefore, we have proved that in the Case 3,

$$1_{\{\alpha_{i_0}\geq\alpha_0\}}\left(\xi^{i_0}\right)^{\mathsf{T}}\gamma_Z\xi^{i_0}\geq 1_{\{\alpha_{i_0}\geq\alpha_0\}}\left(c\epsilon^{\frac{2-\beta}{2}}-G_\epsilon\right),$$

where $\mathrm{E}[|G_\epsilon|^q]\leq c_q\epsilon^{(\frac{2-\beta}{2}+\frac{\gamma}{2})q}$. This completes Case 3.

Putting together the results of the Cases 1, 2 and 3, we see that for $0<\epsilon\leq\epsilon_0$,

$$1_{\{\alpha_{i_0}\geq\alpha_0\}}\left(\xi^{i_0}\right)^{\mathsf{T}}\gamma_Z\xi^{i_0}\geq 1_{\{\alpha_{i_0}\geq\alpha_0\}}\,Z,$$

where

$$\begin{aligned}Z=\min&\left(c\epsilon^{\frac{2-\beta}{2}+\eta}-V_\epsilon,\,c\epsilon^{\frac{2-\beta}{2}}-J_\epsilon\right)\mathbf{1}_{\{t-s>\epsilon\}}+\left(c\epsilon^{\theta\frac{2-\beta}{2}}-W_\epsilon\right)\mathbf{1}_{\left\{t-s\leq\epsilon,\,\epsilon\geq\frac{|x-y|^2}{\delta_0}\right\}}\\&+\left(c\epsilon^{\frac{2-\beta}{2}}-G_\epsilon\right)\mathbf{1}_{\left\{t-s\leq\epsilon<\frac{|x-y|^2}{\delta_0}\right\}},\end{aligned}$$

where for any $q\geq 1$,

$$\begin{aligned}&\mathrm{E}\left[|V_\epsilon|^q\right]\leq C\epsilon^{\left(\frac{2-\beta}{2}+\frac{\gamma}{2}\right)q},\qquad \mathrm{E}\left[|J_\epsilon|^q\right]\leq C\epsilon^{\left(\frac{2-\beta}{2}+\eta\right)q},\\&\mathrm{E}\left[|W_\epsilon|^q\right]\leq C\epsilon^{\theta\frac{2-\beta}{2}q+\frac{\gamma}{2}q\min(\theta,1-2\theta)},\qquad \mathrm{E}\left[|G_\epsilon|^q\right]\leq C\epsilon^{\left(\frac{2-\beta}{2}+\frac{\gamma}{2}\right)q}.\end{aligned}$$

Therefore,

$$\begin{aligned}Z\geq\min\Big(&c\epsilon^{\frac{2-\beta}{2}+\eta}-V_\epsilon,\ c\epsilon^{\frac{2-\beta}{2}}-J_\epsilon\mathbf{1}_{\{t-s>\epsilon\}}-G_\epsilon\mathbf{1}_{\{t-s\leq\epsilon<\frac{|x-y|^2}{\delta_0}\}},\\&c\epsilon^{\theta\frac{2-\beta}{2}}-W_\epsilon\mathbf{1}_{\{t-s\leq\epsilon,\,\epsilon\geq\frac{|x-y|^2}{\delta_0}\}}\Big).\end{aligned}$$

Note that all the constants are independent of i_0. Then using [6, Proposition 3.5] (extended to the minimum of three terms instead of two), we deduce that for all $p\geq 1$, there is $C>0$ such that

$$E\left[\left(1_{\{\alpha_{i_0}\geq\alpha_0\}}\left(\xi^{i_0}\right)^{\mathsf{T}}\gamma_Z\xi^{i_0}\right)^{-p}\right]\leq E\left[1_{\{\alpha_{i_0}\geq\alpha_0\}}\,Z^{-p}\right]\leq E\left[Z^{-p}\right]\leq C.$$

Since this applies to any $p\geq 1$, we can use Hölder's inequality to deduce (5.13). This proves Proposition 5.7. □

The following result is analogous to [6, Theorem 6.3].

