Detecting Price Jumps in the Presence of Market Microstructure Noise

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Abstract

In this paper we design a test to detect the arrivals of jumps in asset prices contaminated by market microstructure noise. This test is defined by means of the truncated two-scales realized volatility estimator, recently introduced in Brownlees, Nualart, and Sun (2016), which is a robust estimator of the realized volatility in the presence of price jumps and market microstructure noise. We derive the asymptotic value of the power of the test given the significance level, and provide conditions for the test to be consistent. Simulations show that the test performs satisfactorily when the sampling frequency is large. In particular, we show that the test performs better than some prevalent jump tests.

Keywords: Jumps, Market Microstructure Noise, High-Frequency Price Data

JEL: C12, C14, C58

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1 Introduction

Discontinuities in the path of asset prices are typically modeled with jumps, and jumps are the focus of a large segment of the literature. For example, Bakshi, Cao, and Chen (1997) show how jumps affect option values. Moreover, Tauchen and Zhou (2011) study the influence of the jump risk factor on the variation of the credit default swap spreads. Accordingly, several contributions in the literature focus on detecting jumps based on high-frequency data, and some examples include the works by Andersen, Bollerslev, and Diebold (2007), Aït-Sahalia and Jacod (2009), Corsi, Pirino, and Reno (2010), Lee and Mykland (2008), Aït-Sahalia and Jacod (2011) and Li, Todorov, Tauchen, and Lin (2016).

In this article, we propose a novel non-parametric jump detection technique. We assume the presence of market microstructure noise in the observed high-frequency prices. In order to explore the jump arrival time, we divide the whole period under consideration into small intervals. Then for each interval we derive a statistic to detect whether there is a jump on it. The statistic is defined in terms of the truncated two-scales realized volatility estimator (TTSRV) proposed by Brownlees et al. (2016), which is a consistent volatility estimator robust to both jumps and noise. We compare the absolute value of the statistic with some threshold. The choice of the threshold depends on the significance level and the asymptotic distribution of the statistic when there is no jump on the interval. If the threshold is small, we reject the null hypothesis and believe there is a jump on the interval. For a generic jump, it is likely to be detected, since it significantly increases the magnitude of the statistic. Once we have identified an interval as containing a jump, the jump size can be consistently estimated by averaging the overlapping local returns that include the interval. The test proposed by Lee and Mykland (2008) has the same goal as ours, since we both aim to detect jump arrival times. Our advantage is that this method is vulnerable to the noise which is common in financial markets according to the work by Hansen and Lunde (2012).

The simulations indicate that when the sampling frequency increases, the test becomes more efficient, and when the sampling frequency is high enough, the test reaches satisfactory performance. This supports our theory that the test is consistent. Moreover,
the simulations show that jumps with larger size are easier to detect, which is in line with the intuition. The simulations also demonstrate the superiority of our test compared to the tests by Lee and Mykland (2008) and Jiang and Oomen (2008) when assuming a moderate size of the variance of the noise.

In the literature there are essentially two approaches for testing jumps: parametric and non-parametric. The parametric approach typically assumes a specific model to characterize jumps. Such models can then be estimated on the basis of observed data. For example, Chernov, Gallant, Ghysels, and Tauchen (2003), and Pan (2002) adopt the Efficient Method of Moments (EMM) and the Generalized Method of Moments (GMM) to measure the jump-related parameters, respectively. Other parametric approaches related to jump models can be found, among others, in Bakshi et al. (1997), Bates (2000), and Piazzesi (2005). Since jump occurrence in financial markets is quite irregular, parametric approaches are likely to commit misspecifications in spite of their intensive computations. Accordingly, the results obtained can be counterintuitive and insignificant. The non-parametric jump tests typically propose statistics to detect the existence of realized jumps, since the jumps can significantly affect the behavior of the statistics. For example, Andersen et al. (2007) consider the difference between the bipower variation (BPV) and the realized volatility (RV) as the statistic to test jumps. When there are no jumps, both the BPV and the RV are consistent estimators of the quadratic variation of the asset price process. Jumps can increase the difference between the BPV and the RV, since the BPV is immune to jumps and the RV is not. Jiang and Oomen (2008) explore jumps also by comparing the RV to another volatility measure, since their difference is sensitive to jumps. Some other non-parametric jump detection methods can be found, among others, in Aït-Sahalia and Jacod (2009), Aït-Sahalia, Jacod, and Li (2012), Corsi et al. (2010) and Lee and Mykland (2008).

