A TRUNCATED TWO–SCALES REALIZED VOLATILITY ESTIMATOR

Christian Brownlees*  Eulalia Nualart*  Yucheng Sun*

June 2016

Abstract

This paper introduces a novel estimator of the integrated volatility of asset prices based on high frequency data that is consistent in the presence of price jumps and market microstructure noise. We begin by introducing a jump signaling indicator based on a local average of intra-daily returns that allows to detect jumps when the price is contaminated by noise. We then combine this technique with the two-scales realized volatility estimator to introduce the so called truncated two-scales realized volatility estimator (TTSRV). We establish consistency of the TTSRV in the presence of finite or infinity activity jumps and noise. In case of finite activity jumps, we also establish the asymptotic distribution of the estimator. A simulation study shows that the TTSRV performs satisfactorily in finite samples and that it out–performs a number of alternative estimators recently proposed in the literature.

Keywords: integrated volatility, two-scales realized volatility estimator, jumps, market microstructure noise

* Department of Economics and Business, Universitat Pompeu Fabra and Barcelona GSE, e-mail: christian.brownlees@upf.edu, eulalia@nualart.es, yucheng.sun@upf.edu
We have benefited from comments by Nour Meddahi, George Tauchen and conference participants to the “Financial Econometrics Conference” of the Toulouse School of Economics (Toulouse, 13 and 14 of May 2016). Christian Brownlees and Eulalia Nualart acknowledge financial support from Spanish Ministry of Science and Technology (Grant MTM2015-67304-P) and from Spanish Ministry of Economy and Competitiveness through the Severo Ochoa Programme for Centres of Excellence in R&D (SEV-2011-0075). Eulalia Nualart’s research is supported in part by the European Union programme FP7-PEOPLE-2012-CIG under grant agreement 333938.
1 Introduction

The volatility of asset prices is a fundamental ingredient for asset pricing, risk management and portfolio allocation. Over the last decade, the financial econometrics literature has developed a new generation of estimators of the daily volatility of asset prices based on intra–daily data typically referred to as realized volatility estimators. The classic realized volatility estimator of Andersen, Bollerslev, Diebold, and Labys (2003) for example is defined as the sum of the squares of high–frequency intra–daily returns. Under appropriate assumptions, this estimator provides a consistent estimate of the quadratic variation of asset prices when prices follow a continuous stochastic model and are directly observed (see e.g. Barndorff-Nielsen and Shephard (2002) and Andersen et al. (2003)).

It is well acknowledged in the literature that asset prices exhibit discontinuities in their sample paths and that are also contaminated by market microstructure noise (see e.g. Barndorff-Nielsen and Shephard (2006) and Hansen and Lunde (2012)). The presence of discontinuities has motivated modelling prices as a combination of a continuous and a jump process. However, allowing for a jump component makes it more challenging to estimate the quadratic variation of the continuous part, which is typically the object of interest from an economic perspective. The presence of market microstructure noise also poses challenges to the estimation of the quadratic variation. In fact, in the presence of noise standard realized volatility estimators are inconsistent as the sampling frequency of the data increases.

These two important stylized facts of asset prices have motivated the development of a number of estimators which are consistent in the presence of jumps, noise or both. An estimator that is robust to price jumps is the truncated realized volatility estimator introduced by Mancini (2008, 2009), which deals with both finite and infinite activity jumps. Moreover, different realized power and multipower variation estimators that are also robust to price jumps have been introduced, see e.g. Barndorff-Nielsen and Shephard (2004) and Corsi, Pirino, and Reno (2010) for the case of finite activity jumps, and Barndorff-Nielsen, Shephard, and Winkel (2006), Woerner (2006), Jacod (2008), and Jacod and Todorov (2014) for infinite activity jumps. Consistent estimators in the pres-
ence of noise are the two-scales realized volatility \cite{Zhang} and the realized kernels \cite{Barndorff-Nielsen}. Finally, contributions that propose estimators that are consistent in the presence of both finite activity jumps and noise include, among others, Podolskij and Vetter \cite{Podolskij}, Fan and Wang \cite{Fan}, Barunik and Vacha \cite{Barunik} and Christensen, Oomen, and Podolskij \cite{Christensen}.

This paper contributes to this latter strand of the literature by developing a novel realized volatility estimator that is consistent in the presence of both finite or infinity activity jumps and noise. We do so by combining a truncation technique in the spirit of \cite{Mancini} to deal with the jumps, together with the idea of local average of intra-daily returns developed in \cite{Zhang} to deal with the market microstructure noise. The standard truncation technique introduced by \cite{Mancini} consists of excluding the intra-daily returns larger than a threshold (in absolute value) from the estimation of the quadratic variation, as these are likely to contain a realization of a jump. However, this jump truncation strategy fails if the efficient price process is contaminated by market microstructure noise. We overcome this hurdle by introducing a truncation technique based on a local average of intra-daily returns. We show that such local average smooths away the effect of the noise and retains the property of being large when a jump is realized. We then estimate the quadratic variation using the two-scales realized volatility estimator after truncating the intervals in which the local average is larger than a threshold. We call our estimator the truncated two-scales realized volatility estimator (TTSRV). We show that this estimator is consistent in the presence of finite or infinite activity jumps. In the case where jumps have finite activity, we derive its asymptotic distribution. A simulation study is used to assess the finite sample properties of the TTSRV estimator. The study shows that when the price process is affected by both jumps and noise, our proposed estimator delivers a significant improvement over other commonly employed realized volatility estimators: the truncated realized volatility \cite{Mancini}, the two-scales realized volatility \cite{Zhang}, the bipower variation \cite{Barndorff-Nielsen} and the modulated bipower variation.
Our work is primarily related to the contributions of Podolskij and Vetter (2009), Fan and Wang (2007), Barunik and Vacha (2015) and Christensen et al. (2010). The estimator proposed in Podolskij and Vetter (2009) is a modified version of the modulated bipower variation (MBV) obtained using an estimator of the variance of the market microstructure noise. The resulting estimator is shown to be consistent in the presence of finite activity jumps and noise, but no asymptotic distribution or convergence rate is established. The simulation study of this work shows that our estimator is significantly more efficient than this new MBV. The estimators proposed in Fan and Wang (2007) and Barunik and Vacha (2015) are based on wavelet techniques. They first use wavelets to detect the locations and sizes of price jumps and then remove those jumps from the price series. The authors then apply noise robust realized volatility estimators to the jump adjusted data. The asymptotic distribution of the wavelet-based two-scales estimator is derived in Fan and Wang (2007). Our truncated two-scales estimator is similar in spirit to this approach since both estimators are based on detecting the jumps first and then applying a noise robust estimator to the jump adjusted data. In fact, the TTSRV has the same asymptotic distribution of the estimator proposed in Fan and Wang (2007). Our estimator however is easier to compute than the wavelet-based one, and we also show that it is robust to both infinite activity jumps and noise. Moreover, we are able to derive a technique to estimate the asymptotic variance of the estimator error, which is absent in Fan and Wang (2007). The estimator proposed in Christensen et al. (2010) is based on intra-daily quantile ranges. It achieves the optimal convergence rate of a realized volatility estimator in the presence of noise and finite activity jumps. However, its efficiency relies on specific assumptions on the dynamics of the spot volatility that are quite restrictive. In fact, the other estimators cited in this section as well as the estimator proposed in this work do not rely on such assumptions. It is important to emphasize that the analysis of all of these estimators has been developed under finite activity jumps only, while in this work we establish the properties of our estimator under both finite and infinite activity jumps. Another strand of the literature relevant to our work is the one that concerns testing for
the presence of price jumps and cojumps (with or without noise), as e.g. in Jacod and Todorov (2009), Jacod, Podolskij, and Vetter (2010), Aït-Sahalia, Jacod, and Li (2012), and Li, Todorov, Tauchen, and Lin (2016). These papers have inspired the jump detection indicator based on local averages used in this work.

The outline of this paper is as follows. Section 2 introduces basic notation and the definition of the TTSRV estimator. In Section 3 we establish the asymptotic properties of the estimator when price jumps have finite activity. In Section 4 we derive the consistency of the estimator when jumps are of infinite activity. Section 5 contains the result of the simulation study. Section 6 concludes.

2 Methodology

We denote by \( (y_t, t \in [0,1]) \) the efficient log–price process of an asset, where 0 typically represents the opening of the trading day and 1 the closing. The process starts at an initial value \( y_0 \in \mathbb{R} \) and its dynamics are given by

\[
dy_t = a_t dt + \sigma_t dB_t + dJ_t, \quad t \in [0,1],
\]

where \( B \) is a standard Brownian motion and \( J \) is a pure jump Lévy process, both defined on a filtered probability space \( (\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, P) \). We assume that the coefficients \( a \) and \( \sigma \) are progressively measurable processes, which ensures that (1) has a unique strong solution \( (y_t, t \in [0,1]) \) adapted and càdlàg (see e.g. Ikeda and Watanabe (1981)).

We assume that the efficient log–price is contaminated by market microstructure noise. That is, rather than the efficient price \( y_t \) the econometrician observes at discrete times its contaminated counterpart \( x_t \). Specifically, we assume that the observed price \( x_t \) is measured at equally spaced timestamps \( 0 = t_0 < t_1 < \cdots < t_m = 1 \) where \( h = t_i - t_{i-1} = \frac{1}{m} \) and is generated as

\[
x_{t_i} = y_{t_i} + u_{t_i}, \quad i = 1, \ldots, m,
\]

where \( u_{t_i} \) denotes the microstructure noise associated to the \( i \)–th trade. We assume that
$u_t$ is a discrete i.i.d. process, independent of the efficient price process and such that $u_t \sim N(0, \eta^2)$ where $\eta$ is a positive constant. In order to simplify the exposition, we denote by $x_i, y_i$ and $u_i$ the processes $x_{t_i}, y_{t_i}$ and $u_{t_i}$.

The theory developed in this work differs depending on whether the pure jump Lévy process $J$ has finite activity (FA), that is, it jumps a.s. a finite number of times on each finite time interval, or it has infinite activity (IA). It is well-known (see e.g. Ikeda and Watanabe [1981] or Sato [1999]), that we can always decompose $J$ as $J_t = J_{1,t} + J_{2,t}$, where

$$J_{1,t} = \int_0^t \int_{|z|>1} z \mu(dz, ds), \quad J_{2,t} = \int_0^t \int_{|z|\leq1} z (\mu(dz, ds) - \nu(dz)ds),$$

where $\mu$ is the Poisson random measure associated to the jumps of $J$, $\mu(dz, ds) - \nu(dz)ds$ is the compensated measure and $\nu$ is the Lévy measure. Observe that $J_2$ is a square integrable martingale with IA, and for each $t$, $\text{var}(J_{2,t}) = t \int_{|x|\leq1} x^2 \nu(dx) < \infty$. On the other hand, $J_1$ is a compound Poisson process with FA, so we can write $J_{1,t} = \sum_{i=1}^{N_t} Y_i$, where $N$ is a Poisson process with constant intensity $\lambda > 0$, and $Y_i$ are i.i.d. random variables independent of $N$. Recall that $N_t$ counts the number of jumps occurred in the interval $[0, t]$, and the $Y_i$’s are the different jump sizes. We set $\Delta N_t = N_t - N_{t-}$ and $N_i = N_{t_i}$, for $i = 1, ..., m$. Observe that we can also consider a slightly more general jump process $J_t = J_{1,t} + J_{2,t}$, where $J_2$ is an IA Lévy pure jump process and $J_1$ is a general FA jump process, that is, $J_{1,t} = \sum_{i=1}^{N_t} Y_i$, where $N$ is a non-explosive counting process with not necessarily constant intensity, and the random variables $Y_i$ are not necessarily i.i.d., nor independent of $N$.

The aim of this paper is to provide a consistent estimator of the integrated volatility

$$\text{IV} = \int_0^1 \sigma_t^2 dt.$$ 

In case of no microstructure noise nor price jumps, IV is the quadratic variation of the price process, and it is well-known that the realized volatility $\sum_{i=1}^m (y_i - y_{i-1})^2$ is a consistent estimator of IV (see e.g. Barndorff-Nielsen and Shephard [2002]). Several strategies
have been put forward in the literature to estimate IV in case of jumps. The approach introduced by Mancini (2008, 2009) consists of excluding from the realized volatility the intervals \( (t_{i-1}, t_i] \) where jumps are likely to have occurred. In order to identify such intervals, Mancini proposes a truncation method that consists of comparing the value of the squared return over each interval with a given threshold. If this value is larger than the threshold, then it is likely that the interval contains a jump. However, in the presence of microstructure noise this method does not consistently detect jumps since large returns can be observed over short intervals because of the noise.

