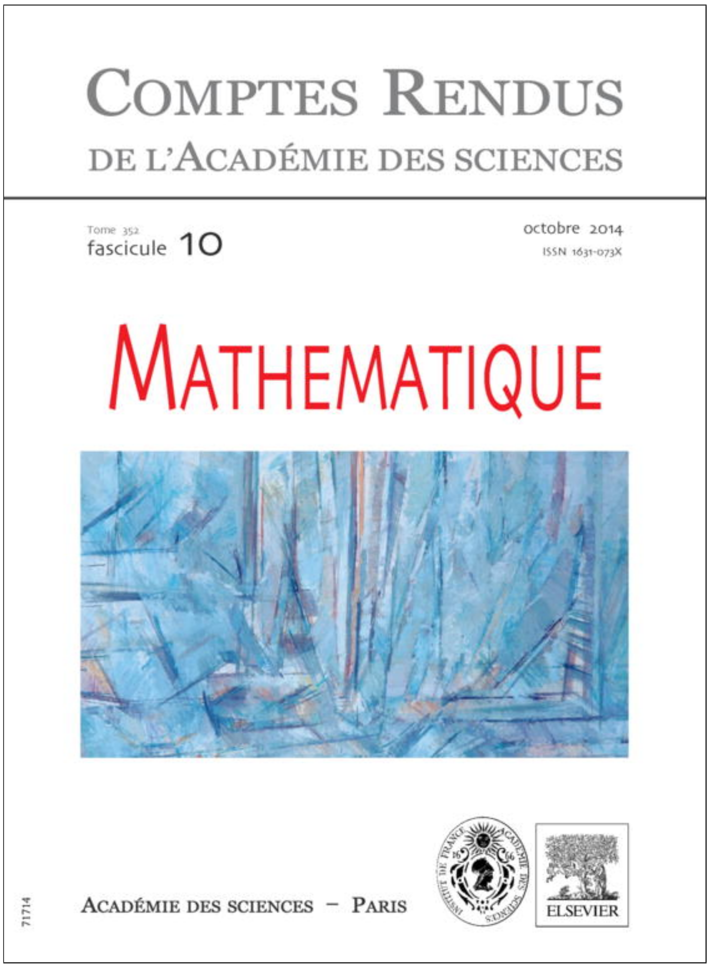
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LAN property for a simple Lévy process

Propriété LAN pour un processus de Lévy simple

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ABSTRACT

In this paper, we consider a simple Lévy process given by a Brownian motion and a compensated Poisson process, whose drift and diffusion parameters as well as its intensity are unknown. Supposing that the process is observed discretely at high frequency, we derive the local asymptotic normality (LAN) property. In order to obtain this result, Malliavin calculus and Girsanov's theorem are applied in order to write the log-likelihood ratio in terms of sums of conditional expectations, for which a central limit theorem for triangular arrays can be applied.

RÉSUMÉ

Dans cet article, nous considérons un processus de Lévy simple donné par un mouvement brownien et un processus de Poisson compensé, dont les paramètres et l'intensité sont inconnus. En supposant que le processus est observé à haute fréquence, nous obtenons la propriété de normalité asymptotique locale. Pour cela, le calcul de Malliavin et le théorème de Girsanov sont appliqués afin d'écrire le logarithme du rapport de vraisemblances comme une somme d'espérances conditionnelles, pour laquelle un théorème central limite pour des suites triangulaires peut être appliqué.

1. Introduction and main result

On a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$, we consider the stochastic process $X^{\theta,\sigma,\lambda} = (X_t^{\theta,\sigma,\lambda})_{t\geq 0}$ in $\mathbb{R}$ defined by

$$X_t^{\theta,\sigma,\lambda} = x_0 + \theta t + \sigma B_t + N_t - \lambda t, \tag{1}$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, $N = (N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$ independent of B, and we denote by $(\widetilde{N}_t^{\lambda})_{t\geq 0}$ the compensated Poisson process $\widetilde{N}_t^{\lambda} := N_t - \lambda t$. The parameters $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$ are

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unknown and Θ, Σ and Λ are closed intervals of $\mathbb{R}, \mathbb{R}_+^*$ and $\mathbb{R}_+^*$, where $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$. Let $\{\widehat{\mathcal{F}}_t\}_{t\geq 0}$ denote the natural filtration generated by B and N. We denote by $\mathrm{P}^{\theta,\sigma,\lambda}$ the probability law induced by the $\widehat{\mathcal{F}}_t$-adapted càdlàg stochastic process $X^{\theta,\sigma,\lambda}$, and by $\mathrm{E}^{\theta,\sigma,\lambda}$ the expectation with respect to $\mathrm{P}^{\theta,\sigma,\lambda}$. Let $\xrightarrow{\mathrm{P}^{\theta,\sigma,\lambda}}$ and $\xrightarrow{\mathcal{L}(\mathrm{P}^{\theta,\sigma,\lambda})}$ denote the convergence in $\mathrm{P}^{\theta,\sigma,\lambda}$-probability and in $\mathrm{P}^{\theta,\sigma,\lambda}$-law, respectively.

