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Introduction to Linear Programming and Integer Linear Programming

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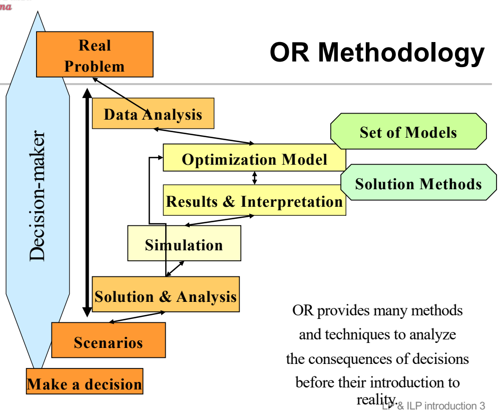
Introduction

- Outline - brief introduction to…
  - Modeling
  - Linear Programming
  - Integer Linear Programming
  - Solution Methods for LP and ILP
  - AMPL

OR Methodology



OR provides many methods and techniques to analyze the consequences of decisions before their introduction to reality.

Linear Programming

- **Linear programming (LP)** is a widely used mathematical modeling technique designed to help managers in planning and decision making relative to resource allocation.
- **Integer Linear Programming (ILP)**
  - An integer programming model is one where one or more of the decision variables has to take on an **integer or binary values** in the final solution.
- *Combinatorial Optimization (next)*

OR Methodology

- Identify a problem
- Get to know the problem context (get Data)
- Build a Model (Mathematical Model)
- Obtain a solution to the model (Algorithm)
- Understand the solution the real context
- Take the decision

OR Methodology

- Identify the problem
  - Describe in detail the problem, and identify all the components.
  - Identify the qualitative and quantitative aspects of the problem.
- Get to know the problem context
  - Identify the relationships of the problem with the context of the organization.
  - Obtain the relevant data.
  - Verify the data.

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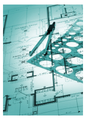
OR Methodology

- Build a model
  - A Model is an abstract representation of a real world system
  - Simplification is the very essence of Modeling
  - Which components should be included in the model?
  - Types of models:



- Linear programming
- Integer Linear Programming
- Combinatorial Optimization
- Networks
- Non-linear
- Simulation
- Etc.

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OR Methodology

- Obtain a solution to the model
  - Which are the adequate solution methods to be applied to the model?
    - Exact Methods
    - Heuristics and Metaheuristics
  - Which is the best software to be applied?
    - Commercial software
    - Algorithm design and code

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OR Methodology

- Understand the solution the real context
  - Does the solution obtained makes sense in the real life?
  - Did we have considered all relevant components?
- Take the decision
  - Evaluate the impact of the solution
  - Evaluate the decision process (model + method)
  - Review frequently the impact of the decision.

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Linear Programming

- Linear programming (LP) is a widely used mathematical modeling technique designed to help managers in planning and decision making relative to resource allocation.

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Linear Programming

- The Linear Programming Models have 4 properties in common:
  - All problems seek to maximize or minimize some quantity (**the objective function**).
  - Restrictions or **constraints** that limit the degree to which we can pursue our objective are present.
  - There must be alternative courses of action from which to choose.
  - The objective and constraints in problems must be expressed in terms of **linear equations or inequalities**.

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Linear Programming

- Formulating a linear program involves developing a mathematical model to represent the managerial problem.
- The steps in formulating a linear program are:
  - **Completely understand** the managerial problem being faced.
  - Identify the objective and the constraints.
  - Define the **decision variables**.
  - Use the decision variables to write mathematical expressions for the **objective function and the constraints**.

Linear Programming Methodology

- Understand well the problem (by words)
  - What is the problem
  - What is the decision we have to take? Can we make quantitative?
  - What are the constraints on the decisions?
  - What is the objective or how we are going to evaluate the solution?
- Define the decision variables (math)
- Define the constraints on the variables (math)
- Define the objective function (math)
- Solve it!!!

Example 2: Call Center

- A call center is hiring personnel since it is expanding to a 24h working period.
  - The call center works 24h a day, and needs personnel to attend the customers every hour.
  - The human recourses and the operations directors have estimate the number of persons in each interval of time.
  - There are 6 intervals of time (4 hours each).
  - The contract of the new employees is for 8h in a row.