In order to obtain a consistent estimator of IV in the presence of jumps and noise we propose the following estimation strategy. We first introduce a jump signaling device that is able to detect the location of jumps in the presence of microstructure noise. We then use the jump signaling device to truncate a noise robust realized volatility estimator.

In order to detect the presence of a jump in a given time interval \( (t_{i-1}, t_i] \), we consider the following measure

\[
\beta_i = \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} (x_j - x_{j-K_1}), \quad \text{for} \quad i = 1, \ldots, m,
\]

where \( K_1 = K_1(m) \) satisfies \( \lim_{m \to \infty} \frac{K_1}{m} = 0 \) and \( \lim_{m \to \infty} K_1 = \infty \). The \( \beta_i \) measure is a local average of overlapping returns. Figure 1 provides a schematic representation of its computation. In the interval \( (t_{i-K_1}, t_{i+K_1-1}] \) we construct \( K_1 \) overlapping intervals of the form of \( (t_j, t_{j+K_1}] \). The \( \beta_i \) measure is the average of the returns on these \( K_1 \) intervals. A jump realized on \( (t_{i-1}, t_i] \) will affect all the returns and will make \( |\beta_i| \) large. On the other
hand, if no jump is realized on \((t_{i-1}, t_i]\) the local averaging smooths away the effect of the noise and \(|\beta_i|\) will be small. In fact, Theorem I below quantifies these facts in terms of a threshold \(r(h)\) as \(m\) is large. Notice the intervals \((t_i-K_1, t_{i+K_1-1}]\) are not likely to contain jumps for large \(m\) since its length \(K_1h\) is assumed to be small for large \(m\).

A classic estimator of IV that is consistent in the presence of microstructure noise is the two-scale realized volatility (TSRV) estimator, which is defined as (see Zhang et al. (2005))

\[
\hat{\sigma}^2 = \frac{1}{K} \sum_{j=K}^{m} (x_j - x_{j-K})^2 - \frac{m - K + 1}{mK} \sum_{j=1}^{m} (x_j - x_{j-1})^2,
\]

where \(K = cm^{2/3}\) and \(c\) is a positive constant. When there is no microstructure noise nor jumps, the first term converges to IV in probability. In the presence of microstructure noise, the second term corrects the bias and \(\hat{\sigma}^2\) becomes a consistent estimator of IV. However, this estimator is not consistent in the presence of jumps. Our strategy consists of truncating the TSRV estimator using the \(\beta_i\) measure to obtain a consistent estimator in the presence of microstructure noise as well as jumps. However, the TSRV estimator is difficult to truncate due to the different scale of both sums. Therefore, we work with the following modified version of the TSRV estimator

\[
\hat{\sigma}^2_{TS} = \frac{1}{K} \sum_{j=K}^{m} (x_j - x_{j-K})^2 - \frac{1}{K} \sum_{j=K}^{m} (x_j - x_{j-1})^2.
\]

We observe that \(\hat{\sigma}^2_{TS}\) is also a consistent estimator of IV in the presence of microstructure noise since the second term is of the same order as the second term in the TSRV. Moreover, Proposition I below shows that both estimators \(\hat{\sigma}^2_{TS}\) and \(\hat{\sigma}^2\) have the same asymptotic distribution derived in Zhang et al. (2005) for \(\hat{\sigma}^2\).

We finally introduce the truncated two-scales realized volatility estimator (TTSRV)

\[
\hat{\sigma}^2_{TTS} = \frac{1}{K} \sum_{j=K}^{m} (x_j - x_{j-K})^2 1_{E_j} - \frac{1}{K} \sum_{j=K}^{m} (x_j - x_{j-1})^2 1_{E_j},
\]

where \(E_j = \{ |\beta_i| \leq r(h), \text{ for all } i = j - K + 1, \ldots, j\} \), where \(r(h)\) is the threshold introduced in Theorem I. In Section 3 we study the asymptotic properties of this estimator.
when $J_2 \equiv 0$ so that $J$ has FA, while in Section 4 we allow $J_2$ to be non-zero.

3 Theory: Finite Activity Jumps

3.1 Jump Detection

In this section, we consider the case where $J$ has FA. That is, $J = J_1$ and $J_1$ is a compound Poisson process with FA or a more general FA jump process as above. In this case, the quadratic variation of $y$ on $[0, 1]$ is given by

$$[y]_1 = IV + \sum_{i=1}^{N_1} Y_i^2.$$ 

The following theorem shows that the absolute values of $\beta_i$ can be used to identify the intervals where no jumps occurred.

**Theorem 1.** Suppose that

1. For all $t \in [0, 1]$, $P(\triangle N_t \neq 0, Y_{N_t} = 0) = 0$.
2. $\limsup_{h \to 0} \sup_{i \in \{1, \ldots, m + K_1 - 1\}} \left| \int_{t_i-K_1}^{t_i} a_s \, ds \right| \leq C(\omega) < \infty \ a.s.$
3. $\limsup_{h \to 0} \sup_{i \in \{1, \ldots, m + K_1 - 1\}} \left| \int_{t_i-K_1}^{t_i} \sigma^2_s \, ds \right| \leq M(\omega) < \infty \ a.s.$
4. $r(h)$ is a deterministic function such that
   \[
   \lim_{h \to 0} r(h) = 0, \quad \lim_{h \to 0} \frac{\log \frac{1}{h}}{K_1} = 0, \quad \text{and} \quad \lim_{h \to 0} \frac{\sqrt{K_1 h \log \frac{1}{h}}}{r(h)} = 0.
   \]

Then, for $P$-almost all $\omega$, there exists $\bar{h}(\omega) > 0$ such that for all $h \leq \bar{h}(\omega)$ we have for all $i = 1, \ldots, m$,

$$1_{\left\{ N_{i-K_1-1} - N_{i-K_1} = 0 \right\}}(\omega) \leq 1_{\left\{ |\beta_i| \leq r(h) \right\}}(\omega) \quad \text{and} \quad 1_{\left\{ |\beta_i| \leq r(h) \right\}}(\omega) \leq 1_{\left\{ N_i - N_{i-1} = 0 \right\}}(\omega).$$
Remark 1. (i) Assumption (1) says that the sizes of the price jumps are a.s. non-zero. Note that any FA Lévy process satisfies this condition, since \( \nu(\{0\}) = 0 \).

(ii) Assumptions (2) and (3) are satisfied by common assumptions on \( a \) and \( \sigma \) such as being a.s. bounded on \([0,1]\). In particular, they are satisfied as soon as \( a \) and \( \sigma \) have càdlàg paths.

(iii) Assumption (4) indicates how to choose the threshold. If e.g. \( K_1 = m^{\alpha_1} \) with \( 0 < \alpha_1 < 1 \), then we can choose \( r(h) = h^{\alpha_2} \) with \( 0 < \alpha_2 < \min\left(\frac{\alpha_1}{2}, \frac{1-\alpha_1}{2}\right) \).

(iv) As in Mancini (2009) and Zhang et al. (2005), the results of this paper can be extended to not necessarily equally spaced observations, but for the sake of conciseness, we leave it to the interested reader.

Theorem 1 quantifies the intuition of the \( \beta_i \) measure given after its definition in terms of a threshold. The first inequality implies that when there are no jumps on the interval \((t_{i-K_1}, t_{i+K_1-1})\) (which recall it is likely to occur when \( m \) is large), \( |\beta_i| \) will be smaller than the threshold for large \( m \). This is because condition (4) implies that the absolute value of the local average of the returns over that interval when there are no jumps goes faster to zero than the threshold. On the other hand, the second inequality implies that if there is a jump on the interval \((t_{i-1}, t_i]\), \( |\beta_i| \) will be larger than the threshold for large \( m \). Therefore, in this case, the intervals \((t_{j-K}, t_j]\) that contain \((t_{i-1}, t_i]\) will be removed from the estimator to eliminate the impact of the jump.

3.2 Consistency and Asymptotic Mixed Normality

When there are no price jumps, the next result shows that \( \hat{\sigma}^2 \) and \( \hat{\sigma}_{TS}^2 \) have the same asymptotic distribution (derived in Zhang et al. (2005, Theorem 4) for \( \hat{\sigma}^2 \)).

Proposition 1. Consider the framework of Section 2 with \( J = 0 \), and assume the drift coefficient \( a \) and the diffusion coefficient \( \sigma \) are a.s. continuous on \([0,1]\), and \( \sigma \) is a.s. bounded away from 0. Then, as \( m \to \infty \),

\[
m^{1/6} \left( \frac{\hat{\sigma}_{TS}^2}{\hat{\sigma}^2_{TS}} - \int_0^1 \sigma_t^2\,dt \right) \xrightarrow{\mathcal{L}} \left( 8c^{-2}\eta^4 + \frac{4}{3}c \int_0^1 \sigma_t^4\,dt \right)^{1/2} N(0,1),
\]
where the convergence is stable in law, and $c$ is the constant such that $K = cm^{2/3}$.

As in [Zhang et al. (2005)], stable convergence means that the left-side converges to the right-side jointly with the $x$ process, and the $N(0, 1)$ random variable is independent of $x$.

We now turn to the analysis of the TTSRV estimator defined in (3). We first show that it is a consistent estimator of IV.

**Theorem 2.** Consider the assumptions of Theorem 1, and those of Proposition 1 on $a$ and $\sigma$. Assume also that $\lim_{m \to \infty} K_1 \log m \frac{m^{1/6}}{m^{1/3}} = 0$. Then as $m \to \infty$,

$$\hat{\sigma}^2_{TTS} \overset{P}{\to} \int_0^1 \sigma_t^2 dt.$$

The proof of this Theorem is divided into two steps. First, since by Theorem 1 price jumps can be detected as $m$ is large and are removed from the computation of the TTSRV estimator, we first show that the difference between the TTSRV computed when there is a jump component in the price and when there is not is a.s. zero when $m$ is large (see the proof of (11) below). Second, we show that when there are no price jumps, the difference between the TTSRV and $\hat{\sigma}^2_{TTS}$ is $O_P(K_1 m^{-1/3} \log m)$ (see the proof of (12) below). Thus, the assumption that $\lim_{m \to \infty} K_1 \log m \frac{m^{1/6}}{m^{1/3}} = 0$ gives the desired result. We can e.g. choose $K_1 = m^{\alpha_1}$ and $r(h) = h^{\alpha_2}$ with $0 < \alpha_1 < \frac{1}{3}$ and $0 < \alpha_2 < \frac{\alpha_1}{2}$.

Using Proposition 1 and the same steps of the proof of Theorem 2 we obtain that asymptotic distribution of the TTSRV, which is the same as in Proposition 1.

**Theorem 3.** Consider the assumptions of Theorem 1, and those of Proposition 1 on $a$ and $\sigma$. Assume also that $\lim_{m \to \infty} K_1 \log m \frac{m^{1/6}}{m^{1/3}} = 0$. Then as $m \to \infty$,

$$m^{1/6} \left( \hat{\sigma}^2_{TTS} - \int_0^1 \sigma_t^2 dt \right) \overset{L}{\to} \left( 8c^{-2}\eta^4 + \frac{4}{3} c \int_0^1 \sigma_t^4 dt \right)^{1/2} N(0, 1),$$

where the convergence is stable in law.

We can e.g. choose $K_1 = m^{\alpha_1}$ and $r(h) = h^{\alpha_2}$ with $0 < \alpha_1 < \frac{1}{6}$ and $0 < \alpha_2 < \frac{\alpha_1}{2}$. 11
3.3 Estimating the Asymptotic Variance

In this section we follow similar steps as in Zhang et al. (2005) in order to estimate the asymptotic variance of the TTSRV estimator, which by Theorem 3 equals \(8c^{-2}\eta^4 + \frac{4}{3}c\int_0^1 \sigma^4_s ds\). More specifically, we apply our truncation technique to the estimator proposed in Zhang et al. (2005).

First, we begin by noting that \(\eta^2\) can be estimated by \(\hat{\eta}^2 = \frac{\sum_{m} (x_i - x_{i-1})^2}{2m}\). In our setting it can be easily checked that \(\hat{\eta}^2\) is still a consistent estimator of \(\eta^2\). The reason is that since we have finite price jumps on \([0,1]\), they will only affect finite terms in \(\sum_{m} (x_i - x_{i-1})^2\), so the jump effect will vanish as \(m \to \infty\).