For fixed $(\theta_0, \sigma_0, \lambda_0) \in \Theta \times \Sigma \times \Lambda$, we consider an equidistant discrete observation of the process $X^{\theta_0,\sigma_0,\lambda_0}$ denoted by $X^n = (X_{t_0}, X_{t_1}, \ldots, X_{t_n})$, where $t_k = k\Delta_n$ for $k \in \{0, \ldots, n\}$, and $\Delta_n \leq 1$. We assume that $n\Delta_n \to \infty$, and $\Delta_n \to 0$, $n \to \infty$.

Given the process $(X_t^{\theta,\sigma,\lambda})_{t\geq 0}$, we denote by $p(\cdot; (\theta, \sigma, \lambda))$ the density of the vector $(X_{t_0}^{\theta,\sigma,\lambda}, \ldots, X_{t_n}^{\theta,\sigma,\lambda})$. In particular, the density of X^n is $p(\cdot; (\theta_0, \sigma_0, \lambda_0))$.

For $(u, v, w) \in \mathbb{R}^3$, set $\theta_n := \theta_0 + \frac{u}{\sqrt{n\Delta_n}}$, $\sigma_n := \sigma_0 + \frac{v}{\sqrt{n}}$, $\lambda_n := \lambda_0 + \frac{w}{\sqrt{n\Delta_n}}$.

The aim of this paper is to prove that the following LAN property for the likelihood at $(\theta_0, \sigma_0, \lambda_0)$ holds.

Theorem 1.1. *For all $z = (u, v, w) \in \mathbb{R}^3$, as $n \to \infty$,*

$$\log \frac{p(X^n; (\theta_n, \sigma_n, \lambda_n))}{p(X^n; (\theta_0, \sigma_0, \lambda_0))} \xrightarrow{\mathcal{L}(\mathrm{P}^{\theta_0,\sigma_0,\lambda_0})} z^{\mathsf{T}} \mathcal{N}\big(0, \Gamma(\theta_0, \sigma_0, \lambda_0)\big) - \frac{1}{2} z^{\mathsf{T}} \Gamma(\theta_0, \sigma_0, \lambda_0) z,$$

where $\mathcal{N}(0, \Gamma(\theta_0, \sigma_0, \lambda_0))$ is a centered $\mathbb{R}^3$-valued Gaussian vector with covariance matrix

$$\Gamma(\theta_0, \sigma_0, \lambda_0) = \frac{1}{\sigma_0^2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 + \frac{\sigma_0^2}{\lambda_0} \end{pmatrix}.$$

Theorem 1.1 extends in the linear case and in the presence of jumps the results of Gobet in [5] and [6] for multidimensional continuous elliptic diffusions. The main idea of these papers is to use the Malliavin calculus in order to obtain an expression for the derivative of the log-likelihood function in terms of a conditional expectation. Some extensions of Gobet's work with the presence of jumps are given, e.g., in [2,4], and [7]. However, in the present note, we estimate the coefficients and jump intensity parameters at the same time. Since we are dealing with a simple Lévy process with finite jumps, the explicit expression of the density could be used in order to derive the LAN property, as, e.g., in [1]. However, the main motivation for this paper is to show some of the important properties and arguments in order to prove the LAN property in the non-linear case, whose proof is non-trivial. In particular, we present four important lemmas of independent interest, which will be key elements in dealing with the non-linear case. The key argument consists in conditioning on the number of jumps within the conditional expectation, which expresses the transition density and the number of jumps in the conditioning random variable. When these two conditions relate to different jumps, one may use a large deviation principle in the estimate. When they are equal, one uses the complementary set. Within all these arguments, the Gaussian-type upper and lower bounds of the density conditioned on the jumps is again strongly used. This idea seems to have many other uses in the set-up of stochastic differential equations driven by a Brownian motion and a jump process. We remark here that a plain Itô–Taylor expansion would not solve the problem as higher moments of the Poisson process do not become smaller as the expansion order increases.

When the LAN property holds true, convolution and minimax theorems can be applied, and one can derive, in particular, lower bounds for the variance of the estimators. Using Proposition 2.1 below, one can check that the maximum likelihood estimators of $(\theta_0, \sigma_0, \lambda_0)$ are consistent and asymptotically normal with the asymptotic covariance matrix $\Gamma(\theta_0, \sigma_0, \lambda_0)^{-1}$ and the rate of convergence $(\sqrt{n\Delta_n}, \sqrt{n}, \sqrt{n\Delta_n})$ (see also [8]).