Example 2: Call Center

	Interval of times					
Period	1 0:00 – 4:00 h	2 4:00 – 8:00 h	3 8:00 – 12:00 h	4 12:00 – 16:00 h	5 16:00 – 20:00 h	6 20:00 – 24:00 h
Employees needed	9	8	3	7	5	4

- Which is the minimal number of employees that should be hired?

Example 2: Call Center

- Describe the problem by words.
  - What is the problem?
  - What is the decision we have to take? Can we make quantitative?
  - What are the constraints on the decisions?
  - What is the objective or how we are going to evaluate the solution?

Example 2: Call Center

- Define the decision variables:
  - Number of employees to be hired to start at the beginning of period 1 (x1)
  - Number of employees to be hired to start at the beginning of period 2 (x2)
  - Number of employees to be hired to start at the beginning of period 3 (x3)
  - Number of employees to be hired to start at the beginning of period 4 (x4)
  - …

Ejemplo 1: Personal laboratorio

- Define the constraints
  - In period 1, there is the need of 9 employees…
  - In period 2, there is the need of 8 employees…
  - In period 3, there is the need of 3 employees…
  - …

	Periods					
	1 **0:00-4:00**	**2** **4:00-8:00**	**3** **8:00-12:00**	**4** **12:00-16:00**	**5** **16:00-20:00**	**6** **20:00-24:00**
0:00	X^1	X^1				
4:00		X^2	X^2			
8:00			X^3	X^3		
12:00				X^4	X^4	
16:00					X^5	X^5
20:00	X^6					X^6
Personnel	≥ 9	≥ 8	≥ 3	≥ 7	≥ 5	≥ 4

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Example 2: Call Center

- The constraints…
  - The number of persons that start at beginning of period 1 + the ones that continue from period 6 must be greater or equal to 9:
    - X6 + X1 ≥ 9
  - The number of persons that start at beginning of period 2 + the ones that continue from period 1 must be greater or equal to 8.
    - X1 + X2 ≥ 8
  - …

Example 2: Call Center

- Define the constraints:
  - X6 + X1 ≥ 9
  - X1 + X2 ≥ 8
  - X2 + X3 ≥ 3
  - X3 + X4 ≥ 7
  - X4 + X5 ≥ 5
  - X5 + X6 ≥ 4
  - Xj ≥ 0, j= 1,...,6.

Example 2: Call Center

- Define the objective function
- Min Z= X1 + X2 + X3 + X4 + X5 + X6

Example 2: Call Center

- The complete model
- Min Z= X1 + X2 + X3 + X4 + X5 + X6
  - Subject to:
    - X6 + X1 ≥ 9
    - X1 + X2 ≥ 8
    - X2 + X3 ≥ 3
    - X3 + X4 ≥ 7
    - X4 + X5 ≥ 5
    - X5 + X6 ≥ 4
    - Xj ≥ 0, j= 1,...,6.

Solution Methods

- Obtain the optimal solution of the LP Model
- **Exact Method**
  - The PL models are "easy" to be solved
  - Simplex Method (Dantzig 1947)
    - Primal simplex
    - Dual simplex
  - Interior Point Method (Karmarkar 1984)
  - ALGORITMS / SOFTWARE
    - Excel Solver
    - Commercial Software
    - Open Software

Solution Methods

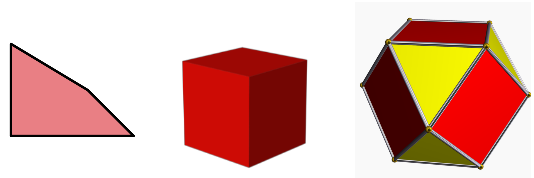
- LINEAR PROGRAMMING SOFTWARE SURVEY
  - https://www.informs.org/ORMS-Today/Public-Articles/June-Volume-44-Number-3/Linear-Programming-Software-Survey
  - June 2017
    - https://www.informs.org/ORMS-Today/OR-MS-Today-Software-Surveys/Linear-Programming-Software-Survey
  - Vendors
    - GLPK (GNU Linear Programming Kit), Gurobi Optimizer, IBM ILOG CPLEX Optimization Studio, LINDO & LINGO, Premium Solver Pro, etc
  - Prices
    - Free to $4000/licence

Simplex Method

- Dantzig (1947)
  - The set of solutions is a convex set.
  - If there is a optimal solution, there exist an optimal solution in a corner point (or extreme point)
  - An extreme point always have at least two adjacent extreme points.
  - If a extreme point has no adjacent extreme points with better value for the objective function, then it is the optimal solution.