Next, following the notation in Section 6 of Zhang et al. (2005), we divide \([0,1]\) into segments \((T_n, T_{n+1}]\), where \(T_n = \frac{nM}{m}\), for \(n = 1, \ldots, m/M\). The value of \(M\) will be specified later and for simplicity we assume \(m/M\) is an integer. We define the TTSRV estimator for the period \([0, T_n]\) as

\[
< \overline{X}, \overline{X} >_{T_n}^K = \frac{1}{K} \sum_{j=K}^{M_n} (x_j - x_{j-K})^2 1_{E_j} - \frac{1}{K} \sum_{j=K}^{M_n} (x_j - x_{j-1})^2 1_{E_j}.
\]

Consider the following truncated version of an estimator introduced in Zhang et al. (2005)

\[
\hat{s}_0^2 = m^{1/3} \sum_{n=1}^{m/M} \left( < \overline{X}, \overline{X} >_{T_n}^{K_2} - < \overline{X}, \overline{X} >_{T_{n-1}}^{K_2} - < \overline{X}, \overline{X} >_{T_n}^{K_3} - < \overline{X}, \overline{X} >_{T_{n-1}}^{K_3} \right)^2,
\]

where \(K_2 = c_2m^{2/3}\), \(K_3 = c_3m^{2/3}\), and \(c_2, c_3\) are positive constants. Zhang et al. (2005) show that the untruncated version of \(\hat{s}_0^2\) converges in probability when there are no price jumps as \(m \to \infty\) to

\[
V = \frac{4}{3} \left( c_2^{1/2} - c_3^{1/2} \right)^2 \int_0^1 \sigma^4_s ds + 8\eta^4(c_2^{-2} + c_3^{-2} - c_2^{-1}c_3^{-1}).
\]

In the presence of jumps, we obtain the analogous result in our setting.
Theorem 4. Assume the hypotheses of Theorem 3, and that 
\[ \lim_{m \to \infty} \frac{K_1(\log m)^2}{m^{1/6}} = 0, \quad \lim_{m \to \infty} \frac{M}{m^{2/3}} = \infty, \quad \text{and} \quad \lim_{m \to \infty} \frac{M}{m^{4/6}\log m} = 0. \]

Then as \( m \to \infty \), \( \hat{s}_0^2 \xrightarrow{P} V \).

Finally, in order to estimate the asymptotic variance of the TTSRV estimator it suffices to combine the estimators \( \hat{\eta}^2 \) and \( s_0^2 \) using the formula provided in Zhang et al. (2005).

Notice that we can choose \( K_1 = m^{\alpha_1} \) and \( M = m^{\alpha_2} \) with \( 0 < \alpha_1 < \frac{1}{6} \) and \( \frac{2}{3} < \alpha_2 < \frac{5}{6} \).

4 Theory: Infinite Activity Jumps

In this section, we allow \( J_2 \) to be a non-zero IA Lévy pure jump process, and \( J_1 \) is assumed to be a general FA jump process with counting process \( N \). As \( J = J_1 + J_2 \), the quadratic variation of the process \( y \) up time 1 becomes
\[
[y]_1 = IV + \sum_{t \leq 1} (\Delta J_{1,t})^2 + \sum_{t \leq 1} (\Delta J_{2,t})^2.
\]

The following theorem shows that in this IA jumps setting, the TTSRV estimator still consistently estimates IV.

Theorem 5. Assume the hypotheses of Theorem 3, and that
\[ \lim_{m \to \infty} \frac{K_3^3 \log m}{r^2(h)m^{1/3}} = 0. \]

Assume also that \( P (N_i - N_{i-1} > 0) = O \left( \frac{1}{m} \right) \) and that \( J_2 \) is independent of \( N \). Then as \( m \to \infty \), \( \hat{\sigma}_{\text{TTS}}^2 \xrightarrow{P} \int_0^1 \sigma_t^2 dt \).

For example, we can set \( K_1 = m^{\alpha_1} \) and \( r(h) = h^{\alpha_2} \), with \( 0 < 3\alpha_1 + 2\alpha_2 < \frac{1}{3} \) and \( 0 < \alpha_2 < \frac{\alpha_1}{2} \). The structure of the proof of this theorem is similar to that of Mancini (2009, Theorem 4). On one hand, we use the measure \( \beta_i \) and the threshold \( r(h) \) in order to cut off the jumps from \( J_1 \). On the other hand, we truncate the jumps in \( J_2 \) with absolute
value larger than $\sqrt{\delta + 16r^2(h)}$, where $\delta > 0$ is arbitrary, and show that the information loss caused by the truncation is negligible.

It is more challenging to establish a central limit theorem result in the infinite activity case, and we leave this problem open for future research. We point out that in [Cont and Mancini (2011)] a central limit theorem for the truncated realized volatility estimator in the case of infinite activity jumps and no noise is established. However, their result relies on an argument which we are not able to use when prices are contaminated by microstructure noise. In particular, their asymptotic result relies on choosing a threshold low enough such that the error caused by the infinite jumps is negligible. In our framework, because of the noise component, the threshold has to be sufficiently large.

5 Simulation Study

In this section we perform a simulation study to assess the performance of the TTSRV estimator. The simulation exercise consists of simulating one day of high frequency data and then applying the TTSRV estimator to estimate the integrated volatility. We consider two different specifications of the prices process, the first one has finite active jumps while in the second the jump activity is infinite. The TTSRV estimator is also benchmarked against a set of alternative estimators proposed in the literature.

We simulate the observed price $x_t$ according to

$$x_t = \int_0^t \sigma_s dB_s + J_t + u_t.$$ 

The spot volatility $\sigma_s$ follows a CIR process

$$d\sigma_s = \kappa(v - \sigma_s) + \tau \sigma_s dW_s,$$

where $v = 9$ is the long run mean of the process, $\tau = 2.74$ is its volatility, $\kappa = 0.1$ is the mean reversion parameter, and $W$ is a standard Brownian motion independent of $B$. The noise $u_t$ is a discrete i.i.d. $N(0, \eta^2)$, where $\eta > 0$. Two different specifications for
the jump process \( J \) are used. In first case, labelled as model 1, \( J \) is a compound Poisson process with a constant intensity \( \lambda = 2 \). The jump sizes are i.i.d. \( N(0, \xi^2) \), where \( \xi > 0 \). In the second case, labelled as model 2, \( J \) is a variance gamma (VG) process, which is a pure jump process with infinite activity and finite variation. The process is defined as

\[
J_s = d_1 G_s + d_2 \overline{W}_{G_s},
\]

where \( G_s \) is a Gamma random variable with shape parameter \( s/b \) and scale parameter \( b > 0 \), and \( \overline{W} \) is a standard Brownian motion independent of \( B, W \) and \( G \). We fix the values of \( d_1 \) and \( d_2 \) to respectively \(-0.8 \) and \( 0.8 \). We assume that a trading day is eight hours long and the observed price \( x_t \) is measured each second (that is, \( m = 28,800 \)). The simulation is carried out using the Euler simulation scheme. Throughout this section we compute the TTSRV for \( K_1 \) set to 4 (which is approximately \( m^{1/7} \)) and \( K \) set to 30. For each model and parameter setting the simulation is replicated 1000 times.

Figure 2 and 3 show the plot of the MSE of the TTSRV estimator as a function of \( r(h) \) in, respectively, model 1 and model 2 for different magnitudes of \( \eta \) (0.05, 0.10, 0.15). For model 1, \( \xi \) is fixed to 2 while for model 2, \( b \) is fixed to 2. The figures show that in both cases the MSE is a decreasing function of \( r(h) \) when \( r(h) \) is small. This is because when the threshold is too small, many intervals that do not contain jumps are truncated, and this increases the variance of the TTSRV estimator. On the other hand, the MSE is
an increasing function of $r(h)$ when $r(h)$ is large. This is because when the threshold is too large, the intervals that contain price jumps are not truncated, and this increases the bias of the the TTSRV estimator.

Figure 4 and 5 show the plot of the MSE of the TTSRV estimator as a function of $r(h)$ in, respectively, model 1 and model 2 for different magnitudes of the jump component. For model 1, $\xi$ is set to 1, 2 or 3 while for model 2, $b$ is set to 1, 2 and 3. In both sets of simulations $\eta$ is fixed to 0.1. The MSE has the same convex shape documented in Figures 2 and 3.

Next we investigate the finite sample distribution of the TTSRV estimator. We do this under model 1 only and for different values of $\xi$. We define the standardized estimation
Theorem 3 implies that if the sample size is sufficiently large, \( z \) should be approximately normally distributed. We compute \( z \) keeping the value of the threshold \( r(h) \) fixed at 0.75.
and $\eta = 0.1$. As mentioned in Mancini (2009, Remark in p.278) the optimality of the threshold varies for each model and one usually chooses the one that performs better in the simulations. Figure 6 shows the histogram and the normal qqplot of $z$ while Table 4 reports summary statistics. The TTSRV has a bias that increases with the standard deviation of the jumps’ size. We note that, comparing with the results of the simulation study of Mancini (2009), the bias is roughly of the same order of the one of the TRV estimator when no noise is present (see also Mancini, 2008). Overall, the approximation provided by the asymptotic theory is adequate.

Last, we compare the efficiency of the TTSRV with other estimators of the integrated volatility: The truncated realized volatility (TRV), the bipower variation (BPV), the modulated bipower variation (MBV) and the TSRV. In this exercise the threshold $r(h)$ of the TTSRV estimator is fixed to 0.75.

The TRV proposed by Mancini (2008, 2009) is a truncated version of the classic realized volatility estimator. It is defined as

$$\hat{\sigma}^2_{\text{TRV}} = \sum_{i=2}^{m} (x_i - x_{i-1})^2 1_{\{|x_i - x_{i-1}| \leq r(h)\}}.$$ 

As we have previously pointed out this estimator is not consistent in the presence of noise. Moreover, the truncation devices used by this estimator does not truncate jumps with high probability in this setting. To minimize the impact of the noise, we use the optimal sampling scheme proposed by Zhang et al. (2005), which shows that the optimal amount of equidistant observations for constructing $\hat{\sigma}^2_{\text{RV}}$ is given by

$$\bar{m} = \left( \frac{1}{4\eta^2} \int_0^1 \sigma_z^4 ds \right)^{1/3}.$$ 

Note however that this is only optimal in case jumps are absent. Also, as mentioned before it is not clear from the literature how to set $r(h)$. Here, we set it to 0.7 as this approximately minimizes the MSE of the estimator in the scenarios considered in this study.

Another important jump robust estimator proposed in the literature is the bipower
variation (BPV), introduced by Barndorff-Nielsen and Shephard (2004). It is defined as

$$\hat{\sigma}_{BPV}^2 = \frac{\pi^2}{m} \sum_{j=3}^{m} (x_j - x_{j-1})(x_{j-1} - x_{j-2})^2.$$ 

The performance of $\hat{\sigma}_{BPV}^2$ is also compromised in the presence of noise. Again, in this case reducing the sampling frequency can relieve the problem, and we use the same low frequency sampling scheme used for computing $\hat{\sigma}_{TRV}^2$.

We also consider the MBV proposed by Podolskij and Vetter (2009). The estimator is defined as

$$\hat{\sigma}_{MBV}^2 = \frac{e_1 e_2 \sigma^2 - v_2 \hat{\eta}^2}{v_1},$$

where

$$\hat{\sigma}^2 = \sum_{n=1}^{M} |X_n^{(K)} X_{n+1}^{(K)}|, \quad \hat{\eta}^2 = \frac{1}{2m} \sum_{i=1}^{m} (x_i - x_{i-1})^2,$$

$$X_n^{(K)} = \frac{1}{m/M - K + 1} \sum_{i=\lceil n/K \rceil m}^{n} (x_{i+K} - x_i),$$

$e_1 > 0$, $e_2 > 1$, $\mu_1 = \sqrt{\frac{2}{\pi}}$, $M = \frac{n}{e_2 K}$, $\bar{K} = e_1 m^{1/2}$, $v_1 = \frac{e_1 (3e_2 - 4 + \max(2 - e_2)3)}{3(e_2 - 1)^2}$ and $v_2 = \frac{2 \min(e_2 - 1, 1)}{e_1 (e_2 - 1)^2}$ (which is an estimator of $\eta^2$). As pointed out by Podolskij and Vetter (2009), the computation of the optimal values of $e_1$ and $e_2$ involves solving polynomial equations of degree higher than two. Following Podolskij and Vetter (2009), we set $e_1 = 0.8$ and $e_2 = 2.3$.