2. Preliminaries

In this section, we introduce the preliminary results needed for the proof of Theorem 1.1. In order to deal with the log-likelihood ratio in Theorem 1.1, we will use the following decomposition

$$\log \frac{p(X^n; (\theta_n, \sigma_n, \lambda_n))}{p(X^n; (\theta_0, \sigma_0, \lambda_0))} = \log \frac{p(X^n; (\theta_n, \sigma_n, \lambda_n))}{p(X^n; (\theta_n, \sigma_0, \lambda_n))} + \log \frac{p(X^n; (\theta_n, \sigma_0, \lambda_n))}{p(X^n; (\theta_n, \sigma_0, \lambda_0))} + \log \frac{p(X^n; (\theta_n, \sigma_0, \lambda_0))}{p(X^n; (\theta_0, \sigma_0, \lambda_0))}. \tag{2}$$

For each one of the above terms, we will use a mean value theorem and then analyze each term. We start as in Gobet [5] by applying the integration by parts formula of the Malliavin calculus on each interval $[t_k, t_{k+1}]$ to obtain an expression for the derivatives of the log-likelihood functions w.r.t. θ and σ. Moreover, using Girsanov's theorem, we obtain an expression for the log-likelihood function w.r.t. λ. In order to avoid confusion with the observed process, we consider on the same probability space $(\Omega, \mathcal{F}, \mathrm{P})$, the flow $Y^{\theta,\sigma,\lambda}(s, x) = (Y_t^{\theta,\sigma,\lambda}(s, x), t \geq s)$, $x \in \mathbb{R}$ on the time interval $[s, \infty)$ and with initial condition $Y_s^{\theta,\sigma,\lambda}(s, x) = x$ satisfying

$$Y_t^{\theta,\sigma,\lambda}(s, x) = x + \theta(t - s) + \sigma(W_t - W_s) + \widetilde{M}_t^{\lambda} - \widetilde{M}_s^{\lambda}, \tag{3}$$

where $W = (W_t)_{t\geq 0}$ is a Brownian motion and $M = (M_t)_{t\geq 0}$ is a Poisson process with intensity λ, where (B, N, W, M) are mutually independent. In particular, we write $Y_t^{\theta,\sigma,\lambda} \equiv Y_t^{\theta,\sigma,\lambda}(0, x_0)$, for all $t \geq 0$. For any $t > s$, we denote by $p^{\theta,\sigma,\lambda}(t -$

$s, x, y)$ the transition density of $Y_t^{\theta,\sigma,\lambda}$ conditioned on $Y_s^{\theta,\sigma,\lambda} = x$. We consider the Malliavin calculus on the Wiener space induced by the Brownian motion W. We denote by $\{\widetilde{\mathcal{F}}_t\}_{t\geq 0}$ the natural filtration generated by W and M, by $\widetilde{\mathrm{P}}_x^{\theta,\sigma,\lambda}$ the probability law induced by $Y^{\theta,\sigma,\lambda}(t_k, x)$, and by $\widetilde{\mathrm{E}}_x^{\theta,\sigma,\lambda}$ the corresponding expectation.

Proposition 2.1. *For all $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$, and $k \in \{0, \ldots, n-1\}$,*

$$\frac{\partial_\theta p^{\theta,\sigma,\lambda}}{p^{\theta,\sigma,\lambda}}(\Delta_n, x, y) = \frac{1}{\sigma}\widetilde{\mathrm{E}}_x^{\theta,\sigma,\lambda}\big[W_{t_{k+1}} - W_{t_k} \bigm| Y_{t_{k+1}}^{\theta,\sigma,\lambda} = y\big],$$
$$\frac{\partial_\sigma p^{\theta,\sigma,\lambda}}{p^{\theta,\sigma,\lambda}}(\Delta_n, x, y) = \frac{1}{\sigma \Delta_n}\widetilde{\mathrm{E}}_x^{\theta,\sigma,\lambda}\big[(W_{t_{k+1}} - W_{t_k})^2 \bigm| Y_{t_{k+1}}^{\theta,\sigma,\lambda} = y\big] - \frac{1}{\sigma},$$
$$\frac{\partial_\lambda p^{\theta,\sigma,\lambda}}{p^{\theta,\sigma,\lambda}}(\Delta_n, x, y) = \widetilde{\mathrm{E}}_x^{\theta,\sigma,\lambda}\left[-\frac{W_{t_{k+1}} - W_{t_k}}{\sigma} + \frac{\widetilde{M}_{t_{k+1}}^\lambda - \widetilde{M}_{t_k}^\lambda}{\lambda} \,\middle|\, Y_{t_{k+1}}^{\theta,\sigma,\lambda} = y\right].$$

We next present the four lemmas mentioned in the introduction. For all $m \geq 0$ and $k \in \{0, \ldots, n-1\}$, consider the events $\widehat{J}_{m,k} := \{N_{t_{k+1}} - N_{t_k} = m\}$ and $\widetilde{J}_{m,k} := \{M_{t_{k+1}} - M_{t_k} = m\}$.