Simplex Method

- In Euclidean space, an object is convex if for every pair of points within the object, every point on the straight line segment that joins them is also within the object.



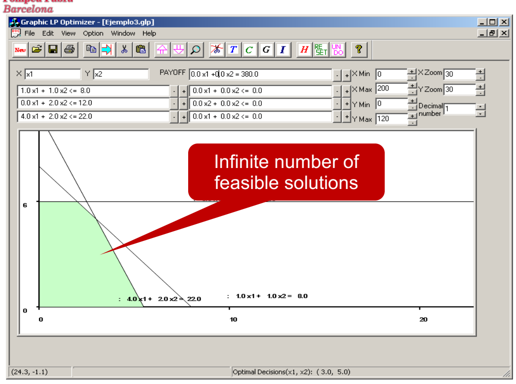
Simplex Method

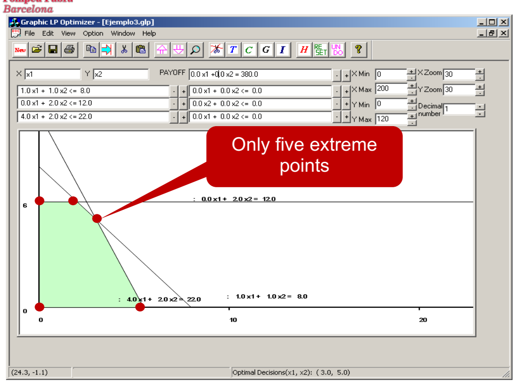
- Basic theorem
- If exist a optimal solution there exist one in a extreme point.
- Simplex method
  - Look only in the extreme point (or corner points of the polytope)
  - There are a finite number of extreme points

Simplex Method

- Example of Linear Programming

$$\begin{aligned}
&\textit{Maximizar} \quad Z = 60x_1 + 40x_2 \\
&sa \\
&x_1 + x_2 \le 8 \\
&2x_2 \le 12 \\
&4x_1 + 2x_2 \le 22 \\
&x_1 \ge 0, \quad x_2 \ge 0
\end{aligned}$$





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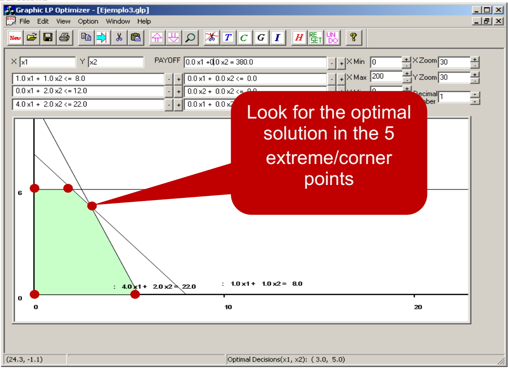

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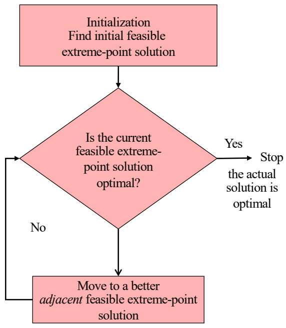
Simplex Method

- An iterative procedure
  - Move from extreme point to extreme point
  - Optimality test in the Simplex Method:
    - If a extreme-corner solution has no adjacent solutions that are better, then it must be an optimal solution