For reference purposes, we also consider the TSRV estimator. Note that when jumps are present this estimator converges in probability to (see Zhang et al., 2005)

$$\int_0^1 \sigma_t^2 dt + \sum_{0 \leq t \leq 1} (\Delta J_t)^2.$$

Figures 7 and 8 show the plot of the MSE of the estimators in models 1 and 2 as function of $\eta$. In model 1, $\xi$ is 0.7 and in model 2, $b$ is also 0.7. Tables 2 and 3 report the MSE of the estimators for selected values of $\eta$. The pictures show that the TTSRV estimator
Figure 7: The figure shows the plot of the MSE of the TTSRV, TSRV, TRV, MBV and BPV estimators in model 1 as a function of the market microstructure noise standard deviation $\eta$.

Figure 8: The figure shows the plot of MSE of the TTSRV, TSRV, TRV, MBV and BPV estimators in model 2 as a function of the market microstructure noise standard deviation $\eta$. 
\[ \eta = 0.01 \eta = 0.02 \eta = 0.03 \eta = 0.04 \eta = 0.05 \]

<table>
<thead>
<tr>
<th>TTSRV</th>
<th>0.220</th>
<th>0.212</th>
<th>0.218</th>
<th>0.207</th>
<th>0.219</th>
<th>0.211</th>
</tr>
</thead>
<tbody>
<tr>
<td>TSRV</td>
<td>1.594</td>
<td>1.570</td>
<td>1.392</td>
<td>1.580</td>
<td>1.531</td>
<td>1.695</td>
</tr>
<tr>
<td>TRV</td>
<td>0.167</td>
<td>0.404</td>
<td>0.727</td>
<td>1.125</td>
<td>1.375</td>
<td>1.940</td>
</tr>
<tr>
<td>MBV</td>
<td>1.878</td>
<td>1.868</td>
<td>2.023</td>
<td>1.848</td>
<td>2.180</td>
<td>1.735</td>
</tr>
<tr>
<td>BPV</td>
<td>0.012</td>
<td>0.319</td>
<td>0.812</td>
<td>1.425</td>
<td>1.964</td>
<td>2.670</td>
</tr>
</tbody>
</table>

Table 2: The table reports the MSE of the TTSRV, TSRV, TRV, MBV and BPV estimators in model 1 for different values of the market microstructure noise standard deviation \( \eta \).

<table>
<thead>
<tr>
<th>( \xi = 0 )</th>
<th>( \xi = 0.2 )</th>
<th>( \xi = 0.4 )</th>
<th>( \xi = 0.6 )</th>
<th>( \xi = 0.8 )</th>
<th>( \xi = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>TTSRV</td>
<td>0.203</td>
<td>0.237</td>
<td>0.250</td>
<td>0.226</td>
<td>0.221</td>
</tr>
<tr>
<td>TSRV</td>
<td>4.850</td>
<td>4.733</td>
<td>4.598</td>
<td>5.362</td>
<td>4.093</td>
</tr>
<tr>
<td>TRV</td>
<td>0.177</td>
<td>0.489</td>
<td>0.836</td>
<td>1.187</td>
<td>1.539</td>
</tr>
<tr>
<td>MBV</td>
<td>1.966</td>
<td>1.991</td>
<td>1.854</td>
<td>2.159</td>
<td>1.899</td>
</tr>
<tr>
<td>BPV</td>
<td>0.010</td>
<td>0.408</td>
<td>0.877</td>
<td>1.638</td>
<td>2.133</td>
</tr>
</tbody>
</table>

Table 3: The table reports the MSE of the TTSRV, TSRV, TRV, MBV and BPV estimators in model 2 for different values of the market microstructure noise standard deviation \( \eta \).

<table>
<thead>
<tr>
<th>( b = 0.01 )</th>
<th>( b = 0.2 )</th>
<th>( b = 0.4 )</th>
<th>( b = 0.6 )</th>
<th>( b = 0.8 )</th>
<th>( b = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>TTSRV</td>
<td>0.454</td>
<td>0.401</td>
<td>0.280</td>
<td>0.244</td>
<td>0.223</td>
</tr>
<tr>
<td>TSRV</td>
<td>0.454</td>
<td>1.089</td>
<td>3.278</td>
<td>4.397</td>
<td>6.085</td>
</tr>
<tr>
<td>TRV</td>
<td>1.405</td>
<td>2.528</td>
<td>2.131</td>
<td>1.970</td>
<td>1.811</td>
</tr>
<tr>
<td>MBV</td>
<td>1.786</td>
<td>1.721</td>
<td>1.744</td>
<td>1.456</td>
<td>2.033</td>
</tr>
<tr>
<td>BPV</td>
<td>3.004</td>
<td>2.868</td>
<td>2.929</td>
<td>2.673</td>
<td>3.009</td>
</tr>
</tbody>
</table>

Table 4: The table reports the MSE of the TTSRV, TSRV, TRV, MBV and BPV estimators in model 1 for different values of the jump size standard deviation \( \xi \).

Table 5: The table reports the MSE of the TTSRV, TSRV, TRV, MBV and BPV estimators in model 2 for different values of the VG process scale parameter \( b \).

dominates the competing estimators almost uniformly over the range of \( \eta \) considered. It only performs worse than the BPV and slightly worse than the TRV estimators when the magnitude of the noise is negligible. The MSE of the TRV and BPV estimators increases steadily as \( \eta \) increases as these estimators are not consistent in the presence of noise. The MSE of the TSRV estimator does not change as the variance of the noise increases, however it is much larger as it is not consistent in the presence of jumps. In this setting, the MBV estimator does not perform particularly well (especially in model 1) despite being robust to both noise and jumps.

Figures 9 and 10 show the plot of the MSE of the estimators under model 1 as function of \( \xi \) and in model 2 as a function of \( b \). In both models \( \eta \) is fixed to 0.05. Tables 4 and 5 report the MSE of the estimators for selected values of, respectively, \( \xi \) and \( b \). Again, the TTSRV achieves the best performance overall. The TSRV performs well in this setting only when the standard deviation of the jump size or the scale of the VG process are so
small that the impact of the jump component in negligible. The TRV, MBV and BPV estimators roughly have the same performance and are not affected by the variability of
the jump component.

6 Conclusions

In this work we introduce a novel estimator of the integrated volatility of asset prices that is consistent in the presence of both finite or infinite activity price jumps and market microstructure noise. We first introduce a jump detection indicator which consistently detects jumps in the presence of noise. We then use our jump detection methodology to construct a truncated version of the two-scales realized volatility estimator that we call truncated two-scales realized volatility (TTSRV) estimator. In the finite activity jumps case, we show that this estimator is consistent as well as asymptotically normal. Because the intervals that contain price jumps will be a.s. truncated from the estimator and the information loss is negligible, the estimator has the same asymptotic properties as the standard two-scales realized volatility estimator in the case of no jumps. Moreover, we introduce an estimator of the asymptotic variance. In the infinite activity jumps case, we show that the TTSRV estimator is consistent. A simulation study is used to compare our proposed approach to other important realized volatility estimators proposed in the literature. The study shows that the TTSRV estimator performs favorably relative to its competitors when the price process contains jumps and is contaminated by noise.
References


A Proofs

Proof of Theorem 1. We start proving the first inequality of Theorem 1. For each $\omega$, set $J_{0,h} = \{ i \in \{1, ..., m \} : N_{i+K_1-1} - N_{i-K_1} = 0 \}$. It suffices to prove that a.s. for small $h$

$$\sup_{i \in J_{0,h}} |\beta_i| \leq 1. \quad (6)$$
To evaluate this sup, using the definition of $\beta_i$, we have that a.s.

\[
\sup_{i \in J_{0,h}} |\beta_i| \leq \sup_{i \in J_{0,h}} \left| \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} \int_{t_j}^{t_{j+1}} a_s ds \right| + \sup_{i \in J_{0,h}} \left| \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} \int_{t_j}^{t_{j+1}} \sigma_s dB_s \right|
\]

\[
+ \sup_{i \in J_{0,h}} \left| \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} (u_j - u_{j-K_1}) \right|
\]

\[
\leq \sup_{i \in \{1, \ldots, m+K_1-1\}} \left| \int_{t_{i-K_1}}^{t_i} a_s ds \right| + \sup_{i \in \{1, \ldots, m+K_1-1\}} \left| \int_{t_{i-K_1}}^{t_i} \sigma_s dB_s \right|
\]

\[
+ \sup_{i \in \{1, \ldots, m\}} \left| \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} (u_j - u_{j-K_1}) \right|
\]

We evaluate the three terms separately. We first write

\[
\sup_{i \in \{1, \ldots, m+K_1-1\}} \left| \int_{t_{i-K_1}}^{t_i} a_s ds \right| = \sup_{i \in \{1, \ldots, m+K_1-1\}} \left| \int_{t_{i-K_1}}^{t_i} a_s ds \right| \times \sqrt{K_1 h \log \frac{1}{h}}.
\]

Then, by assumptions (2) and (4), a.s. for small $h$ the left hand side is bounded by 1.

For the second term, we write

\[
\sup_{i \in \{1, \ldots, m+K_1-1\}} \left| \int_{t_{i-K_1}}^{t_i} \sigma_s dB_s \right| \leq \sup_{i \in \{1, \ldots, m+K_1-1\}} \left| \int_{t_{i-K_1}}^{t_i} \sigma_s dB_s \right| \times \sqrt{2K_1 h M \log \frac{1}{K_1} \log \frac{1}{K_1 h}}.
\]

where $M$ is the random constant of assumption (3).

Since $\int_{t_{i-K_1}}^{t_i} \sigma_s dB_s$ is a time changed Brownian motion, by Karatzas and Shreve (1999, p.114 theorem 9.25), we have that a.s.

\[
\limsup_{h \to 0} \sup_{i \in \{1, \ldots, m+K_1-1\}} \left| \int_{t_{i-K_1}}^{t_i} \sigma_s dB_s \right| \leq 1.
\]

Moreover, by assumption (3) and the monotonicity of the function $f(x) = x \log \frac{1}{x}$, we get that a.s.

\[
\limsup_{h \to 0} \sup_{i \in \{1, \ldots, m+K_1-1\}} \sqrt{2K_1 h M \log \frac{1}{K_1 h}} \leq 1.
\]
Finally, since $K_1 h \to 0$ as $h \to 0$, and $M(\omega) < \infty$, we have that a.s.

$$\lim_{h \to 0} \sqrt{2K_1 h M \log \frac{1}{K_1 h M}} = \sqrt{M}.$$ 

Therefore, using assumption (4), we conclude that a.s. for small $h$ the left hand side of (7) is bounded by 1.

For the last term, we write

$$\sup_{i \in \{1, \ldots, m\}} \left| \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} (u_j - u_{j-K_1}) \right| r(h) \leq 2 \eta \left( \sqrt{2 + \epsilon} \right).$$

Now, since $\frac{1}{K_1} \sum_{j=i}^{i+K_1-1} (u_j - u_{j-K_1}) \leq N \left( 0, \frac{2 \eta^2}{K_1} \right)$, we have that for all $x > 0$,

$$P \left( \sup_{i \in \{1, \ldots, m\}} \left| \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} (u_j - u_{j-K_1}) \right| \geq x \right) \leq 2m \exp \left( -\frac{x^2 K_1}{2 \eta^2} \right).$$

Therefore, for all $\epsilon > 0$,

$$\sum_{m \geq 1} P \left( \sup_{i \in \{1, \ldots, m\}} \left| \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} (u_j - u_{j-K_1}) \right| \geq \sqrt{(2 + \epsilon) \eta^2} \right) \leq 2 \sum_{m \geq 1} m^{-(1+\epsilon)} < \infty.$$

Thus, using Borell-Cantelli Lemma and letting $\epsilon \downarrow 0$, we obtain that a.s.

$$\limsup_{h \to 0} \frac{\sup_{i \in \{1, \ldots, m\}} \left| \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} (u_j - u_{j-K_1}) \right| \sqrt{\log \frac{1}{K_1}}}{r(h)} \leq 2 \eta,$$

which together with assumption (4) implies that a.s. for $h$ small the left hand side of (8) is bounded by 1. This concludes the proof of (6).