Lemma 2.1. *For all $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$, $k \in \{0, \ldots, n-1\}$, and $m \geq 0$,*

$$\widetilde{\mathrm{P}}_x^{\theta,\sigma,\lambda}\big(\widetilde{J}_{m,k} \bigm| Y_{t_{k+1}}^{\theta,\sigma,\lambda} = y\big) = \frac{e^{-(y-x-m-(\theta-\lambda)\Delta_n)^2/(2\sigma^2\Delta_n)}\frac{(\lambda\Delta_n)^m}{m!}}{\sum_{i=0}^\infty e^{-(y-x-i-(\theta-\lambda)\Delta_n)^2/(2\sigma^2\Delta_n)}\frac{(\lambda\Delta_n)^i}{i!}}.$$

For all $j, p \geq 0$ and $k \in \{0, \ldots, n-1\}$, we introduce the random variable

$$S_j^p := \mathbf{1}_{\widehat{J}_{j,k}}\widetilde{\mathrm{E}}_{X_{t_k}}^{\theta,\sigma,\lambda}\big[(M_{t_{k+1}} - M_{t_k})^p \mathbf{1}_{\widetilde{J}_{j,k}^c} \bigm| Y_{t_{k+1}}^{\theta,\sigma,\lambda} = X_{t_{k+1}}\big].$$

Lemma 2.2. *For all $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$, $j, p \geq 0$ and $k \in \{0, \ldots, n-1\}$,*

$$S_j^p = \mathbf{1}_{\widehat{J}_{j,k}}\frac{\sum_{m=0:m\neq j}^\infty m^p e^{-(\sigma_0(B_{t_{k+1}}-B_{t_k})+j-m+(\theta_0-\theta-\lambda_0+\lambda)\Delta_n)^2/(2\sigma^2\Delta_n)}\frac{(\lambda\Delta_n)^m}{m!}}{\sum_{i=0}^\infty e^{-(\sigma_0(B_{t_{k+1}}-B_{t_k})+j-i+(\theta_0-\theta-\lambda_0+\lambda)\Delta_n)^2/(2\sigma^2\Delta_n)}\frac{(\lambda\Delta_n)^i}{i!}}. \tag{4}$$

We next fix $\alpha \in (0, \frac{1}{2})$, and analyze S_j^p in two separate cases as follows:

$$S_j^p = S_j^p \mathbf{1}_{\{|B_{t_{k+1}}-B_{t_k}|\leq\Delta_n^\alpha\}} + S_j^p \mathbf{1}_{\{|B_{t_{k+1}}-B_{t_k}|>\Delta_n^\alpha\}} =: S_{1,j}^p + S_{2,j}^p.$$

Furthermore, we write $S_{1,j}^p = S_{1,1,j}^p + S_{1,2,j}^p$, and $S_{2,j}^p = S_{2,1,j}^p + S_{2,2,j}^p$, where $S_{1,1,j}^p$ and $S_{2,1,j}^p$ contain the terms $\sum_{m<j}$, and $S_{1,2,j}^p$ and $S_{2,2,j}^p$ contain the terms $\sum_{m>j}$ in (4).

Lemma 2.3. *Assume that $|\theta - \theta_0| \leq \frac{C}{\sqrt{n\Delta_n}}$ and $|\lambda - \lambda_0| \leq \frac{C}{\sqrt{n\Delta_n}}$, for some constant $C > 0$. Then for all $\sigma \in \Sigma$, $j, p \geq 0$, $k \in \{0, \ldots, n-1\}$, and for n large enough,*

$$S_{1,1,j}^p \leq \mathbf{1}_{\widehat{J}_{j,k}}\frac{j!}{(\lambda\Delta_n)^j}\sum_{m<j} m^p e^{-\frac{(j-m)^2}{4\sigma^2\Delta_n}}\frac{(\lambda\Delta_n)^m}{m!}, \qquad S_{1,2,j}^p \leq \mathbf{1}_{\widehat{J}_{j,k}} e^{-\frac{1}{4\sigma^2\Delta_n}}\sum_{\ell>0}(\ell+j)^p\frac{(\lambda\Delta_n)^\ell}{\ell!},$$
$$S_{2,1,j}^p \leq j^p \mathbf{1}_{\widehat{J}_{j,k}}\mathbf{1}_{\{|B_{t_{k+1}}-B_{t_k}|>\Delta_n^\alpha\}}, \qquad S_{2,2,j}^p \leq \mathbf{1}_{\widehat{J}_{j,k}}\mathbf{1}_{\{|B_{t_{k+1}}-B_{t_k}|>\Delta_n^\alpha\}}\sum_{\ell=0}^\infty(\ell+j+1)^p\frac{(\lambda\Delta_n)^\ell}{\ell!}.$$