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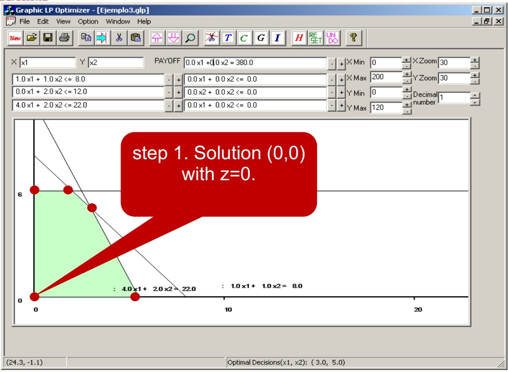

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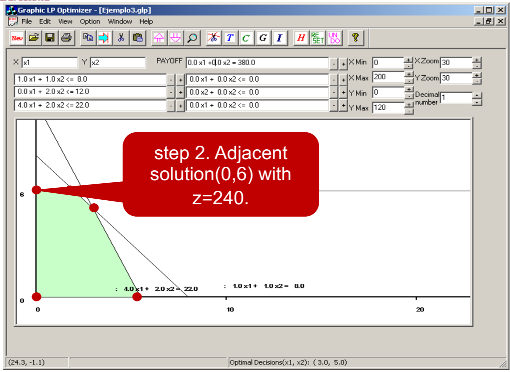

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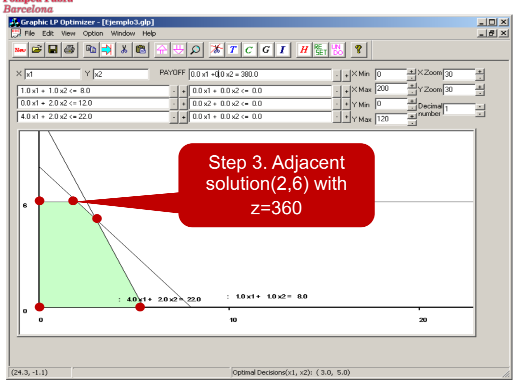

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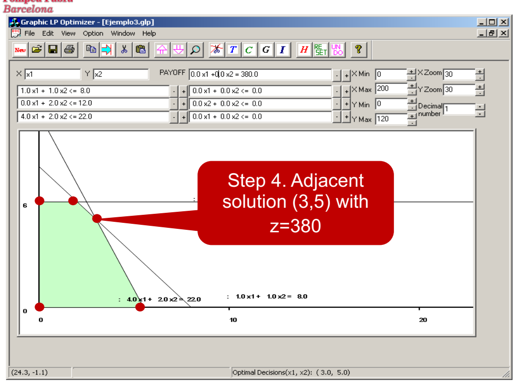
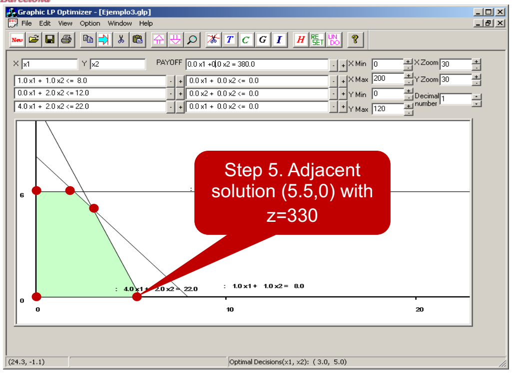
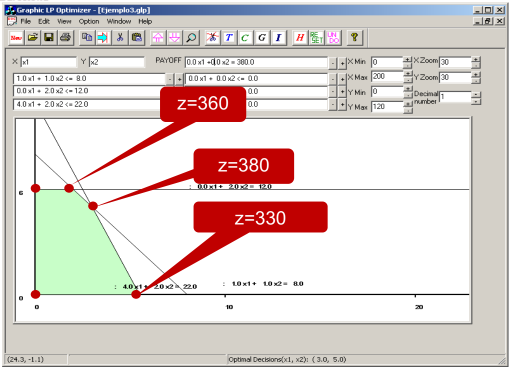
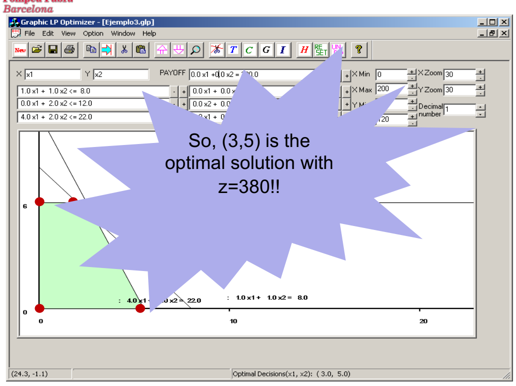
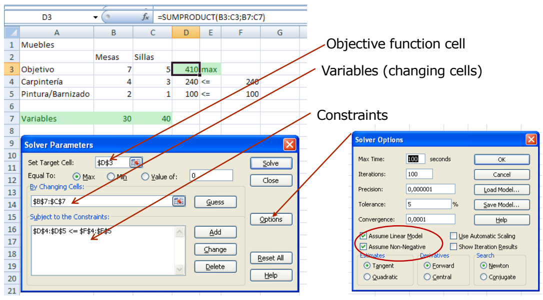