We next show the second inequality of Theorem 1. For each $\omega$, set $J_{1,h} = \{ i \in \{1, \ldots, m\} : N_i - N_{i-1} > 0 \}$. It suffices to prove that a.s. for small $h$

$$\inf_{i \in J_{1,h}} |\beta_i| > \frac{r(h)}{r(h)}.$$  

(9)
In order to evaluate this infimum, observe that for all \( i \in J_{1,h}, |\beta_i| = |A_i + B_i| \), where
\[
A_i = \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} \left( \int_{t_{j-K_1}}^{t_j} a_s ds + \int_{t_{j-K_1}}^{t_j} \sigma_s dB_s + (u_j - u_{j-K_1}) \right)
\]
and
\[
B_i = \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} \sum_{\ell=1}^{N_{j-N_{j-K_1}}} Y_\ell.
\]

In the first part of the theorem we have shown that a.s. the term \( \frac{A_i}{r(h)} \) tends to 0 as \( h \to 0 \) uniformly with respect to \( i \). On the other hand, we observe that
\[
t_{i+K_1-1} - t_{i-K_1} \leq 2K_i h \to 0 \quad \text{as} \quad h \to 0.
\]

Therefore, a.s. for \( h \) small we have that for each \( i, N_{i+K_1-1} - N_{i-K_1} \leq 1 \). Thus, a.s.
\[
\lim_{h \to 0} \inf_{i \in J_{1,h}} \{ |\beta_i| \} \geq \lim_{h \to 0} \frac{\inf_{i \in J_{1,h}} \{ |B_i| \}}{r(h)} \geq \frac{\inf_{i \in N_1} \{ Y_i \}}{r(h)} = \infty,
\]
where in the last equality we have used assumptions (1) and (4). This proves \( \Box \).

**Proof of Proposition 7** For simplicity, we assume that the drift is zero. Since we are assuming that \( J = 0 \), we have that
\[
\hat{\sigma}_T^2 - \int_0^1 \sigma^2_s ds = A + B + C,
\]
where
\[
A = \left( \frac{1}{K} \sum_{i=K}^m \left( \int_{t_{i-K}}^{t_i} \sigma_s dB_s \right)^2 - \int_0^1 \sigma^2_s ds \right) - \frac{1}{K} \sum_{i=K}^m \left( \int_{t_{i-1}}^{t_i} \sigma_s dB_s \right)^2,
\]
\[
B = 2 \frac{1}{K} \sum_{i=K}^m \left( \int_{t_{i-K}}^{t_i} \sigma_s dB_s \right) (u_i - u_{i-K}) - \frac{2}{K} \sum_{i=K}^m \left( \int_{t_{i-1}}^{t_i} \sigma_s dB_s \right) (u_i - u_{i-1}),
\]
\[
C = \frac{1}{K} \sum_{i=0}^{K-2} u_i^2 - \frac{1}{K} \sum_{i=m-K+1}^{m-1} u_i^2 - \frac{2}{K} \sum_{i=K}^m u_{i-K} u_i + \frac{2}{K} \sum_{i=K}^m u_{i-1} u_i.
\]

According to Zhang et al. (2005) Theorem 3, we have that as \( m \to \infty \)
\[
m^{1/6} \left( \frac{1}{K} \sum_{i=K}^m \left( \int_{t_{i-K}}^{t_i} \sigma_s dB_s \right)^2 - \int_0^1 \sigma^2_s ds \right) \overset{\mathcal{L}}{\to} \sqrt{\frac{4c}{3}} \int_0^1 \sigma^4_s ds \text{ N}(0,1),
\]
where the convergence is stable in law.

On the other hand, since \( \sum_{i=K}^m \left( \int_{t_{i-1}}^{t_i} \sigma_s dB_s \right)^2 = O_p(1) \) and \( K = cm^{2/3} \), we get that as \( m \to \infty \),
\[
m^{1/6} \frac{1}{K} \sum_{i=K}^m \left( \int_{t_{i-1}}^{t_i} \sigma_s dB_s \right)^2 \overset{P}{\to} 0.
\]
Moreover, equations (A.8) and (A.13) in [Zhang et al. (2005)] imply that as $m \to \infty$,

\[
m^{1/6} \frac{2}{K} \sum_{i=K}^{m} \left( \int_{t_{i-1}}^{t_i} \sigma_s dB_s \right) (u_i - u_{i-1}) \xrightarrow{P} 0,
\]

and

\[
m^{1/6} \frac{2}{K} \sum_{i=K}^{m} \left( \int_{t_i}^{t_{i+K}} \sigma_s dB_s \right) (u_i - u_{i-K}) \xrightarrow{P} 0.
\]

Finally, straightforward computations as in [Zhang et al. (2005)] show that as $m \to \infty$,

\[
m^{1/6} \frac{2}{C} \xrightarrow{F} 2\sqrt{2}c^{-1} \eta^2 N(0, 1).
\]

Since $A$ and $C$ are independent, this concludes the desired result.

**Proof of Theorem 3.** For simplicity we assume that the drift is zero. Set

\[
\overline{x}_t = x_t - J_{1,t} = \int_0^t \sigma_s dB_s + u_t,
\]

and consider the corresponding estimators

\[
\hat{\sigma}_T^2 = \frac{1}{K} \sum_{j=K}^{m} (\overline{x}_j - \overline{x}_{j-K})^2 - \frac{1}{K} \sum_{j=K}^{m} (\overline{x}_j - \overline{x}_{j-1})^2,
\]

and

\[
\hat{\sigma}_{TTS}^2 = \frac{1}{K} \sum_{j=K}^{m} (\overline{x}_j - \overline{x}_{j-K})^2 1_{E_j} - \frac{1}{K} \sum_{j=K}^{m} (\overline{x}_j - \overline{x}_{j-1})^2 1_{E_j},
\]

where recall that $E_j = \{|\beta_i| \leq r(h), \text{ for all } i = j - K + 1, \ldots, j\}$ and

\[
\beta_j = \frac{1}{K_1} \sum_{i=j}^{j+K_1-1} (x_i - x_{i-K_1}).
\]

Since as $m \to \infty$, $\hat{\sigma}_T^2$ converges in probability to IV, it suffices to show that as $m \to \infty$,

\[
\hat{\sigma}_{TTS}^2 - \hat{\sigma}_T^2 \xrightarrow{P} 0,
\]

and

\[
\hat{\sigma}_T^2 - \hat{\sigma}_{TTS}^2 \xrightarrow{P} 0.
\]

**Proof of (11).** By the second inequality of Theorem 1, we have that a.s. as $m \to \infty$, for all $j \in \{K, \ldots, m\}$,

\[
1_{E_j} = 1_{\{|\beta_{j-K+1}| \leq r(h)\}} 1_{\{|\beta_{j-K+2}| \leq r(h)\}} \cdots 1_{\{|\beta_j| \leq r(h)\}} \leq 1_{\{N_j - N_{j-K} = 0\}}.
\]

Thus, a.s. as $m \to \infty$, for all $j \in \{K, \ldots, m\}$,

\[
\left( (\overline{x}_j - \overline{x}_{j-K})^2 - (x_j - x_{j-K})^2 \right) 1_{E_j} = 0.
\]
Similarly, a.s. as \( m \to \infty \)

\[
\left( (\bar{x}_j - \bar{x}_{j-1})^2 - (x_j - x_{j-1})^2 \right) 1_{E_j} = 0.
\]

Therefore, a.s. as \( m \to \infty \), \( \tilde{\sigma}_{\text{TS}}^2 - \tilde{\sigma}_{\text{TTS}}^2 = 0 \), which proves (11).

**Proof of (12).** We have

\[
\tilde{\sigma}_{\text{TS}}^2 - \tilde{\sigma}_{\text{TTS}}^2 = \frac{1}{K} \sum_{j=K}^{m} \left( (\bar{x}_j - \bar{x}_{j-K})^2 - (\bar{x}_j - \bar{x}_{j-1})^2 \right) 1_{E_j},
\]

where

\[
1_{E_j} = 1 \left\{ \sup_{i \in \{j-K+1, \ldots, j\}} |\beta_i| > r(h) \right\}
\]

\[
= 1_{|\beta_{j-K+1}| > r(h)} + \sum_{i=j-K+2}^{j} \left( 1_{|\beta_i| > r(h)} \prod_{p=j-K+1}^{i-1} 1_{|\beta_p| \leq r(h)} \right).
\]

Therefore,

\[
|\tilde{\sigma}_{\text{TS}}^2 - \tilde{\sigma}_{\text{TTS}}^2| \leq \sup_{i \in \{K, \ldots, m\}} \sum_{\ell \in \{0, \ldots, K-1\}} \frac{1}{K} \sum_{j=i}^{\min(i+\ell, m)} \left( (\bar{x}_j - \bar{x}_{j-K})^2 - (\bar{x}_j - \bar{x}_{j-1})^2 \right) \sum_{i=1}^{m} 1_{|\beta_i| > r(h)}.
\]

By the first inequality of Theorem [1] a.s. as \( m \to \infty \),

\[
\sum_{i=1}^{m} 1_{|\beta_i| > r(h)} \leq \sum_{i=1}^{m} 1_{\{N_{i+K_1-1} - N_{i-K_1} > 0\}} \leq (2K_1 - 1) (N_{m+K_1-1} - N_{1-K_1}),
\]

and \( N_{m+K_1-1} - N_{1-K_1} \) is a.s. finite.

Without loss of generality, we assume that \( i \in \{K, \ldots, m - K + 1\} \). Otherwise, the proof follows along the same lines. In this case, for all \( i \in \{K, \ldots, m - K + 1\} \) and \( \ell \in \{0, \ldots, K - 1\} \), \( \min(i + \ell, m) = i + \ell \). Moreover, we have that

\[
\frac{1}{K} \sum_{j=i}^{i+\ell} \left( (\bar{x}_j - \bar{x}_{j-K})^2 - (\bar{x}_j - \bar{x}_{j-1})^2 \right) = D_1 + D_2 + D_3,
\]

where

\[
D_1 = \frac{1}{K} \sum_{j=i}^{i+\ell} \left( \int_{t_{j-K}}^{t_j} \sigma_s dB_s \right)^2 - \left( \int_{t_{j-1}}^{t_j} \sigma_s dB_s \right)^2,
\]

\[
D_2 = \frac{2}{K} \sum_{j=i}^{i+\ell} \left( \int_{t_{j-K}}^{t_j} \sigma_s dB_s \right) (u_j - u_{j-K}) - \left( \int_{t_{j-1}}^{t_j} \sigma_s dB_s \right) (u_j - u_{j-1}),
\]

\[
D_3 = \frac{1}{K} \sum_{j=i}^{i+\ell} \left( (u_j - u_{j-K})^2 - (u_j - u_{j-1})^2 \right).
\]
We now claim that for all \( k = 1, 2, 3 \), as \( m \to \infty \),

\[
K_1 \sup_{\ell \in \{0, \ldots, K-1\}} |D_{k\ell}| \overset{P}{\to} 0.
\]

We start with \( D_1 \). Since \( \ell < K \), \( |D_1| \) is not larger than \( \sup_{j \in \{K, \ldots, m\}} \left( \int_{t_{j-K}}^{t_j} \sigma_s dB_s \right)^2 \) which is \( \text{O}_P \left( m^{-1/3} \log m \right) \). Because \( u \) is independent of \( \sigma \) and \( B \), and \( \frac{4}{K^2} \sum_{j=2}^{i+\ell} \left( \int_{t_{j-K}}^{t_j} \sigma_s dB_s \right)^2 \) is \( \text{O}_P \left( m^{-1} \right) \), we get that \( \sup |D_2| \) is \( \text{O}_P \left( m^{-1/2} \log m \right) \). Finally, since \( u_i \) is a \( N(0, \eta^2) \) i.i.d. sequence, \( \sup |D_3| \) is \( \text{O}_P \left( m^{-1/3} \log m \right) \). Therefore, since \( K_1 m^{-1/3} \log m \to 0 \) as \( m \to \infty \), (12) holds true.

Proof of Theorem 4. Using the same notation as in the proof of Theorem 2, we have that Proposition 1 applies to \( \hat{\sigma}^2_{TTS} \). Thus, it suffices to show that as \( m \to \infty \),

\[
m^{1/6} (\hat{\sigma}^2_{TTS} - \hat{\sigma}^2_{\hat{\sigma}^2_{TTS}}) \overset{P}{\to} 0,
\]

and

\[
m^{1/6} (\sigma^2_{TTS} - \hat{\sigma}^2_{TTS}) \overset{P}{\to} 0.
\]

(13) follows the fact that a.s. \( \sigma^2_{TTS} - \hat{\sigma}^2_{TTS} = 0 \) as \( m \to \infty \), as shown in the proof of (11). On the other hand, in the proof of (12), we have shown that \( \sigma^2_{TTS} - \sigma^2_{\hat{\sigma}^2_{TTS}} = O_P \left( K_1 m^{-1/3} \log m \right) \), which implies (14) since \( K_1 m^{-1/3} \log m \to 0 \) as \( m \to \infty \).