Theorem 5.8 *Fix $\eta, T>0$. Assume* **P1** *and* **P2**. *Let $I\times J\subset(0,T]\times\mathbb{R}^k$ be a closed non-trivial rectangle. For any $(s,y),(t,x)\in I\times J$, $s\leq t$, $(s,y)\neq(t,x)$, $k\geq 0$,* *and* $p > 1$,

$$\|(\gamma_Z^{-1})_{m,l}\|_{k,p} \leq \begin{cases} c_{k,p,\eta,T}\left(|t-s|^{\frac{2-\beta}{2}} + \|x-y\|^{2-\beta}\right)^{-\eta} & if\ (m,l) \in (\mathbf{1}), \\ c_{k,p,\eta,T}\left(|t-s|^{\frac{2-\beta}{2}} + \|x-y\|^{2-\beta}\right)^{-1/2-\eta} & if\ (m,l) \in (\mathbf{2})\ or\ (\mathbf{3}), \\ c_{k,p,\eta,T}\left(|t-s|^{\frac{2-\beta}{2}} + \|x-y\|^{2-\beta}\right)^{-1-\eta} & if\ (m,l) \in (\mathbf{4}). \end{cases}$$

Proof As in the proof of [6, Theorem 6.3], we shall use Propositions 5.3–5.5. Set $\Delta = |t-s|^{1/2} + \|x-y\|$.

Suppose first that $k = 0$. Since the inverse of a matrix is the inverse of its determinant multiplied by its cofactor matrix, we use Proposition 5.5 with η replaced by $\tilde{\eta} = \frac{\eta}{2d(2-\beta)}$ and Proposition 5.3 with $\gamma \in (0, 2-\beta)$ such that $2-\beta-\gamma = \frac{\eta}{2(d-\frac{1}{2})}$ to see that for $(m,l) \in (\mathbf{2})$ or $(\mathbf{3})$,

$$\begin{aligned} \|\left(\gamma_Z^{-1}\right)_{m,l}\|_{0,p} &\leq c_{p,\eta,T}\, \Delta^{-d(2-\beta)(1+\tilde{\eta})}\, \Delta^{\gamma\left(d-\frac{1}{2}\right)} \\ &= c_{p,\eta,T}\, \Delta^{-\frac{2-\beta}{2}}\, \Delta^{(2-\beta-\gamma)\frac{1}{2}}\, \Delta^{-d\left(2-\beta-\gamma\right)-\tilde{\eta}d(2-\beta)} \\ &= c_{p,\eta,T}\, \Delta^{-\frac{2-\beta}{2}}\, \Delta^{-\left(d-\frac{1}{2}\right)(2-\beta-\gamma)-\tilde{\eta}d(2-\beta)} \\ &= c_{p,\eta,T}\, \Delta^{-\frac{2-\beta}{2}}\, \Delta^{-\eta}. \end{aligned}$$

This proves the statement for $(m,l) \in (\mathbf{2})$ or $(\mathbf{3})$. The other two cases are handled in a similar way.

For $k \geq 1$, we proceed recursively as in the proof of [6, Theorem 6.3], using Proposition 5.4 instead of 5.3. □

Remark 5.9 In [6, Theorem 6.3], in the case where $d = 1$ and $s = t$, a slightly stronger result, without the exponent η, is obtained. Here, when $s = t$, the right-hand sides of (5.8) and (5.12) can be improved respectively to $C\|x-y\|^{-(2-\beta)d}$ and $C\|x-y\|^{-(2-\beta)2dp}$. Indeed, when $s = t$, Case 1 in the proof of Proposition 5.6 does not arise, and this yields the improvement of (5.12), and, in turn, the improvement of (5.8). However, this does not lead to an improvement of the result of Theorem 5.8 when $s = t$, because the exponent η there is also due to the fact that $\gamma < 2-\beta$ in Proposition 5.3.

In the next subsection, we will establish the estimate of Theorem 1.6(b). For this, we will use the following expression for the density of a nondegenerate random vector that is a consequence of the integration by parts formula of Malliavin calculus.