Excel Solver



Example 2: Call Center

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- Which is the minimal number of employees that should be hired?

Example 2: Call Center

- The complete model
- Min Z= X1 + X2 + X3 + X4 + X5 + X6
  - Subject to:
    - $X6 + X1 \geq 9$
    - $X1 + X2 \geq 8$
    - $X2 + X3 \geq 3$
    - $X3 + X4 \geq 7$
    - $X4 + X5 \geq 5$
    - $X5 + X6 \geq 4$
    - $Xj \geq 0$, j= 1,...,6.

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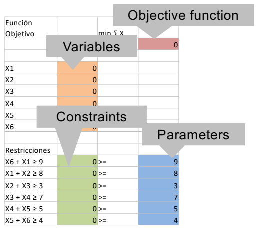
Excel Solver

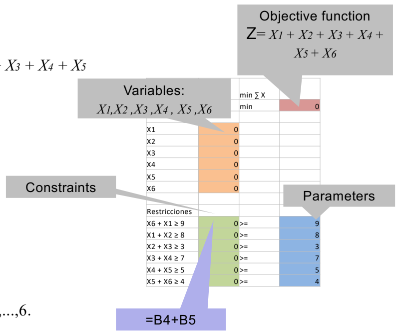
- Excel file
- Specify where to locate …
  - parameters
  - variables
  - constraints
  - Objective function



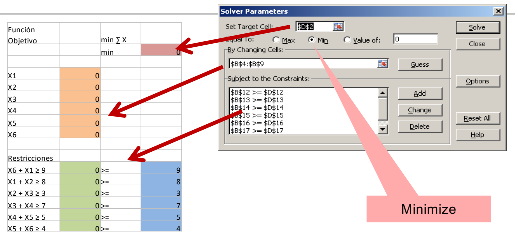
Excel Solver

- Min Z= $X_1 + X_2 + X_3 + X_4 + X_5 + X_6$
- Sujeto a:

$$X_6 + X_1 \geq 9$$
$$X_1 + X_2 \geq 8$$
$$X_2 + X_3 \geq 3$$
$$X_3 + X_4 \geq 7$$
$$X_4 + X_5 \geq 5$$
$$X_5 + X_6 \geq 4$$
$$X_j \geq 0, \quad j = 1,\ldots,6.$$



Excel Solver



Example 2: Call Center

- Excel file
  - callcenter.xlsx

Objective Function		min ∑ X	
		min	19
X1	8		
X2	0		
X3	5		
X4	2		
X5	3		
X6	1		
Constraints			
X6 + X1 ≥ 9	9	>=	9
X1 + X2 ≥ 8	8	>=	8
X2 + X3 ≥ 3	5	>=	3
X3 + X4 ≥ 7	7	>=	7
X4 + X5 ≥ 5	5	>=	5
X5 + X6 ≥ 4	4	>=	4

Example 2: Call Center

	Interval of times					
Period	1 0:00 – 4:00 h	2 4:00 – 8:00 h	3 8:00 – 12:00 h	4 12:00 – 16:00 h	5 16:00 – 20:00 h	6 20:00 – 24:00 h
Employees needed	9	8	3	7	5	4
Empoyees working	8+1	0+8	5+0	2+5	3+2	1+3

Integer Linear Programming

- Not every problem faced by businesses can easily be modeled as linear programming model…
- A large number of decision problems can be solved only if variables have integer values.
- Examples?