Proof of Theorem 5. As in the proof of Theorem 2, we denote by \( <X, X>^K_{T_n} \) and \( \hat{s}^2_0 \) the estimators \( <X, X>^K_{T_n} \) and \( \hat{s}^2_0 \), where \( x \) is replaced with \( \hat{x} \) (no jumps), and by \( <\hat{X}, \hat{X}>^K_{T_n} \) and \( \hat{s}^2_0 \) where \( x \) is replaced with \( \hat{x} \) and there is no truncation.

Following similarly as in the proof of (11), we can easily check that a.s. as \( m \to \infty \), \( \hat{s}^2_0 = s^2_0 \). Moreover, by Zhang et al. (2005), \( \hat{s}^2_0 \overset{P}{\to} V \) as \( m \to \infty \). Thus, it suffices to show that \( \hat{s}^2_0 - s^2_0 \overset{P}{\to} 0 \) as \( m \to \infty \).

Following similarly as in the proof of (12), we have that

\[
\left| <\hat{X}, \hat{X}>^K_{T_n} - <X, X>^K_{T_{n-1}} - ( <\hat{X}, \hat{X}>^K_{T_n} - <\hat{X}, \hat{X}>^K_{T_{n-1}}) \right|
\leq \sup_{i \in \{M(n-1)+1, \ldots, M_n\}} \left| \frac{1}{K} \sum_{j=i}^{i+\ell} \hat{X}_j^K \right| \sum_{i=M(n-1)+1}^{M_n} 1_{|\beta_i|>r(h)},
\]

where \( \hat{X}_j^K = (\hat{x}_j - \hat{x}_{j-K})^2 - (\hat{x}_j - \hat{x}_{j-1})^2 \), and a.s. as \( m \to \infty \)

\[
\sum_{i=M(n-1)+1}^{M_n} 1_{|\beta_i|>r(h)} \leq 2K_1 (N_{Mn+K_1-1} - N_{M(n-1)+1-K_1}).
\]

Therefore,

\[
|\hat{s}^2_0 - s^2_0| \leq m^{1/3} \sum_{k=1}^{7} A_k,
\]

31
where

\[
A_1 = \sup_{i \in \{K_2, \ldots, m\}, \ell \in \{0, \ldots, K_2\}} \left| \frac{1}{K_2} \sum_{j=1}^{i+\ell} X_j \right|^2 4K_1^2 \sum_{n=1}^{m/M} (N_{Mn+K_1-1} - N_{M(n-1)+1-K_1})^2
\]

\[
A_2 = \sup_{i \in \{K_3, \ldots, m\}, \ell \in \{0, \ldots, K_3\}} \left| \frac{1}{K_3} \sum_{j=1}^{i+\ell} X_j \right|^2 4K_1^2 \sum_{n=1}^{m/M} (N_{Mn+K_1-1} - N_{M(n-1)+1-K_1})^2
\]

\[
A_3 = 2 \sup_{n \in \{1, \ldots, m/M\}} \left| \frac{1}{K_2} \sum_{j=1}^{m/M} X_j \right| \times 2K_1 \sum_{j=1}^{m/M} (N_{Mj+K_1-1} - N_{M(j-1)+1-K_1})
\]

\[
A_4 = 2 \sup_{n \in \{1, \ldots, m/M\}} \left| \frac{1}{K_3} \sum_{j=1}^{m/M} X_j \right| \times 2K_1 \sum_{j=1}^{m/M} (N_{Mj+K_1-1} - N_{M(j-1)+1-K_1})
\]

\[
A_5 = 2 \sup_{n \in \{1, \ldots, m/M\}} \left| \frac{1}{K_2} \sum_{j=1}^{m/M} X_j \right| \times 2K_1 \sum_{j=1}^{m/M} (N_{Mj+K_1-1} - N_{M(j-1)+1-K_1})
\]

\[
A_6 = 2 \sup_{n \in \{1, \ldots, m/M\}} \left| \frac{1}{K_3} \sum_{j=1}^{m/M} X_j \right| \times 2K_1 \sum_{j=1}^{m/M} (N_{Mj+K_1-1} - N_{M(j-1)+1-K_1})
\]

\[
A_7 = 2 \sup_{i \in \{K_2, \ldots, m\}, \ell \in \{0, \ldots, K_2\}} \left| \frac{1}{K_2} \sum_{j=1}^{i+\ell} X_j \right|^2 4K_1^2 \sum_{n=1}^{m/M} (N_{Mn+K_1-1} - N_{M(n-1)+1-K_1})^2.
\]

Also similarly as in the proof of (12), we have that

\[
\sup_{i \in \{K_2, \ldots, m\}, \ell \in \{0, \ldots, K_2\}} \left| \frac{1}{K_2} \sum_{j=1}^{i+\ell} X_j \right|^2 = O_P \left( m^{-2/3} (\log m)^2 \right).
\]

As \( m \) is large, there will be at most 1 jump in \((t_{M(n-1)+1-K_1}, t_{Mn+K_1-1})\), which is contained
in at most 2 intervals of this form. Thus,

\[
\sum_{n=1}^{m/M} (N_{Mn+K_1-1} - N_{M(n-1)+1-K_1})^2 = \sum_{n=1}^{m/M} (N_{Mn+K_1-1} - N_{M(n-1)+1-K_1}) \\
\leq 2(N_{m+K_1-1} - N_{1-K_1}).
\]

which is a.s. finite. Finally, since \(K_1^2 = \Theta \left( \frac{m^{1/3}}{\log m^2} \right) \), we conclude that \(m^{1/3}A_1 \xrightarrow{P} 0 \) as \(m \to \infty \). A similar argument holds for \(A_2 \).

In order to treat the other terms, observe that by Fan, Li, and Yu (2012, Theorem 1), for all \(x \in [0, a_1] \) and \(m \) large,

\[
P \left( \sup_{n \in \{1, \ldots, m/M\}} \left| \sum_{k=1}^{K_2} - \sum_{k=1}^{K_2} \int_{T_n-1}^{T_n} \sigma_s^2 ds \right| \geq x \right) \leq \alpha_2 \exp \left( -\alpha_3 x^2 m^{1/3} \right).
\]

Since \([T_n-1, T_n] = O \left( m^{-1/6} \log m \right) \), we conclude that

\[
\sup_{n \in \{1, \ldots, m/M\}} \left| \sum_{k=1}^{K_2} - \sum_{k=1}^{K_2} \int_{T_n-1}^{T_n} \sigma_s^2 ds \right| = O_P \left( m^{-1/6} \log m \right).
\]

Finally, since \(K_1 = \Theta \left( \frac{m^{1/3}}{\log m^2} \right) \), we obtain that \(m^{1/3}A_3 \xrightarrow{P} 0 \) as \(m \to \infty \). A similar argument holds for the rest of the terms, which concludes the desired result. \(\Box\)

We need the following preliminary lemma for the proof of Theorem 5.

**Lemma 1.** Under the assumptions of Theorem 5, there exists a subsequence \(m_k \) such that a.s. for any \(\delta > 0 \), as \(k \to \infty \),

\[
\sup_{i \in \{1, \ldots, m_k\}} \sup_{t \in [t_{i-1}, t_i]} (\triangle J_{2,i})^2 1_{A_{i,k}} \leq \delta + 16\rho^2(h_k),
\]

where \(A_{i,k} := \left\{ \left| \frac{1}{K_1} \sum_{j=1}^{j+K_1-1} (J_{2,j - J_{2,j-K_1}}) \right| \leq 2\rho(h_k) \right\} \) and \(h_k = \frac{1}{m_k} \).

**Proof.** Set \(R^m(t) = \sum_{t_{i-1} \leq t \leq t_i} (J_{2,i} - J_{2,i-1}) \). Then according to Metivier (1982, Theorem 25.1), there exists a subsequence \(m_k \) for which a.s. as \(k \to \infty \),

\[
R^m_k(t) \to [J_2]_t \quad \text{uniformly in } t \in [0, 1],
\]

where \([J_2]_t = \sum_{s \leq t} (\triangle J_{2,s})^2 \) (see Cont and Tankov (2004)). Thus, defining \(f_k(t) = R^m_k(t) - [J_2]_t \), we have a.s. as \(k \to \infty \),

\[
\sup_{i \in \{1, \ldots, m_k\}} \left| (J_{2,i} - J_{2,i-1})^2 - \sum_{s \in [t_{i-1}, t_i]} (\triangle J_{2,s})^2 \right| = \sup_{i \in \{1, \ldots, m_k\}} |f_k(t_i) - f_k(t_{i-1})| \leq 2 \sup_{t \in [0, 1]} |f_k(t)| \to 0.
\]

Therefore, given arbitrary \(\delta > 0 \), as \(k \to \infty \), we have

\[
\sup_{i \in \{1, \ldots, m_k\}} \left| (J_{2,i} - J_{2,i-1})^2 - \sum_{s \in [t_{i-1}, t_i]} (\triangle J_{2,s})^2 \right| < \delta.
\]

33
Therefore, we are left to show that a.s. as $k \to \infty$,
\begin{equation}
\sup_{i \in \{1, \ldots, m_k\}} (J_{2,i} - J_{2,i-1})^2 1_{A_{i,k}} \leq 16r^2(h_k).
\end{equation}

We write,
\begin{align*}
|J_{2,i} - J_{2,i-1}| 1_{A_{i,k}} & \left(1_{\{|J_{2,i} - J_{2,i-1}| \leq 4r(h_k)\}} + 1_{\{|J_{2,i} - J_{2,i-1}| > 4r(h_k)\}} \right) \\
& \leq |J_{2,i} - J_{2,i-1}| \left(1_{\{|J_{2,i} - J_{2,i-1}| \leq 4r(h_k)\}} + 1_{\{|J_{2,i} - J_{2,i-1}| > 4r(h)\}} \left(1_{B_{i,k}^c} + 1_{B_{i,k} 1_{A_{i,k}}} \right) \right),
\end{align*}

where
\begin{equation*}
B_{i,k} = \left\{ \sup_{\ell \in \{i-K_1+1, \ldots, i+K_1-1\}} |J_{2,\ell} - J_{2,\ell-1}| \leq \frac{r(h_k)}{K_1} \right\}.
\end{equation*}

Observe that for any $j \in \{i, \ldots, i+K_1-1\}$ it holds that
\begin{equation*}
|J_{2,j} - J_{2,j-K_1}| \geq |J_{2,i} - J_{2,i-1}| - K_1 \sup_{\ell \in \{i-K_1+1, \ldots, i+K_1-1\}} |J_{2,\ell} - J_{2,\ell-1}|.
\end{equation*}

Thus, on the event $\{|J_{2,i} - J_{2,i-1}| > 4r(h_k)\} \cap B_{i,k}$ we have that
\begin{equation*}
|J_{2,j} - J_{2,j-K_1}| \geq 4r(h_k) - \frac{r(h_k)}{K_1} \cdot K_1 = 3r(h_k).
\end{equation*}

On the other hand, on that same event, it holds that
\begin{equation*}
|J_{2,i} - J_{2,i-1}| > \left| \sum_{\ell \in \{j-K_1+1, \ldots, j\}} (J_{2,\ell} - J_{2,\ell-1}) \right|,
\end{equation*}

which implies that all the increments $J_{2,j} - J_{2,j-K_1}$ have the same sign as $J_{2,i} - J_{2,i-1}$.

Thus, we conclude that on the event $\{|J_{2,i} - J_{2,i-1}| > 4r(h_k)\} \cap B_{i,k}$,
\begin{equation*}
\frac{1}{K_1} \sum_{j=i}^{i+K_1-1} (J_{2,j} - J_{2,j-K_1}) \geq \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} |J_{2,j} - J_{2,j-K_1}| \geq 3r(h_k).
\end{equation*}

This implies that $1_{\{|J_{2,i} - J_{2,i-1}| > 4r(h_k)\}} 1_{B_{i,k} 1_{A_{i,k}}} = 0$.