Corollary 5.10 [17, Corollary 3.2.1] *Let* $F = (F^1, \ldots, F^d) \in (\mathbb{D}^\infty)^d$ *be a nondegenerate random vector and let* $p_F(z)$ *denote the density of* F *(see Theorem 3.1). Then for every subset* σ *of the set of indices* $\{1, \ldots, d\}$,

$$p_F(z) = (-1)^{d-|\sigma|}\mathrm{E}\left[1_{\left\{F^i > z^i, i\in\sigma,\ F^i < z^i, i \notin \sigma\right\}} H_{(1,\ldots,d)}(F,1)\right],$$

where $|\sigma|$ *is the cardinality of* σ, *and*

$$H_{(1,\ldots,d)}(F,1) = \delta\left((\gamma_F^{-1}DF)^d \delta\left((\gamma_F^{-1}DF)^{d-1}\delta(\cdots\delta((\gamma_F^{-1}DF)^1)\cdots)\right)\right).$$

The following result is similar to [6, (6.3)].

Proposition 5.11 *Fix $\eta, T > 0$. Assume* **P1** *and* **P2**. *Let $I \times J \subset (0, T] \times \mathbb{R}^k$ be a closed non-trivial rectangle. For any $(s, y), (t, x) \in I \times J$, $s \le t$, $(s, y) \neq (t, x)$, and $k \ge 0$,*

$$\|H_{(1,\ldots,2d)}(Z, 1)\|_{0,2} \le C_T \left(|t-s|^{\frac{2-\beta}{2}} + \|x-y\|^{2-\beta}\right)^{-(d+\eta)/2},$$

where Z is the random vector defined in (5.7).

Proof The proof is similar to that of [6, (6.3)] using the continuity of the Skorohod integral δ (see [16, Proposition 3.2.1] and [17, (1.11) and p.131]) and Hölder's inequality for Malliavin norms (see [26, Proposition 1.10, p.50]); the only change is that γ in Proposition 5.1 must be chosen sufficiently close to $2 - \beta$. □

5.3 Proof of Theorem 1.6(b)

Fix $T > 0$ and let $I \times J \subset (0, T] \times \mathbb{R}^k$ be a closed non-trivial rectangle. Let $(s, y), (t, x) \in I \times J$, $s \le t$, $(s, y) \neq (t, x)$, and $z_1, z_2 \in \mathbb{R}^d$. Let p_Z be the density of the random vector Z defined in (5.7). Then

$$p_{s,y;\,t,x}\big(z_1, z_2\big) = p_Z\big(z_1, z_2 - z_1\big).$$

Apply Corollary 5.10 with $\sigma = \{i \in \{1, \ldots, d\} : z_2^i - z_1^i \ge 0\}$ and Hölder's inequality to see that

$$\begin{aligned} p_Z\big(z_1, z_1 - z_2\big) \le \prod_{i=1}^{d} &\left(\mathrm{P}\left\{|u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i|\right\}\right)^{\frac{1}{2d}} \\ &\times \|H_{\left(1,\ldots,2d\right)}(Z, 1)\|_{0,2}. \end{aligned} \tag{5.30}$$

When $\|z_1 - z_2\| = 0$,

$$\frac{|t-s|^{\gamma/2} + \|x-y\|^{\gamma}}{\|z_1 - z_2\|} \wedge 1 = 1,$$

since the numerator is positive because $(s, y) \neq (t, x)$. Therefore, (1.5) follows from Proposition 5.11 in this case.

Assume now that $\|z_1 - z_2\| \neq 0$. Then there is $i \in \{1, \ldots, d\}$, and we may as well assume that $i = 1$, such that $0 < |z_1^1 - z_2^1| = \max_{i=1,\ldots,d} |z_1^i - z_2^i|$. Then

$$\begin{aligned} &\prod_{i=1}^{d} \left(\mathrm{P}\left\{|u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i|\right\}\right)^{\frac{1}{2d}} \\ &\quad \le \left(\mathrm{P}\left\{|u_1(t,x) - u_1(s,y)| > |z_1^1 - z_2^1|\right\}\right)^{\frac{1}{2d}}. \end{aligned}$$

Using Chebyshev's inequality and (2.6), we see that this is bounded above by

$$c\left[\frac{|t-s|^{\gamma/2}+\|x-y\|^{\gamma}}{|z_1^1-z_2^1|^2}\wedge 1\right]^{\frac{p}{2d}} \leq \tilde{c}\left[\frac{|t-s|^{\gamma/2}+\|x-y\|^{\gamma}}{\|z_1-z_2\|^2}\wedge 1\right]^{\frac{p}{2d}}. \tag{5.31}$$

The two inequalities (5.30) and (5.31), together with Proposition 5.11, prove Theorem 1.6(b). □

As mentioned in Remark 1.7, in the case where $b\equiv 0$, one can establish the following exponential upper bound.