For all $p \geq 0$ and $k \in \{0, \ldots, n-1\}$, set

$$M_{1,p} := \sum_{j=0}^\infty j^p \mathrm{E}^{\theta_0,\sigma_0,\lambda_0}\big[\mathbf{1}_{\widehat{J}_{j,k}}\widetilde{\mathrm{E}}_{X_{t_k}}^{\theta,\sigma,\lambda}\big[\mathbf{1}_{\widetilde{J}_{j,k}^c} \bigm| Y_{t_{k+1}}^{\theta,\sigma,\lambda} = X_{t_{k+1}}\big] \bigm| \widehat{\mathcal{F}}_{t_k}\big],$$
$$M_{2,p} := \sum_{j=0}^\infty \mathrm{E}^{\theta_0,\sigma_0,\lambda_0}\big[\mathbf{1}_{\widehat{J}_{j,k}}\widetilde{\mathrm{E}}_{X_{t_k}}^{\theta,\sigma,\lambda}\big[(M_{t_{k+1}} - M_{t_k})^p \mathbf{1}_{\widetilde{J}_{j,k}^c} \bigm| Y_{t_{k+1}}^{\theta,\sigma,\lambda} = X_{t_{k+1}}\big] \bigm| \widehat{\mathcal{F}}_{t_k}\big].$$

Lemma 2.4. *Assume that* $|\theta_0-\theta|\le\frac{C}{\sqrt{n\Delta_n}}$ *and* $|\lambda_0-\lambda|\le\frac{C}{\sqrt{n\Delta_n}}$, *for some constant* $C>0$. *Then, for all* $\sigma\in\Sigma$, $p\ge 0$, *and* n *large enough, there exist constants* $C_1,C_2>0$ *such that for any* $\alpha\in(0,\frac{1}{2})$, $k\in\{0,\dots,n-1\}$,

$$M_{1,p}+M_{2,p}\le C_1 e^{-\frac{1}{C_2\Delta_n^{1-2\alpha}}}.$$

We next recall a convergence in probability result, and a central limit theorem for triangular arrays of random variables. For each $n\in\mathbb{N}$, let $(Z_{k,n})_{k\ge1}$ and $(\zeta_{k,n})_{k\ge1}$ be two sequences of random variables defined on the filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathrm{P})$, and assume that they are $\mathcal{F}_{t_{k+1}}$-measurable.

Lemma 2.5. *(See [3, Lemma 9].) Assume that* $\sum_{k=0}^{n-1}\mathrm{E}[Z_{k,n}|\mathcal{F}_{t_k}]\xrightarrow{\mathrm{P}}0$, *and* $\sum_{k=0}^{n-1}\mathrm{E}[Z_{k,n}^2|\mathcal{F}_{t_k}]\xrightarrow{\mathrm{P}}0$, *as* $n\to\infty$. *Then* $\sum_{k=0}^{n-1}Z_{k,n}\xrightarrow{\mathrm{P}}0$, *as* $n\to\infty$.

Lemma 2.6. *Assume that there exist real numbers* M *and* $V>0$ *such that as* $n\to\infty$,

$$\sum_{k=0}^{n-1}\mathrm{E}[\zeta_{k,n}|\mathcal{F}_{t_k}]\xrightarrow{\mathrm{P}}M,\qquad\sum_{k=0}^{n-1}\Big(\mathrm{E}\big[\zeta_{k,n}^2\mid\mathcal{F}_{t_k}\big]-\big(\mathrm{E}[\zeta_{k,n}|\mathcal{F}_{t_k}]\big)^2\Big)\xrightarrow{\mathrm{P}}V,\quad\textit{and}\quad\sum_{k=0}^{n-1}\mathrm{E}\big[\zeta_{k,n}^4|\mathcal{F}_{t_k}\big]\xrightarrow{\mathrm{P}}0.$$

Then as $n\to\infty$, $\sum_{k=0}^{n-1}\zeta_{k,n}\xrightarrow{\mathcal{L}(\mathrm{P})}\mathcal{N}+M$, *where* $\mathcal{N}$ *is a centered Gaussian variable with variance* V.

3. Proof of Theorem 1.1

Set $\theta(\ell):=\theta_n(\ell,u):=\theta_0+\frac{\ell u}{\sqrt{n\Delta_n}}$, $\sigma(\ell):=\sigma_n(\ell,v):=\sigma_0+\frac{\ell v}{\sqrt{n}}$, $\lambda(\ell):=\lambda_n(\ell,w):=\lambda_0+\frac{\ell w}{\sqrt{n\Delta_n}}$, for $\ell\in[0,1]$. Applying the Markov property and Proposition 2.1 to each term in (2), we obtain that