On the other hand, since $J_2$ has independent and stationary increments with zero expectation and variance $ch$, using Chebyshev’s inequality, we get that
\begin{equation*}
P\left( \sup_{i \in \{1, \ldots, m_k\}} 1_{\{|J_{2,i} - J_{2,i-1}| > 4r(h_k)\}} 1_{B_{i,k}^c} = 1 \right) \leq C \frac{K_1^3 h_k}{r^4(h_k)} \left( \frac{K_1^3 \log m}{r^2(h) m^{1/3}} \right)^2 \frac{1}{K_1^3 m^{1/3} \log^2 m},
\end{equation*}

which tends to zero as $k \to \infty$ by hypothesis (5). Therefore, a.s. as $k \to \infty$,
\begin{equation*}
\sup_{i \in \{1, \ldots, m_k\}} |J_{2,i} - J_{2,i-1}| 1_{A_{i,k}} \leq \sup_{i \in \{1, \ldots, m_k\}} |J_{2,i} - J_{2,i-1}| 1_{\{|J_{2,i} - J_{2,i-1}| \leq 4r(h_k)\}},
\end{equation*}

which shows (15).
COROLLARY 1. Under the assumptions of Theorem 5 as \( m \to \infty \)

\[
Z_m := \frac{1}{K} \sum_{j=K}^{m} (J_{2,j} - J_{2,j-K})^2 1_{R_{j,K}} \to 0.
\]

where \( R_{j,K} = \left\{ \sup_{i\in\{j-K+1,\ldots,j\}} \frac{1}{K_i} \sum_{j=i}^{j+i+K_i-1} (J_{2,j} - J_{2,j-K_i}) \leq 2r(h) \right\}. \)

Proof. By Lemma 1, there exists a subsequence \( m_k \) such that a.s. for any \( \delta > 0 \), as \( k \to \infty \),

\[
\sup_{t\in[t_{j-K_k},t_{j}]} |\Delta J_{2,t}| 1_{R_{j,K_k}} \leq \sqrt{\delta + 16r^2(h_k)},
\]

where \( K_k = cm_k^{2/3} \). Therefore, a.s. for any \( \delta \) and large \( k \),

\[
Z_{m_k} \leq \frac{1}{K_k} \sum_{j=K_k}^{m_k} (J_{2,j} - J_{2,j-K_k})^2 \left\{ \sup_{t\in[t_{j-K_k},t_{j}]} |\Delta J_{2,t}| \leq \sqrt{\delta + 16r^2(h_k)} \right\}.
\]

Define the process \( Y \) by

\[
Y_t = \int_0^t \int_{|x| \leq \sqrt{\delta + 16r^2(h_k)}} x (\mu(dx, ds) - \nu(dx)ds) - t \int_{\sqrt{\delta + 16r^2(h_k)} \leq |x| \leq 1} x \nu(dx).
\]

Then, we have that a.s. for any \( \delta \) and large \( k \),

\[
Z_{m_k} \leq \frac{1}{K_k} \sum_{j=K_k}^{m_k} (Y_j - Y_{j-K_k})^2.
\]

On the other hand, observe that

\[
\frac{1}{K_k} \sum_{j=K_k}^{m_k} (Y_j - Y_{j-K_k})^2 \leq \sup_{i\in\{0,\ldots,K_k-1\}} \sum_{j=0}^{c_{i,m_k}} (Y_{i+j,K_k} - Y_{i+(j-1)K_k})^2,
\]

where \( c_{i,m_k} \) is the largest positive integer not larger than \( \frac{m_k-i}{K_k} \).

For any \( 0 \leq t \leq 1 \), define \( \ell_{m_k} \) as the largest positive integer not larger than \( \frac{tm_k}{K_k} \),

\( \Pi_{m_k} = \{0, t - \ell_{m_k} \frac{K_k}{m_k}, t - (\ell_{m_k} - 1) \frac{K_k}{m_k}, \ldots, t - \frac{K_k}{m_k} \}, \)

and

\[
L^{m_k}(t) = \sum_{s_i \in \Pi_{m_k}} \left( Y_{s_i} - Y_{(s_i - \frac{K_k}{m_k})^{0/\ell}} \right)^2.
\]

Observe that the partition \( \Pi_{m_k} \) is different from Metivier (1982, Theorem 25.1), but since \( s_i - (s_i - \frac{K_k}{m_k})^{0/\ell} \to 0 \) uniformly in \( s_i \in [0,1] \) as \( k \to \infty \), we have the same conclusion as that theorem, that is, there exists a subsequence \( m_{k\ell} \) such that a.s. as \( \ell \to \infty \)

\[
\sup_{t\in[0,1]} (L^{m_{k\ell}}(t) - [Y]_t) \to 0.
\]
This implies that a.s. as $\ell \to \infty$

$$\sup_{i \in \{0, \ldots, K_{k\ell} - 1\}} \left| \sum_{j=0}^{c_i m_{k\ell}} \left( Y_{i+j K_{k\ell}} - Y_{i+(j-1) K_{k\ell}} \right)^2 - [Y]_{i+c_i m_{k\ell}} K_{k\ell} \right| \to 0.$$  

Since $\delta$ is arbitrary and $r(h_{k\ell}) \to 0$ as $\ell \to \infty$, we get that a.s. as $\delta \to 0$ and $\ell \to \infty$,

$$[Y]_{i+c_i m_{k\ell}} \leq \int_0^1 \int_{|x| \leq \sqrt{\delta + 16 r^2(h_{k\ell})}} x^2 \mu(ds, dx) \to 0.$$  

Therefore, a.s. as $\ell \to \infty$ and $\delta \to 0$,

$$\sup_{i \in \{0, \ldots, K_{k\ell} - 1\}} \sum_{j=0}^{c_i m_{k\ell}} \left( Y_{i+j K_{k\ell}} - Y_{i+(j-1) K_{k\ell}} \right)^2 \to 0.$$  

Thus, we have shown that there exists a subsequence $m_{k\ell}$ such that as $\ell \to \infty$, $Z_{m_{k\ell}}$ tends to zero in probability. So we can extract a subsequence for which it tends to zero a.s. Repeating our reasoning, from each subsequence $Z_{m_k}$ we can extract a subsequence tending to zero a.s. Thus, $Z_{m_k}$ tends to zero in probability as $k \to \infty$. Repeating again the same reasoning, we conclude that $Z_m$ tends to zero in probability as $m \to \infty$.  

**Proof of Theorem** For simplicity we assume the drift is zero. Set $\bar{\pi}_t = x_t - J_{2, \ell}, \bar{\beta}_j = \frac{1}{K} \sum_{j=j}^{j+K-1} (\bar{\pi}_t - \bar{\pi}_{t-K}), \bar{\pi}_t = x_t - J_{1, \ell}, \bar{\beta}_j = \frac{1}{K} \sum_{j=j}^{j+K-1} (\bar{\pi}_t - \bar{\pi}_{t-K}), \bar{\pi}_t = x_t - J_{1, \ell}$, and $\bar{\beta}_j = \frac{1}{K} \sum_{j=j}^{j+K-1} (\bar{\pi}_t - \bar{\pi}_{t-K})$.

Then,

$$\left| \hat{\sigma}_{TTS}^2 - \int_0^1 \sigma_t^2 dt \right| \leq Y_1 + Y_2 + Y_3 + Y_4,$$

where

$$Y_1 = \frac{1}{K} \sum_{j=K}^m \left( (\bar{\pi}_j - \bar{\pi}_{j-K})^2 - (\bar{\pi}_j - \bar{\pi}_{j-1})^2 \right) 1_{A_j} - \int_0^1 \sigma_t^2 dt$$

$$Y_2 = \frac{1}{K} \sum_{j=K}^m \left( (\bar{\pi}_j - \bar{\pi}_{j-K})^2 - (\bar{\pi}_j - \bar{\pi}_{j-1})^2 \right) (1_{E_j} - 1_{A_j})$$

$$Y_3 = \frac{2}{K} \sum_{j=K}^m (J_{2, j} - J_{2, j-1})^2 1_{E_j} + \frac{2}{K} \sum_{j=K}^m (J_{2, j} - J_{2, j-1})^2 1_{E_j}$$

$$Y_4 = \frac{2}{K} \sum_{j=K}^m (\bar{\pi}_j - \bar{\pi}_{j-K})(J_{2, j} - J_{2, j-K}) 1_{E_j} + \frac{2}{K} \sum_{j=K}^m (\bar{\pi}_j - \bar{\pi}_{j-1})(J_{2, j} - J_{2, j-1}) 1_{E_j},$$

$$A_j = \left\{ \sup_{i \in \{j-K+1, \ldots, j\}} |\beta_i| \leq 2r(h) \right\} \text{ and } E_j = \{ |\beta_i| \leq r(h), i = j - K + 1, \ldots, j \}.$$  

According to Theorem $3$, $Y_1 \xrightarrow{p} 0$ as $m \to \infty$. We are left to proof that the other terms also converge to zero in probability as $m \to \infty$.  

\[ \text{36} \]
Proof that $Y_2 \xrightarrow{P} 0$. Observe that

$$1_{E_j} - 1_{A_j} = 1_{E_j \cap A_j^c} - 1_{A_j \cap E_j^c}.$$ 

Now, using the fact that

$$|\beta_\ell| \geq |\beta_{\ell+1}| - \frac{1}{K_1} \sum_{i=\ell}^{\ell+K_1-1} (J_{2,i} - J_{2,i-K_1}),$$

on the event $E_j \cap A_j^c$, we get that

$$\sup_{\ell \in \{j-K+1,\ldots,j+K_1\}} |J_{2,\ell} - J_{2,\ell-K_1}| \geq r(h),$$

which implies the event

$$C_j = \left\{ \sup_{\ell \in \{j-K-K_1+1,\ldots,j-K_1\}} |J_{2,\ell} - J_{2,\ell+1}| \geq \frac{r(h)}{K_1} \right\}.$$

Therefore, we have shown that

$$E_j \cap A_j^c \subset C_j \cap A_j^c. \quad (16)$$

By the first inequality of Theorem 1, we have a.s. as $m \to \infty$ for all $j \in \{K,\ldots,m\}$

$$A_j^c \subset D_j = \{N_{j+K_1} - N_{j-K_1} > 0\}. \quad (17)$$

Next, using Chebyshev's inequality, the fact that $J_2$ is independent of $N$ and that $K > K_1$ for large $m$, we get that

$$P \left( \sup_{j \in \{K,\ldots,m\}} 1_{C_j \cap D_j} = 1 \right) \leq m^{-1/3} \frac{K_1^2}{r^2(h)} = \frac{K_1^3 \log m}{r^2(h)m^{1/3} K_1 \log m}, \quad (18)$$

which tends to zero as $m \to \infty$ by hypothesis (5).

Consequently, we conclude that as $m \to \infty$

$$\left| \frac{1}{K} \sum_{j=K}^{m} \left( (\bar{x}_j - \bar{x}_{j-K})^2 - (\bar{x}_j - \bar{x}_{j-1})^2 \right) 1_{E_j \cap A_j^c} \right| \xrightarrow{P} 0.$$ 

Thus, we are left to show that as $m \to \infty$

$$\left| \frac{1}{K} \sum_{j=K}^{m} \left( (\bar{x}_j - \bar{x}_{j-K})^2 - (\bar{x}_j - \bar{x}_{j-1})^2 \right) 1_{A_j \cap E_j^c} \right| \xrightarrow{P} 0. \quad (19)$$

By the second inequality of Theorem 1 a.s. as $m \to \infty$ for all $j \in \{K,\ldots,m\}$,

$$A_j \subset \{N_j - N_{j-K} = 0\}, \quad (20)$$

37
which implies that a.s. as \( m \to \infty \)

\[
\frac{1}{K} \sum_{j=K}^{m} \left( (\bar{x}_j - \bar{x}_{j-K})^2 - (\bar{x}_j - \bar{x}_{j-1})^2 \right) 1_{A_j \cap E^c_{j}}
\]

\[
= \frac{1}{K} \sum_{j=K}^{m} \left( (\bar{x}_j - \bar{x}_{j-K})^2 - (\bar{x}_j - \bar{x}_{j-1})^2 \right) 1_{A_j \cap E^c_{j}} \left( 1_{D_j} + 1_{D^c_{j}} \right).
\]

Similar computations as in the proof of Theorem 2 show that

\[
\sup_{j \in \{K, \ldots, m\}} \frac{1}{K} \left| (\bar{x}_j - \bar{x}_{j-K})^2 - (\bar{x}_j - \bar{x}_{j-1})^2 \right| = O_P \left( m^{-2/3} \log m \right).
\]

On the other hand, using (20), we obtain that

\[
\sum_{j=K}^{m} 1_{A_j \cap E^c_{j} \cap D_j} \leq \sum_{j=K}^{m} 1_{\{N_j - N_{j-K} = 0\}} 1_{D_j}
\]

\[
\leq \sum_{j=K}^{m} 1_{\{N_j + K - 1 - N_j > 0\}} + \sum_{j=K}^{m} 1_{\{N_{j-K} - N_{j-K-1} > 0\}}
\]

\[
\leq K_1 (N_{m+K} - N_K) + K_1 (N_{m-K} - N_{1-K}).
\]

Therefore,

\[
\frac{1}{K} \sum_{j=K}^{m} \left( (\bar{x}_j - \bar{x}_{j-K})^2 - (\bar{x}_j - \bar{x}_{j-1})^2 \right) 1_{A_j \cap E^c_{j} \cap D_j}
\]

\[
\leq \sup_{j \in \{K, \ldots, m\}} \frac{1}{K} \left| (\bar{x}_j - \bar{x}_{j-K})^2 - (\bar{x}_j - \bar{x}_{j-1})^2 \right| \sum_{j=K}^{m} 1_{A_j \cap E^c_{j} \cap D_j}
\]

\[
= O_P \left( K_1 m^{-2/3} \log m \right) = O_P \left( \frac{K_1^2 \log m}{r^2(h) m^{1/3} K^2 \log m^{1/3}} \right),
\]

which tends to 0 as \( m \to \infty \) by hypothesis (5).