Lemma 5.12 *Let $\tilde{u}$ be the solution of (1.1) with $b\equiv 0$. Fix $T>0$ and $\gamma\in(0,2-\beta)$. Assume* **P1**. *Let $I\times J\subset(0,T]\times\mathbb{R}^k$ be a closed non-trivial rectangle. Then there exist constants $c, c_T>0$ such that for any $(s,y),(t,x)\in I\times J$, $s\leq t$, $(s,y)\neq(t,x)$, $z_1,z_2\in\mathbb{R}^d$,*

$$\prod_{i=1}^{d}\left(\mathrm{P}\left\{|\tilde{u}_i(t,x)-\tilde{u}_i(s,y)|>|z_1^i-z_2^i|\right\}\right)^{\frac{1}{2d}}\leq c\exp\left(-\frac{\|z_1-z_2\|^2}{c_T\left(|t-s|^{\gamma/2}+\|x-y\|^{\gamma}\right)}\right).$$

Proof Consider the continuous one-parameter martingale $(M_a=(M_a^1,\ldots,M_a^d),\ 0\leq a\leq t)$ defined by

$$M_a^i=\begin{cases}\displaystyle\int_0^a\int_{\mathbb{R}^k}\big(S(t-r,x-v)-S(s-r,y-v)\big)\textstyle\sum_{j=1}^d\sigma_{ij}\big(\tilde{u}(r,v)\big)\,M^j(dr,dv) & \\ \hfill \text{if } 0\leq a\leq s, & \\ \displaystyle\int_0^s\int_{\mathbb{R}^k}(S(t-r,x-v)-S(s-r,y-v))\textstyle\sum_{j=1}^d\sigma_{ij}(\tilde{u}(r,v))\,M^j(dr,dv) & \\ \displaystyle\qquad+\int_s^a\int_{\mathbb{R}^k}S(t-r,x-v)\textstyle\sum_{j=1}^d\sigma_{ij}(\tilde{u}(r,v))\,M^j(dr,dv) & \\ \hfill \text{if } s\leq a\leq t, & \end{cases}$$

for all $i=1,\ldots,d$, with respect to the filtration $(\mathscr{F}_a,\ 0\leq a\leq t)$. Notice that

$$M_0^i=0,\quad M_t^i=\tilde{u}^i(t,x)-\tilde{u}^i(s,y).$$

Moreover, because the M^i are independent and white in time, $\langle M^i\rangle_t=\mathscr{M}_1^i+\mathscr{M}_2^i$, where

$$\mathscr{M}_1^i=\sum_{j=1}^d\int_0^s dr\left\|(S(t-r,x-\cdot)-S(s-r,y-\cdot))\sigma_{ij}(\tilde{u}(r,\cdot))\right\|_{\mathscr{H}}^2,$$
$$\mathscr{M}_2^i=\sum_{j=1}^d\int_s^t dr\left\|S(t-r,x-\cdot)\sigma_{ij}(\tilde{u}(r,\cdot))\right\|_{\mathscr{H}}^2.$$

Using the fact that the coefficients of σ are bounded and Lemma 6.1, we get that

$$\mathscr{M}_2^i \leq c|t-s|^{\frac{2-\beta}{2}}.$$

On the other hand, we write $\mathscr{M}_1^i \leq 2(\mathscr{M}_{1,1}^i + \mathscr{M}_{1,2}^i)$, where

$$\mathscr{M}_{1,1}^i = \sum_{j=1}^{d} \int_0^s dr \left\| (S(t-r, x-\cdot) - S(t-r, y-\cdot))\sigma_{ij}(\tilde{u}(r,\cdot)) \right\|_{\mathscr{H}}^2,$$
$$\mathscr{M}_{1,2}^i = \sum_{j=1}^{d} \int_0^s dr \left\| (S(t-r, y-\cdot) - S(s-r, y-\cdot))\sigma_{ij}(\tilde{u}(r,\cdot)) \right\|_{\mathscr{H}}^2.$$