$$\begin{aligned}\log\frac{p(X^n;(\theta_n,\sigma_0,\lambda_0))}{p(X^n;(\theta_0,\sigma_0,\lambda_0))}&=\sum_{k=0}^{n-1}\log\frac{p^{\theta_n,\sigma_0,\lambda_0}}{p^{\theta_0,\sigma_0,\lambda_0}}(\Delta_n,X_{t_k},X_{t_{k+1}})\\&=\sum_{k=0}^{n-1}\frac{u}{\sqrt{n\Delta_n}}\int_0^1\frac{\partial_\theta p^{\theta(\ell),\sigma_0,\lambda_0}}{p^{\theta(\ell),\sigma_0,\lambda_0}}(\Delta_n,X_{t_k},X_{t_{k+1}})\mathrm{d}\ell\\&=\sum_{k=0}^{n-1}\frac{u}{\sqrt{n\Delta_n}}\frac{1}{\sigma_0}\int_0^1\widetilde{\mathrm{E}}_{X_{t_k}}^{\theta(\ell),\sigma_0,\lambda_0}\big[W_{t_{k+1}}-W_{t_k}\mid Y_{t_{k+1}}^{\theta(\ell),\sigma_0,\lambda_0}=X_{t_{k+1}}\big]\mathrm{d}\ell,\\\log\frac{p(X^n;(\theta_n,\sigma_n,\lambda_n))}{p(X^n;(\theta_n,\sigma_0,\lambda_n))}&=\sum_{k=0}^{n-1}\frac{v}{\sqrt{n}}\int_0^1\frac{\partial_\sigma p^{\theta_n,\sigma(\ell),\lambda_n}}{p^{\theta_n,\sigma(\ell),\lambda_n}}(\Delta_n,X_{t_k},X_{t_{k+1}})\mathrm{d}\ell\\&=\sum_{k=0}^{n-1}\frac{v}{\sqrt{n}}\int_0^1\bigg(\frac{1}{\sigma(\ell)\Delta_n}\widetilde{\mathrm{E}}_{X_{t_k}}^{\theta_n,\sigma(\ell),\lambda_n}\big[(W_{t_{k+1}}-W_{t_k})^2\mid Y_{t_{k+1}}^{\theta_n,\sigma(\ell),\lambda_n}=X_{t_{k+1}}\big]-\frac{1}{\sigma(\ell)}\bigg)\mathrm{d}\ell,\end{aligned}$$

and

$$\begin{aligned}\log\frac{p(X^n;(\theta_n,\sigma_0,\lambda_n))}{p(X^n;(\theta_n,\sigma_0,\lambda_0))}&=\sum_{k=0}^{n-1}\frac{w}{\sqrt{n\Delta_n}}\int_0^1\frac{\partial_\lambda p^{\theta_n,\sigma_0,\lambda(\ell)}}{p^{\theta_n,\sigma_0,\lambda(\ell)}}(\Delta_n,X_{t_k},X_{t_{k+1}})\mathrm{d}\ell\\&=\sum_{k=0}^{n-1}\frac{w}{\sqrt{n\Delta_n}}\int_0^1\widetilde{\mathrm{E}}_{X_{t_k}}^{\theta_n,\sigma_0,\lambda(\ell)}\bigg[-\frac{W_{t_{k+1}}-W_{t_k}}{\sigma_0}+\frac{\widetilde{M}_{t_{k+1}}^{\lambda(\ell)}-\widetilde{M}_{t_k}^{\lambda(\ell)}}{\lambda(\ell)}\,\bigg|\,Y_{t_{k+1}}^{\theta_n,\sigma_0,\lambda(\ell)}=X_{t_{k+1}}\bigg]\mathrm{d}\ell.\end{aligned}$$

Now using Eq. (3), we obtain the following expansion of the log-likelihood ratio

$$\log\frac{p(X^n;(\theta_n,\sigma_n,\lambda_n))}{p(X^n;(\theta_0,\sigma_0,\lambda_0))}=\sum_{k=0}^{n-1}(\xi_{k,n}+H_{k,n}+\eta_{k,n}+M_{k,n}+\beta_{k,n}-R_{k,n}),$$

where

$$\xi_{k,n} := \frac{u}{\sqrt{n\Delta_n}}\frac{1}{\sigma_0^2}\left(\sigma_0(B_{t_{k+1}} - B_{t_k}) - \frac{u\Delta_n}{2\sqrt{n\Delta_n}}\right),$$

$$H_{k,n} := \frac{u}{\sqrt{n\Delta_n}}\frac{1}{\sigma_0^2}\left(\widetilde{N}^{\lambda_0}_{t_{k+1}} - \widetilde{N}^{\lambda_0}_{t_k} - \int_0^1 \widetilde{\mathrm{E}}^{\theta(\ell),\sigma_0,\lambda_0}_{X_{t_k}}\left[\widetilde{M}^{\lambda_0}_{t_{k+1}} - \widetilde{M}^{\lambda_0}_{t_k} \,\middle|\, Y^{\theta(\ell),\sigma_0,\lambda_0}_{t_{k+1}} = X_{t_{k+1}}\right]\mathrm{d}\ell\right),$$