Thus, we are left to show that as \( m \to \infty \)

\[
\frac{1}{K} \sum_{j=K}^{m} \left( (\bar{x}_j - \bar{x}_{j-K})^2 - (\bar{x}_j - \bar{x}_{j-1})^2 \right) 1_{A_j \cap E^c_{j} \cap D^c_{j}} \xrightarrow{p} 0. \tag{21}
\]

By (17), we have that a.s. as \( m \to \infty \) for all \( j \in \{K, \ldots, m\}, \)

\[
A_j \cap E^c_{j} \cap D^c_{j} = E^c_{j} \cap D^c_{j}.
\]

We next show that a.s. as \( m \to \infty \) for all \( j \in \{K, \ldots, m\}, \)

\[
E^c_{j} \cap D^c_{j} = F_j := \left\{ \sup_{i \in \{j-K+1, \ldots, j\}} |\hat{\beta}_i| > r(h) \right\}. \tag{22}
\]

Observe that

\[
F_j = (E^c_{j} \cap D^c_{j}) \cup (F_j \cap D_j).
\]
Moreover,
\[
\sup_{j \in \{K, \ldots, m\}} \left( F_j \cap D_j \right) \leq \left\{ \sup_{j \in \{1, \ldots, m\}} \left| \tilde{\beta}_j \right| > \frac{r(h)}{2} \right\} \cup
\sup_{j \in \{K, \ldots, m\}} \left\{ \sup_{i \in \{j - K - K_1 + 1, \ldots, j + K_1 - 2\}} \left| J_{2, i + 1} - J_{2, i} \right| \geq \frac{r(h)}{2K_1} \right\} \cap D_j.
\]

Therefore, from the second inequality of Theorem 1 and (18), we obtain (22).

Following similarly as in the proof of (12), we have that
\[
\left| \frac{1}{K} \sum_{j=K}^{m} \left( (\tilde{x}_j - \tilde{x}_{j-K})^2 - (\tilde{x}_j - \tilde{x}_{j-1})^2 \right) 1_{F_j} \right| \leq \sup_{i \in \{K, \ldots, m\}} \sum_{j=K}^{i+\ell,m} \left( (\tilde{x}_j - \tilde{x}_{j-K})^2 - (\tilde{x}_j - \tilde{x}_{j-1})^2 \right) \leq \sum_{i=K}^{m} 1_{\left\{ |\tilde{\beta}_i| > r(h) \right\}},
\]
and as \( m \to \infty \),
\[
\sup_{i \in \{K, \ldots, m\}} \sum_{j=K}^{i+\ell,m} \left( (\tilde{x}_j - \tilde{x}_{j-K})^2 - (\tilde{x}_j - \tilde{x}_{j-1})^2 \right) = O_p \left( m^{-1/3} \log m \right).
\]

Moreover, by the definition of \( \tilde{\beta}_i \), we have a.s. as \( m \to \infty \) that
\[
\sum_{i=K}^{m} 1_{\left\{ |\tilde{\beta}_i| > r(h) \right\}} \leq \sum_{i=K}^{m} 1_{\left\{ 1_{\sum_{t=1}^{K_1+1}(J_{2, t} - J_{2, t-K_1}) > \frac{r(h)}{2K_1}} \right\}} \leq \sum_{i=K}^{m} 1_{\left\{ \sup_{t \in \{i - K_1, \ldots, i + K_1 - 2\}} |J_{2, t+1} - J_{2, t}| > \frac{r(h)}{2K_1} \right\}} \leq 2K_1 \sum_{i=K-K_1}^{m+K_1-2} 1_{\left\{ |J_{2, i+1 - J_{2, i}| > \frac{r(h)}{2K_1} \right\}}. \tag{23}
\]

Next, using Chebyshev’s inequality, we get that
\[
E \left( K_1 m^{-1/3} \log m \sum_{i=K-K_1}^{m+K_1-2} 1_{\left\{ |J_{2, i+1} - J_{2, i}| > \frac{r(h)}{2K_1} \right\}} \right) \leq C \frac{K_1^3 \log m}{m^{1/3} \gamma^2(h)},
\]
which tends to zero as \( m \to \infty \) by hypothesis (5). This shows (21) and \( Y_2 \overset{p}{\to} 0 \).

Proof of \( Y_3 \overset{p}{\to} 0 \). We only treat the first term since the second can be treated similarly. Consider the event \( R_{j,K} \) of Corollary 1. We write
\[
(E_j \cap R_{j,K}^c) \subset (E_j \cap R_{j,K}^c \cap D_j^c) \cup (R_{j,K}^c \cap D_j).
\]
Hence, by the definition of $\beta_i$ and $\tilde{\beta}_i$,
\[
\sup_{j \in \{K, \ldots, m\}} (E_j \cap R_{j,K}^c) \leq \sup_{i \in \{1, \ldots, m\}} \{ |\tilde{\beta}_i| > r(h) \} \cup \sup_{j \in \{K, \ldots, m\}} \left( \left\{ \sup_{\ell \in \{j-K+1, \ldots, j+K-2\}} |J_{2,\ell+1} - J_{2,\ell}| \geq \frac{r(h)}{2K} \right\} \cap D_j \right).
\]

Therefore, by the first inequality of Theorem 1 and (18), we obtain that as $m \to \infty$,
\[
P \left( \sup_{j \in \{K, \ldots, m\}} 1_{E_j \cap R_{j,K}^c} = 1 \right) \to 0.
\]

Thus, we conclude that a.s. as $m \to \infty$
\[
\frac{1}{K} \sum_{j=K}^{m} (J_{2,j} - J_{2,j-K})^2 1_{E_j \cap (R_{j,K}^c \cup R_{j,K})} \leq \frac{1}{K} \sum_{j=K}^{m} (J_{2,j} - J_{2,j-K})^2 1_{R_{j,K}},
\]

which converges to zero in probability as $m \to \infty$ according to Corollary 1.

**Proof of $Y_4 \to 0$.** We only treat the first term since the second can be treated similarly. By Cauchy-Schwarz inequality,
\[
\left| \frac{1}{K} \sum_{j=K}^{m} \left( \int_{t_j}^{t_{j-K}} \sigma_u dB_u \right) (J_{2,j} - J_{2,j-K}) 1_{E_j} \right| \leq \sqrt{\frac{1}{K} \sum_{j=K}^{m} \left( \int_{t_j}^{t_{j-K}} \sigma_u dB_u \right)^2} \sqrt{\frac{1}{K} \sum_{j=K}^{m} (J_{2,j} - J_{2,j-K})^2 1_{E_j} \to 0,}
\]

as $m \to \infty$, since the first term converges in probability to $\sqrt{\int_0^1 \sigma_u^2 du}$ according to (10), and the second term converges to zero in probability as shown in the proof of $Y_3 \to 0$.

By (24), we have a.s. as $m \to \infty$,
\[
\frac{1}{K} \sum_{j=K}^{m} (u_j - u_{j-K})(J_{2,j} - J_{2,j-K}) 1_{E_j} = \frac{1}{K} \sum_{j=K}^{m} (u_j - u_{j-K})(J_{2,j} - J_{2,j-K}) 1_{E_j \cap R_{j,K}} \leq |I_1| + |I_2|,
\]

where
\[
I_1 = \frac{1}{K} \sum_{j=K}^{m} (u_j - u_{j-K})(J_{2,j} - J_{2,j-K}) 1_{R_{j,K}},
\]
\[
I_2 = \frac{1}{K} \sum_{j=K}^{m} (u_j - u_{j-K})(J_{2,j} - J_{2,j-K}) 1_{E_{j \cap R_{j,K}}},
\]

Now, for each $j \in \{K, \ldots, m\}$, set $d_{j,K}$ as the largest positive integer not larger than $\frac{j}{K}$, and we separate the sum in $I_1$ between odd and even terms. Because the $u_i$ are i.i.d.
$N(0, \eta^2)$ independent of $J_2$, the random variable

$$\frac{1}{K} \sum_{\text{odd } d_j} (u_j - u_j - K)(J_{2,j} - J_{2,j} - K) \frac{1}{R_{j,K}} \sqrt{\frac{1}{K^2} \sum_{\text{odd } d_j} (J_{2,j} - J_{2,j} - K)^2} \frac{1}{R_{j,K}}$$

is $N(0, \eta^2)$. By Corollary 1 as $m \to \infty$,

$$\frac{1}{K^2} \sum_{\text{odd } d_j} (J_{2,j} - J_{2,j} - K)^2 \frac{1}{R_{j,K}} \to 0.$$

Therefore, as $m \to \infty$,

$$\frac{1}{K} \sum_{\text{odd } d_j} (u_j - u_j - K)(J_{2,j} - J_{2,j} - K) \frac{1}{R_{j,K}} \to 0.$$

A similar argument follows for the even terms. Thus, we conclude that as $m \to \infty$, $I_1$ tends to zero in probability. We next treat $I_2$. Following similarly as in the proof of (12), we have that

$$|I_2| \leq \sup_{i \in \{K, \ldots, m\}} \left| \frac{1}{K} \sum_{j=i}^{\min(i+\ell, m)} (u_j - u_j - K)(J_{2,j} - J_{2,j} - K) \frac{1}{R_{j,K}} \sum_{i=1}^{m} \{ \left| \hat{\beta}_i \right| > \tau(h) \} \right|.$$

Because the $u_i$ are i.i.d. $N(0, \eta^2)$ independent of $J_2$, and using Corollary 1, we have that as $m \to \infty$,

$$\sup_{i \in \{K, \ldots, m\}} \left| \frac{1}{K} \sum_{j=i}^{\min(i+\ell, m)} (u_j - u_j - K)(J_{2,j} - J_{2,j} - K) \frac{1}{R_{j,K}} \right| = O_p \left( \sqrt{\frac{\log m}{K}} \right).$$

On the other hand, using (23), we get that

$$\sum_{i=1}^{m} \{ \left| \hat{\beta}_i \right| > \tau(h) \} \leq m \sum_{i=1}^{m} \left( \{ N_{i+K-1} - N_{i-K} > 0 \} + \{ \left| \hat{\beta}_i \right| > \tau(h) \} \right) \leq 2K_1(N_{m+K_1-1} - N_{1-K_1}) + 2K_1 \sum_{i=K-K_1}^{m+K_1-2} \{ \left| J_{2,i+1} - J_{2,i} \right| > \frac{\tau(h)}{K_1} \}.$$

Next, using Chebyshev’s inequality, we have that

$$E \left( K_1 \sqrt{\frac{\log m}{K}} \sum_{i=K-K_1}^{m+K_1-2} \{ \left| J_{2,i+1} - J_{2,i} \right| > \frac{\tau(h)}{K_1} \} \right) \leq C \sqrt{\frac{\log m}{K}} \frac{K_1^3}{K} \frac{1}{m^{1/3}} \frac{1}{(\tau(h))^2} \frac{1}{\sqrt{\log m}},$$

which tends to zero as $m \to \infty$ by hypothesis (5). Since $K_1 \sqrt{\frac{\log m}{K}}$ also converges to zero as $m \to \infty$ by hypothesis (5), we conclude that $I_2$ tends to 0 in probability as $m \to \infty$. 

41
Thus, we have shown that as $m \to \infty$,

$$\frac{1}{K} \sum_{j=K}^{m} (u_j - u_{j-2K})(J_{2,j} - J_{2,j-2K})1_{E_j} \xrightarrow{p} 0.$$ 

We are left to show that as $m \to \infty$

$$\frac{1}{K} \sum_{j=K}^{m} (J_{1,j} - J_{1,j-2K})(J_{2,j} - J_{2,j-2K})1_{E_j} \xrightarrow{p} 0.$$ 

This follows easily since on the event $E_j \cap A_j^c$ it tends to zero by (16), (17) and (18), and on $E_j \cap A_j$ all the terms of the sum are zero by (20). Hence, we conclude that $Y_4 \xrightarrow{p} 0$. \qed