In order to bound these two terms, we will use the factorisation method. Using the semigroup property of S and the Beta function, it yields that, for any $\alpha \in (0, 1)$,

$$S(t-r, x-z) = \frac{\sin(\pi\alpha)}{\pi} \int_r^t d\theta \int_{\mathbb{R}^k} d\eta\, \psi_\alpha(t-\theta, x-\eta) S(\theta-r, \eta-z)(\theta-r)^{-\alpha},$$

where $\psi_\alpha(t, x) = S(t, x)t^{\alpha-1}$. Hence, using the boundedness of the coefficients of σ, we can write

$$\mathscr{M}_{1,1}^i \leq c \sum_{j=1}^{d} \int_0^s dr \left\| \int_r^t d\theta \int_{\mathbb{R}^k} d\eta\, |\psi_\alpha(t-\theta, x-\eta) - \psi_\alpha(t-\theta, y-\eta)| \right.$$
$$\left. \times S(\theta-r, \eta-\cdot)(\theta-r)^{-\alpha} \right\|_{\mathscr{H}}^2,$$

and $M_{1,2}^i \leq c(M_{1,2,1}^i + M_{1,2,2}^i)$, where

$$\mathscr{M}_{1,2,1}^i = \sum_{j=1}^{d} \int_0^s dr \left\| \int_r^s d\theta \int_{\mathbb{R}^k} d\eta\, |\psi_\alpha(t-\theta, y-\eta) - \psi_\alpha(s-\theta, y-\eta)| \right.$$
$$\left. \times S(\theta-r, \eta-\cdot)(\theta-r)^{-\alpha} \right\|_{\mathscr{H}}^2,$$
$$\mathscr{M}_{1,2,2}^i = \sum_{j=1}^{d} \int_0^s dr \left\| \int_s^t d\theta \int_{\mathbb{R}^k} d\eta\, \psi_\alpha(t-\theta, y-\eta) S(\theta-r, \eta-\cdot)(\theta-r)^{-\alpha} \right\|_{\mathscr{H}}^2.$$

Using Hölder's inequality, (5.3), (5.4) and Lemma 5.2, we get that for any $\alpha \in (0, \frac{2-\beta}{4})$ and $\gamma \in (0, 4\alpha)$,

$$
\begin{aligned}
\mathscr{M}^i_{1,1} &\le c \sup_{(r,z)\in[0,T]\times\mathbb{R}^k} \|S(r-*, z-\cdot)(r-*)^{-\alpha}\|^2_{\mathscr{H}^d_r} \\
&\qquad \times \left(\int_0^t dr \int_{\mathbb{R}^k} dz\, |\psi_\alpha(t-r, x-z) - \psi_\alpha(t-r, y-z)| \right)^2 \\
&\le c_T(\alpha)\|x-y\|^\gamma, \\
\mathscr{M}^i_{1,2,1} &\le c \sup_{(r,z)\in[0,T]\times\mathbb{R}^k} \|S(r-*, z-\cdot)(r-*)^{-\alpha}\|^2_{\mathscr{H}^d_r} \\
&\qquad \times \left(\int_0^t dr \int_{\mathbb{R}^k} dz\, |\psi_\alpha(t-r, y-z) - \psi_\alpha(s-r, y-z)| \right)^2 \\
&\le c_T(\alpha)\|t-s\|^{\gamma/2}, \\
\mathscr{M}^i_{1,2,2} &\le c \sup_{(r,z)\in[0,T]\times\mathbb{R}^k} \|S(r-*, z-\cdot)(r-*)^{-\alpha}\|^2_{\mathscr{H}^d_r} \left(\int_s^t dr \int_{\mathbb{R}^k} dz\, \psi_\alpha(t-r, y-z) \right)^2 \\
&\le c_T(\alpha)\|t-s\|^{\gamma/2}.
\end{aligned}
$$

Thus, we have proved that for any $\gamma \in (0, 2-\beta)$,

$$\langle M^i \rangle_t \le c_T \left(|t-s|^{\gamma/2} + \|x-y\|^\gamma \right).$$