Integer Programming

- An integer programming model is one where one or more of the decision variables has to take on an integer value in the final solution.
- An integer variable that can only take a value equal to 0 or to 1 is called binary variable
  - Pure integer programming where all variables have integer values.
  - Mixed-integer programming where some but not all of the variables will have integer values.
  - Zero-one integer programming are special cases in which all the decision variables must have integer solution values of 0 or 1.

Integer Programming Example: Car rental Flipcar

- Flipcar, a car rental company, is planning to offer electrical cars to its clients.
- The choice is limited to two models, and the manager wishes to decide on how many cars of each type to buy, if
  - There is limited parking space for charging the cars
  - There is a budget limit
  - The maintenance services of the company has limited capacity
- The goal is to maximize profits.

Integer programming

$$\text{Max } Z = X_1 + 1.4X_2$$

s.a.

$$\begin{aligned} X_1 + 0.5X_2 &\le 6 \\ 0{,}5X_1 + X_2 &\le 5.5 \\ X_1 + X_2 &\le 6.8 \\ X_1, X_2 &>= 0 \text{ and integer} \end{aligned}$$

Being

X_1 = Number of electrical cars of type 1 to buy

X_2 = Number of electrical cars of type 2 to buy

Approaches for solving the problem

- Solve the LP relaxation
  - Forget about the integer constraints

$$\text{Max } Z = X_1 + 1.4X_2$$

s.a.

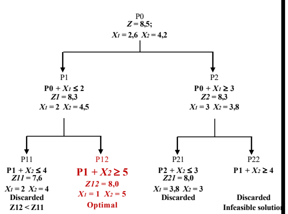
$$\begin{aligned} X_1 + 0.5X_2 &\le 6 \\ 0{,}5X_1 + X_2 &\le 5.5 \\ X_1 + X_2 &\le 6.8 \\ X_1, X_2 &\ge 0 \end{aligned}$$

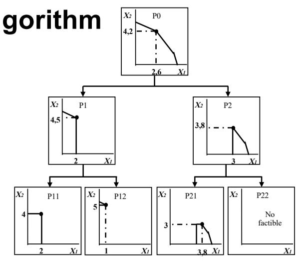
  - What is the solution?
  - Can you round the solution?

Approaches for solving the problem

- Solve the LP relaxation and round-off values
  - Rounding may provide with a non-optimal solution.
  - Rounding may provide non-feasible solutions.
- Enumerate all possible solutions
  - Enumeration is generally not possible for large problems.
- Use Branch-and-bound algorithm
- …

Branch and Bound algorithm





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Product-mix planning in an Airplane Plant

- You're the manager of an airplane plant and you want to determine the best product-mix of your six models to produce. The six models currently under production are the Rocket, Meteor, Streak, Comet, Jet, and Biplane. Each plane has a known profit contribution. There is also a fixed cost associated with the production of any plane in a period.

Source: Optimization Modeling with Lingo.

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Product-mix planning in an Airplane Plant

- The profit and fixed costs are given in the following table:

Plane	Profit	Setup
Rocket	30	35
Meteor	45	20
Streak	24	60
Comet	26	70
Jet	24	75
Biplane	30	30

Source: Optimization Modeling with Lingo.

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Product-mix planning in an Airplane Plant

- Each plane is produced using six raw materials—steel, copper, plastic, rubber, glass, and paint.
- The units of these raw materials required by the planes as well as the total availability of the raw materials are:

	Rocket	Meteor	Streak	Comet	Jet	Biplane	Available
Steel	1	4	0	4	2	1	800
Copper	4	5	3	0	1	0	1160
Plastic	0	3	8	0	1	0	1780
Rubber	2	0	1	2	1	5	1050
Glass	2	4	2	2	2	4	1360
Paint	1	4	1	4	3	4	1240

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Product-mix planning in an Airplane Plant

- The problem is to determine the final mix of products that maximizes net profit (gross profit – setup costs) without exceeding the availability of any raw material.
- Your brand new Meteor model has the highest profit per unit of anything you've ever manufactured and the lowest setup cost.
- Maybe you should build nothing but Meteors? Then again, maybe not.