$$\eta_{k,n} := \frac{v}{\sqrt{n}}\int_0^1 \frac{1}{\Delta_n}\left(\frac{\sigma_0^2}{\sigma(\ell)^3}(B_{t_{k+1}} - B_{t_k})^2 - \frac{\Delta_n}{\sigma(\ell)}\right)\mathrm{d}\ell,$$

$$\begin{aligned}M_{k,n} := \frac{v}{\sqrt{n}}\int_0^1 \frac{1}{\Delta_n}\frac{1}{\sigma(\ell)^3}\Big\{&\big(\theta_0\Delta_n + \widetilde{N}^{\lambda_0}_{t_{k+1}} - \widetilde{N}^{\lambda_0}_{t_k}\big)^2 + 2\sigma_0(B_{t_{k+1}} - B_{t_k})\big(\theta_0\Delta_n + \widetilde{N}^{\lambda_0}_{t_{k+1}} - \widetilde{N}^{\lambda_0}_{t_k}\big)\\ &- \widetilde{\mathrm{E}}^{\theta_n,\sigma(\ell),\lambda_n}_{X_{t_k}}\Big[\big(\theta_n\Delta_n + \widetilde{M}^{\lambda_n}_{t_{k+1}} - \widetilde{M}^{\lambda_n}_{t_k}\big)^2\\ &+ 2\sigma(\ell)(W_{t_{k+1}} - W_{t_k})\big(\theta_n\Delta_n + \widetilde{M}^{\lambda_n}_{t_{k+1}} - \widetilde{M}^{\lambda_n}_{t_k}\big) \,\Big|\, Y^{\theta_n,\sigma(\ell),\lambda_n}_{t_{k+1}} = X_{t_{k+1}}\Big]\Big\}\mathrm{d}\ell,\end{aligned}$$

$$\begin{aligned}\beta_{k,n} := &-\frac{w}{\sqrt{n\Delta_n}}\frac{1}{\sigma_0^2}\left(\sigma_0(B_{t_{k+1}} - B_{t_k}) + \frac{w\Delta_n}{2\sqrt{n\Delta_n}} - \frac{u\Delta_n}{\sqrt{n\Delta_n}}\right)\\ &+ \frac{w}{\sqrt{n\Delta_n}}\int_0^1 \widetilde{\mathrm{E}}^{\theta_n,\sigma_0,\lambda(\ell)}_{X_{t_k}}\left[\frac{\widetilde{M}^{\lambda(\ell)}_{t_{k+1}} - \widetilde{M}^{\lambda(\ell)}_{t_k}}{\lambda(\ell)} \,\middle|\, Y^{\theta_n,\sigma_0,\lambda(\ell)}_{t_{k+1}} = X_{t_{k+1}}\right]\mathrm{d}\ell,\end{aligned}$$

$$R_{k,n} := \frac{w}{\sqrt{n\Delta_n}}\frac{1}{\sigma_0^2}\int_0^1\left(N_{t_{k+1}} - N_{t_k} - \widetilde{\mathrm{E}}^{\theta_n,\sigma_0,\lambda(\ell)}_{X_{t_k}}\left[M_{t_{k+1}} - M_{t_k} \,\middle|\, Y^{\theta_n,\sigma_0,\lambda(\ell)}_{t_{k+1}} = X_{t_{k+1}}\right]\right)\mathrm{d}\ell.$$

We next show that the random variables $\xi_{k,n}, \eta_{k,n}, \beta_{k,n}$ are the terms that contribute to the limit in Theorem 1.1, and $H_{k,n}, M_{k,n}$ and $R_{k,n}$ are the negligible contributions. That is,

Lemma 3.1. *As $n \to \infty$,*

$$\sum_{k=0}^{n-1}(H_{k,n} + M_{k,n} - R_{k,n}) \xrightarrow{\mathrm{P}^{\theta_0,\sigma_0,\lambda_0}} 0, \qquad \sum_{k=0}^{n-1}\mathrm{E}^{\theta_0,\sigma_0,\lambda_0}\big[\xi^4_{k,n} + \eta^4_{k,n} + \beta^4_{k,n} \,\big|\, \widehat{\mathcal{F}}_{t_k}\big] \xrightarrow{\mathrm{P}^{\theta_0,\sigma_0,\lambda_0}} 0,$$

$$\sum_{k=0}^{n-1}\mathrm{E}^{\theta_0,\sigma_0,\lambda_0}[\xi_{k,n} + \eta_{k,n} + \beta_{k,n}|\widehat{\mathcal{F}}_{t_k}] \xrightarrow{\mathrm{P}^{\theta_0,\sigma_0,\lambda_0}} -\frac{u^2}{2\sigma_0^2} - \frac{v^2}{2}\frac{2}{\sigma_0^2} - \frac{w^2}{2\sigma_0^2}\left(1 + \frac{\sigma_0^2}{\lambda_0}\right) + \frac{uw}{\sigma_0^2},$$