By the exponential martingale inequality [16, A.5],

$$\mathrm{P}\left\{ |\tilde{u}^i(t,x) - \tilde{u}^i(s,y)| > |z^i_1 - z^i_2| \right\} \le 2\exp\left(-\frac{|z^i_1 - z^i_2|^2}{c_T\left(|t-s|^{\gamma/2} + \|x-y\|^\gamma\right)} \right),$$

which implies the desired result. □

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6 Appendix

Lemma 6.1 *There is $C > 0$ such that for any $0 < \epsilon \le s \le t$ and $x \in \mathbb{R}^k$,*

$$\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}^k} d\xi\, \|\xi\|^{\beta-k}\, |\mathscr{F}S(t-r, x-\cdot)(\xi)|^2 = C((t-s+\epsilon)^{\frac{2-\beta}{2}} - (t-s)^{\frac{2-\beta}{2}}).$$

Moreover, there exists $\tilde{C} > 0$ *such that the above integral is bounded above by* $\tilde{C}\epsilon^{\frac{2-\beta}{2}}$*, and if* $t-s \leq \epsilon$*, then there exists* $\bar{C} > 0$ *such that the above integral is bounded below by* $\bar{C}\epsilon^{\frac{2-\beta}{2}}$.

Proof Using (2.3) and changing variables [$\tilde{r} = t - r$, $\tilde{\xi} = \xi\sqrt{r}$] yields

$$
\begin{aligned}
\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}^k} d\xi \, \|\xi\|^{\beta-k} \, |\mathscr{F}S(t-r, x-\cdot)(\xi)|^2
&= \int_{t-s}^{t-s+\epsilon} dr \, r^{-\beta/2} \int_{\mathbb{R}^k} d\xi \, \|\xi\|^{\beta-k} \, e^{-\|\xi\|^2} \\
&= C \int_{t-s}^{t-s+\epsilon} dr \, r^{-\beta/2} \\
&= C((t-s+\epsilon)^{\frac{2-\beta}{2}} - (t-s)^{\frac{2-\beta}{2}}).
\end{aligned}
$$

If $\epsilon < t-s$, then the last integral is bounded above by $C(t-s)^{-\beta/2}\epsilon \leq C\epsilon^{\frac{2-\beta}{2}}$. On the other hand, if $t-s \leq \epsilon$, then the last integral is bounded above by

$$
\int_{0}^{2\epsilon} dr \, r^{-\beta/2} \leq C\epsilon^{\frac{2-\beta}{2}}.
$$

Finally, if $t-s \leq \epsilon$, then

$$
\int_{t-s}^{t-s+\epsilon} dr \, r^{-\beta/2} \geq \epsilon(t-s+\epsilon)^{-\beta/2} \geq \epsilon(2\epsilon)^{-\beta/2} = c\epsilon^{\frac{2-\beta}{2}}.
$$

□

Lemma 6.2 *Assume* **P1**. *For all* $T > 0$ *and* $q \geq 1$*, there exists a constant* $c = c(q, T) \in (0, \infty)$ *such that for every* $0 < \epsilon \leq s \leq t \leq T$, $x \in \mathbb{R}^k$*, and* $a > 0$*,*

$$
\begin{aligned}
W := \mathrm{E}\left[\sup_{\xi \in \mathbb{R}^d : \|\xi\| \leq a} \left(\int_{s-\epsilon}^{s} dr \sum_{l=1}^{d} \left\| \sum_{i=1}^{d} a_i(l, r, t, x)\xi_i \right\|_{\mathscr{H}}^2\right)^q\right] \\
\leq c\, a^{2q} \, (t-s+\epsilon)^{\frac{2-\beta}{2}q} \epsilon^{\frac{2-\beta}{2}q},
\end{aligned}
$$

where $a_i(l, r, t, x)$ *is defined in (4.1).*

Proof Use (4.1) and the Cauchy-Schwarz inequality to get

$$W \le c\, a^{2q}\left(\mathrm{E}\left[\left(\int_{s-\epsilon}^{s} dr\, \|W_1\|^2_{\mathscr{H}^d}\right)^q\right] + \mathrm{E}\left[\left(\int_{s-\epsilon}^{s} dr\, \|W_2\|^2_{\mathscr{H}^d}\right)^q\right]\right), \tag{6.1}$$

where

$$W_1 = \sum_{i,j=1}^{d} \int_r^t \int_{\mathbb{R}^k} S(t-\theta, x-\eta) D_r(\sigma_{i,j}(u(\theta,\eta)))\, M^j(d\theta, d\eta),$$

$$W_2 = \sum_{i=1}^{d} \int_r^t d\theta \int_{\mathbb{R}^k} d\eta\, S(t-\theta, x-\eta) D_r(b_i(u(\theta,\eta))).$$