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Product-mix planning in an Airplane Plant

- Model

Q_i = Quantity to be produced of airplane i; (i = 1(Rocket),...,6 (Biplane))

$$B_i = \begin{cases} 1 \text{ if plane i is build.} \\ 0 \text{ if plane is not build.} \end{cases} \text{for (i = 1(Rocket),...,6 (Biplane))}$$

$$MAX \quad Z = \sum_{i=1}^{6} \left(profit_i * Q_i - setup_i * B_i \right)$$

$$s.t.$$

$$\sum_{i=1}^{6} usage_{ij} * Q_i \leq available_j \quad \text{for } j = 1,\dots,6 \quad \text{(raw materials)}$$

$$Q_i \leq M * B_i \quad \text{for i} = 1,\dots,6 \quad \text{(planes)}$$

$$Q_i \geq 0 \text{ and integer; } B_i \text{ binary}$$

What is the value of M?

	Rocket	Meteor	Streak	Comet	Jet	Biplane	Available
Steel	800	200	10000	200	400	800	800
Copper	290	232	386.6667	10000	1160	10000	1160
Plastic	10000	593.3333	222.5	10000	1780	10000	1780
Rubber	525	10000	1050	525	1050	210	1050
Glass	680	340	680	680	680	340	1360
Paint	1240	310	1240	310	413.3333	310	1240
400	290	200	222.5	200	400	210	

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Binary variables

- Binary variables add a new layer of flexibility in modeling real situations where appears selections, choices or other conditions.
- There are several prototype problems such as:
  - Location problems
  - Routing problems
  - Set covering problems
  - Scheduling problems
  - and many, many more …

Algebraic Modeling Languages

- High-level computer programming languages for describing and solving mathematical optimization problems.
- Advantages
  - Computer model resembles mathematical notation
  - Faster modeling cycles
  - Modeling and future maintenance becomes easier and more reliable.
  - Direct change of solver
  - No need to know specifics from each solver
  - Easy deployment for real-sized large-scale problems

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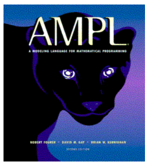
Mathematical Programming Languages

- AIMMS
- AMPL
- GAMS
- LINGO
- OPL ide (CPLEX)
- MOSEL-XPRESS
- CMPL COIN-OR (open source)
- …



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AMPL

- www.ampl.com
  - http://ampl.com/products/ampl/ampl-for-students/
  - Student version is limited to 300 variables and 300 constraints and objectives
- Download the student version (amplcml.zip)
  - AMPL program (ampl.exe)
  - Solver Gurobi (gurobi.exe) LP & MIP
  - Solver CPLEX (cplex.exe) LP & MIP & QP
  - Solver MINOS (minos.exe) NLP

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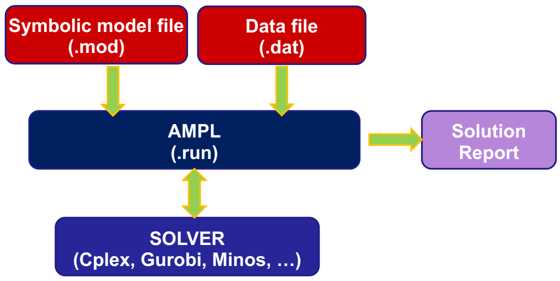
AMPL workflow



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Example: Call Center

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  - The call center works 24h a day, and needs personnel to attend the customers every hour.
  - The human recourses and the operations directors have estimate the number of persons in each interval of time.
  - There are 6 intervals of time (4 hours each).
  - The contract of the new employees is for 8h in a row.

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- Which is the minimal number of employees that should be hired?

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Example: Call Center

- The complete model
- Min Z= X1 + X2 + X3 + X4 + X5 + X6
  - Subject to:
    - $X6 + X1 \geq 9$
    - $X1 + X2 \geq 8$
    - $X2 + X3 \geq 3$
    - $X3 + X4 \geq 7$
    - $X4 + X5 \geq 5$
    - $X5 + X6 \geq 4$
    - $Xj \geq 0$, j= 1,...,6.