$$\begin{aligned}&\sum_{k=0}^{n-1}\Big(\mathrm{E}^{\theta_0,\sigma_0,\lambda_0}\big[\xi^2_{k,n} + \eta^2_{k,n} + \beta^2_{k,n} \,\big|\, \widehat{\mathcal{F}}_{t_k}\big] - \mathrm{E}^{\theta_0,\sigma_0,\lambda_0}[\xi_{k,n}|\widehat{\mathcal{F}}_{t_k}]^2 - \mathrm{E}^{\theta_0,\sigma_0,\lambda_0}[\eta_{k,n}|\widehat{\mathcal{F}}_{t_k}]^2 - \mathrm{E}^{\theta_0,\sigma_0,\lambda_0}[\beta_{k,n}|\widehat{\mathcal{F}}_{t_k}]^2\Big)\\ &\quad\xrightarrow{\mathrm{P}^{\theta_0,\sigma_0,\lambda_0}} \frac{u^2}{\sigma_0^2} + 2\frac{v^2}{\sigma_0^2} + \frac{w^2}{\sigma_0^2}\left(1 + \frac{\sigma_0^2}{\lambda_0}\right),\end{aligned}$$

$$\sum_{k=0}^{n-1}\Big(\mathrm{E}^{\theta_0,\sigma_0,\lambda_0}\big[\xi_{k,n}\eta_{k,n} \,\big|\, \widehat{\mathcal{F}}_{t_k}\big] - \mathrm{E}^{\theta_0,\sigma_0,\lambda_0}[\xi_{k,n}|\widehat{\mathcal{F}}_{t_k}]\mathrm{E}^{\theta_0,\sigma_0,\lambda_0}[\eta_{k,n}|\widehat{\mathcal{F}}_{t_k}]\Big) \xrightarrow{\mathrm{P}^{\theta_0,\sigma_0,\lambda_0}} 0,$$

$$\sum_{k=0}^{n-1}\Big(\mathrm{E}^{\theta_0,\sigma_0,\lambda_0}[\xi_{k,n}\beta_{k,n}|\widehat{\mathcal{F}}_{t_k}] - \mathrm{E}^{\theta_0,\sigma_0,\lambda_0}[\xi_{k,n}|\widehat{\mathcal{F}}_{t_k}]\mathrm{E}^{\theta_0,\sigma_0,\lambda_0}[\beta_{k,n}|\widehat{\mathcal{F}}_{t_k}]\Big) \xrightarrow{\mathrm{P}^{\theta_0,\sigma_0,\lambda_0}} -\frac{uw}{\sigma_0^2},$$

$$\sum_{k=0}^{n-1}\Big(\mathrm{E}^{\theta_0,\sigma_0,\lambda_0}\big[\eta_{k,n}\beta_{k,n} \,\big|\, \widehat{\mathcal{F}}_{t_k}\big] - \mathrm{E}^{\theta_0,\sigma_0,\lambda_0}[\eta_{k,n}|\widehat{\mathcal{F}}_{t_k}]\mathrm{E}^{\theta_0,\sigma_0,\lambda_0}[\beta_{k,n}|\widehat{\mathcal{F}}_{t_k}]\Big) \xrightarrow{\mathrm{P}^{\theta_0,\sigma_0,\lambda_0}} 0.$$

Finally, Lemma 2.6 applied to $\zeta_{k,n} = \xi_{k,n} + \eta_{k,n} + \beta_{k,n}$ concludes the proof of Theorem 1.1.

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References

[1] Y. Aït-Sahalia, J. Jacod, Fisher's information for discretely sampled Lévy processes, Econometrica 76 (2008) 727–761.
[2] E. Clément, S. Delattre, A. Gloter, Asymptotic lower bounds in estimating jumps, Bernoulli 20 (2014) 1059–1096.
[3] V. Genon-Catalot, J. Jacod, On the estimation of the diffusion coefficient for multi-dimensional diffusion processes, Ann. Inst. Henri Poincaré B, Probab. Stat. 29 (1993) 119–151.
[4] A. Gloter, E. Gobet, LAMN property for hidden processes: the case of integrated diffusions, Ann. Inst. Henri Poincaré B, Probab. Stat. 44 (2008) 104–128.
[5] E. Gobet, LAMN property for elliptic diffusions: a Malliavin calculus approach, Bernoulli 7 (2001) 899–912.
[6] E. Gobet, LAN property for ergodic diffusions with discrete observations, Ann. Inst. Henri Poincaré 38 (2002) 711–737.
[7] R. Kawai, LAN property for Ornstein–Uhlenbeck processes with jumps under discrete sampling, J. Theor. Probab. 26 (2013) 932–967.
[8] T. Ogihara, N. Yoshida, Quasi-likelihood analysis for the stochastic differential equation with jumps, Stat. Inference Stoch. Process. 14 (2011) 189–229.