Then

$$\mathrm{E}\left[\left(\int_{s-\epsilon}^{s} dr\, \|W_1\|^2_{\mathscr{H}^d}\right)^q\right] = \mathrm{E}\left[\|W_1\|^{2q}_{L^2([s-\epsilon,s],\mathscr{H}^d)}\right].$$

We then apply [21, (6.8) in Theorem 6.1] (see also [18, (3.13)]) to see that this is

$$\le \left(\int_{s-\epsilon}^{t} dr \int_{\mathbb{R}^k} \mu(d\xi)\, |\mathscr{F} S(r)(\xi)|^2\right)^{q-1}$$
$$\times \int_{s-\epsilon}^{t} d\rho \int_{\mathbb{R}^k} \mu(d\xi)\, |\mathscr{F} S(t-\rho)(\xi)|^2 \sup_{\eta\in\mathbb{R}^k} \mathrm{E}\left[\|D_{\cdot,*} u(\rho,\eta)\|^{2q}_{L^2([s-\epsilon,s],\mathscr{H}^d)}\right]. \tag{6.2}$$

According to [21, Lemma 8.2],

$$\sup_{\eta\in\mathbb{R}^k} \mathrm{E}\left[\|D_{\cdot,*} u(\rho,\eta)\|^{2q}_{L^2([s-\epsilon,s],\mathscr{H}^d)}\right] \le C\left(\int_{s-\epsilon}^{s\wedge\rho} dr \int_{\mathbb{R}^k} \mu(d\xi) |\mathscr{F} S(\rho-r)(\xi)|^2\right)^q,$$

and we have

$$\int_{\mathbb{R}^k} \mu(d\xi)\, |\mathscr{F} S(r)(\xi)|^2 = \int_{\mathbb{R}^k} \frac{d\xi}{\|\xi\|^{k-\beta}} e^{-r\|\xi\|^2} = r^{-\frac{\beta}{2}} \int_{\mathbb{R}^k} \frac{dv}{\|v\|^{k-\beta}} e^{-\|v\|^2} = c_0 r^{-\frac{\beta}{2}}. \tag{6.3}$$

For $\rho \le s$,

$$\int_{s-\epsilon}^{s\wedge\rho} dr \int_{\mathbb{R}^k} \mu(d\xi)\, |\mathscr{F}S(\rho-r)(\xi)|^2 = c_0 \int_0^{\rho-s+\epsilon} dr\, r^{-\frac{\beta}{2}} = c(\rho-s+\epsilon)^{\frac{2-\beta}{2}} \le c\epsilon^{\frac{2-\beta}{2}},$$

and for $s \le \rho$,

$$\begin{aligned}\int_{s-\epsilon}^{s\wedge\rho} dr \int_{\mathbb{R}^k} \mu(d\xi)\, |\mathscr{F}S(\rho-r)(\xi)|^2 &= c_0 \int_{\rho-s}^{\rho-s+\epsilon} dr\, r^{-\frac{\beta}{2}} \\ &= c\left((\rho-s+\epsilon)^{\frac{2-\beta}{2}} - (\rho-s)^{\frac{2-\beta}{2}}\right) \\ &= c\epsilon \int_0^1 (\rho-s+\epsilon\nu)^{-\frac{\beta}{2}} d\nu \le c\epsilon \int_0^1 (\epsilon\nu)^{-\frac{\beta}{2}} d\nu \\ &= c\epsilon^{\frac{2-\beta}{2}}.\end{aligned}$$