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Example: Call Center

- Model callcenter.mod

```
set EMPL := 1 .. 6;
var X {j in EMPL} >= 0;
minimize Total:  sum {j in EMPL} X[j];
subject to Cap:X[6] + X[1] >= 9;
subject to Cap1:X[1] + X[2] >= 8;
subject to Cap2:X[2] + X[3] >= 3;
subject to Cap3:X[3] + X[4] >= 7;
subject to Cap4:X[4] + X[5] >= 5;
subject to Cap5:X[5] + X[6] >= 4;
```

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Example: Call Center

- AMPL console
  - ampl: reset;
  - ampl: model callcenter.mod;
  - option solver cplex;
  - ampl: solve;
  - CPLEX 12.8.0.0: optimal solution; objective 19
    6 dual simplex iterations (0 in phase I)

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High Note Sound Company

$$\begin{aligned} \max\ & 50x_1 + 120x_2 \\ \text{s. to}\ & 2x_1 + 4x_2 \leq 80 \\ & 3x_1 + x_2 \leq 60 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- .mod file (highnote.mod)

```
var x1 >= 0; # Number of CD players to produce
var x2 >= 0; # Number of stereo receivers to produce

maximize Profit: 50*x1 + 120*x2;

subject to Electrician: 2*x1 + 4*x2 <= 80;
subject to Technician: 3*x1 + x2 <= 60;
```

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AMPL commands file

- .run file (highnote.run)

```
reset;
option solver cplex;
option cplex_options 'sensitivity';

model highnote.mod;
solve;

display x1,x2;
display Profit;
```

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AMPL

- AMPL console

```
ampl: include highnote.run;
CPLEX 12.6.1.0: sensitivity
CPLEX 12.6.1.0: optimal solution; objective 2400
1 dual simplex iterations (1 in phase I)
suffix up OUT;
suffix down OUT;
suffix current OUT;
x1 = 0
x2 = 20
Profit = 2400
ampl:
```

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Good modeling practices

- Use meaningful names («production» not «x»)
- Use comments to describe any declaration
- Organize programs in a clear flow

High Note Sound Company

- highnote2.mod

```
# Symbolic data
set PRODUCT;
set PROCESS;

param profit{i in PRODUCT};
param hours{j in PROCESS, i in PRODUCT};
param maxHours{j in PROCESS};

# Symbolic model
# Variables
var production{i in PRODUCT} >= 0;
# Objective
maximize Profit: sum{i in PRODUCT} profit[i]*production[i];

# Constraints
subject to ResourcesAvailable{j in PROCESS}:
sum{i in PRODUCT} hours[j,i]*production[i] <= maxHours[j];
```


High Note Sound Company

- highnote2.dat

```
data;
set PRODUCT:= CDplayer stereoRecv;
set PROCESS:= Electrician Technician;

param profit:=
  CDplayer    50
  stereoRecv 120;


param hours:
              CDplayer  stereoRecv:=
Electrician      2          4
Technician       3          1;


param maxHours:=
  Electrician 80
  Technician  60;
```


High Note Sound Company

- highnote2.run

```
# High Note Sound Company - Example 2
# Commands file    ---------------------------------

reset;
option solver cplex;

model highnote2.mod;
data highnote2.dat;
expand;

solve;

display production;
display Profit;
```


Some AMPL keywords

- Declarations:
  - set
  - param
  - var
- Objectives:
  - maximize
  - minimize
- Constraints:
  - subject to

AMPL commands

- display <value>;
  - <value> is any set, parameter or variable of the model
  - Arranges the information into lists and tables.
- expand;
  - Displays the current instantiated model.
- show;
  - Displays the names of the sets, parameters, variables, constraints, and objective function of current LP model.
- reset;
  - Cleans the current working LP problem.
  - reset data; Keeps the models but cleans the data information.

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Real Problems..

- Many real problems do not have this property…
- Many real problems have a large-scale dimension
  - Large number of variables
  - Large number of constraints
  - Need to be solved in very short time…
- Metaheuristics need to be applied!
- Examples
  - * Location problems
  - * Routing problems
  - * Set covering problems
  - * Scheduling problems

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