Therefore, from (6.2) and (6.3) above,

$$\mathrm{E}\left[\|W_1\|^{2q}_{L^2([s-\epsilon,s],\mathscr{H}^d)}\right] \le c(t-s+\epsilon)^{\frac{2-\beta}{2}q} \epsilon^{\frac{2-\beta}{2}q}. \tag{6.4}$$

We now examine the second term in (6.1). Notice that

$$\begin{aligned}\int_{s-\epsilon}^{s} dr\, \|W_2\|^2_{\mathscr{H}^d} &\le C \sum_{i=1}^d \int_{s-\epsilon}^{s} dr \Big\langle \int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}^k} d\eta\, 1_{\{\theta>r\}} S(t-\theta, x-\eta) D_r(b_i(u(\theta,\eta))), \\ &\qquad \int_{s-\epsilon}^{t} d\tilde{\theta} \int_{\mathbb{R}^k} d\tilde{\eta}\, 1_{\{\tilde{\theta}>r\}} S(t-\tilde{\theta}, x-\tilde{\eta}) D_r(b_i(u(\tilde{\theta},\tilde{\eta})))\Big\rangle_{\mathscr{H}^d} \\ &= C \sum_{i=1}^d \int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}^k} d\eta \int_{s-\epsilon}^{t} d\tilde{\theta} \int_{\mathbb{R}^k} d\tilde{\eta}\, S(t-\theta, x-\eta) S(t-\tilde{\theta}, x-\tilde{\eta}) \\ &\qquad \times \int_{s-\epsilon}^{s} dr\, \langle D_r(b_i(u(\theta,\eta))), D_r(b_i(u(\tilde{\theta},\tilde{\eta})))\rangle_{\mathscr{H}^d}.\end{aligned}$$

The dr-integral is equal to

$$\langle D(b_i(u(\theta,\eta))), D(b_i(u(\tilde{\theta},\tilde{\eta})))\rangle_{\mathscr{H}^d_{s-\epsilon,s}}.$$

Therefore, we can apply Hölder's inequality to see that

$$
\begin{aligned}
&\mathrm{E}\left[\left(\int_{s-\epsilon}^{s} dr\, \|W_2\|^2_{\mathscr{H}^d}\right)^q\right] \\
&\quad \le C\sum_{i=1}^{d}\left(\int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}^k} d\eta \int_{s-\epsilon}^{t} d\tilde{\theta} \int_{\mathbb{R}^k} d\tilde{\eta}\, S(t-\theta, x-\eta) S(t-\tilde{\theta}, x-\tilde{\eta})\right)^{q-1} \\
&\qquad \times \int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}^k} d\eta \int_{s-\epsilon}^{t} d\tilde{\theta} \int_{\mathbb{R}^k} d\tilde{\eta}\, S(t-\theta, x-\eta) S(t-\tilde{\theta}, x-\tilde{\eta}) \\
&\qquad \times E\left[\langle D(b_i(u(\theta,\eta))), D(b_i(u(\tilde{\theta},\tilde{\eta})))\rangle^q_{\mathscr{H}^d_{s-\epsilon,s}}\right].
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, we see that the expectation above is bounded by

$$
\begin{aligned}
&\mathrm{E}\left[\left\| D(b_i(u(\theta,\eta)))\right\|^q_{\mathscr{H}^d_{s-\epsilon,s}} \left\| D(b_i(u(\theta,\tilde{\eta})))\right\|^q_{\mathscr{H}^d_{s-\epsilon,s}}\right] \\
&\quad \le \sup_{\theta,\,\eta} \mathrm{E}\left[\left\| D(b_i(u(\theta,\eta)))\right\|^{2q}_{\mathscr{H}^d_{s-\epsilon,s}}\right].
\end{aligned}
\tag{6.5}
$$

Arguing as for the term W_1 and using **P1**, we bound the expectation by $c\epsilon^{\frac{2-\beta}{2}q}$, and the remaining integrals are bounded by $(t-s+\epsilon)^q$, so that

$$
\mathrm{E}\left[\left(\int_{s-\epsilon}^{s} dr\, \|W_2\|^2_{\mathscr{H}^d}\right)^q\right] \le C(t-s+\epsilon)^q \epsilon^{\frac{2-\beta}{2}q}.
$$

Together with (6.1) and (6.4), this completes the proof. □

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