However, most of the approaches concerning non-parametric jump tests are not able to pin down the precise jump arrival time or size. That is, they can only test whether there are jumps on a given period, without further information on their number, arrival time or size. Another obstacle of jump detection is market microstructure noise. It makes
the observed price different from the efficient one, and turns most non-parametric jump tests invalid. A notable exception appears in Aït-Sahalia et al. (2012), where a statistic robust to noise is proposed. Jiang and Oomen (2008) also propose a noise robust statistic, but they require the spot volatility to be constant, which is hardly true in reality. In this paper, we introduce a methodology that allows to detect jump times and sizes in the presence of noise.

The rest of the article is organized as follows: In Section 2 we define the basic setting, the statistic and discuss its asymptotic distribution. In Section 3 we obtain the limiting probabilities of testing errors. In Section 4 we perform Monte Carlo simulations to evaluate the efficiency of our test and compare it to the tests proposed by Lee and Mykland (2008) and Jiang and Oomen (2008). Section 5 concludes.

2 Framework and Asymptotic Theory

We use $y_t$ to denote the value of the efficient log-price of some asset at time $t$, where $t \in [0,1]$. The process $y_t$ starts at $y_0 \in \mathbb{R}$, and is defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, P)$. Its dynamics are characterized by

$$dy_t = a_t dt + \sigma_t dB_t + dJ_t, \quad t \in [0,1]. \quad (1)$$

Here $B_t$ is a standard Brownian motion. The drift $a_t$ and the spot volatility $\sigma_t$ are progressively measurable processes on $[0,1]$. $J_t$ is a finite activity jump process independent of $(\sigma, B)$, of the form $J_t = \sum_{i=1}^{N_t} Y_i$. $N_t$ is a non-explosive counting process, and $Y_i$ are i.i.d. random variables and independent of $N$. Thus the solution $(y_t, t \in [0,1])$ to equation (1) is unique in the strong sense, adapted and càdlàg (see for example Ikeda and Watanabe (1981)).

In addition, we assume that due to the presence of microstructure noise in the market, the efficient price $y_t$ is not observable. That is, the observed transaction price denoted as $x_t$ is different from $y_t$. To this extent, we assume that the timestamps when we observe the transaction prices are $0 = t_0 < t_1 < \cdots < t_m = 1$ where $t_i = \frac{i}{m}$, and $x_{t_i}$ and $y_{t_i}$ are
related by
\[ x_{ti} = y_{ti} + u_{ti}, \quad i = 1, \ldots, m, \]
where \( u_{ti} \) is the noise component for the \( i \)th trade. \( u_{ti} \) is assumed to be a discrete i.i.d process, independent of the process \( y_{ti} \) and such that \( u_{ti} \sim N(0, \eta^2) \), where \( \eta \) is a positive constant. We use \( x_{ti}, y_{ti} \) and \( u_{ti} \) to denote respectively the processes \( x_{ti}, y_{ti} \) and \( u_{ti} \) to simplify the notation.

Moreover, we assume the following conditions on the dynamics of \( a_{ti} \) and \( \sigma_{ti} \).

**Assumption 1.** \( \sigma_{ti} \) is positive, and there exists constants \( a > 0, \sigma^+ > \sigma^- > 0 \) such that \( |a_{ti}| < a, \sigma^- < \sigma_{ti} < \sigma^+ \) for all \( t \in (0, 1] \).

**Assumption 2.** For any constants \( c > 0, \epsilon > 0, 0 < \alpha < 1 \),
\[
\sup_i \sup_{t_i \leq u \leq t_i + cm^{\alpha-1}} |\sigma_u - \sigma_{ti}| = O_P(m^{\frac{1}{2}(\alpha-1)+\epsilon}), \tag{2}
\]
as \( m \to \infty \).

Intuitively, Assumption 2 implies that the spot volatility \( \sigma_{ti} \) does not change much over a short time interval. It is reasonable in view of Lemma 2 in Mykland and Zhang (2006), which shows that (2) holds when \( \sigma_{ti} \) is a general Itô process. For example, in Podolskij and Vetter (2009) and Christensen, Oomen, and Podolskij (2010), they assume that the dynamics of \( \sigma_{ti} \) are:
\[
d\sigma_{ti} = b_{ti}dt + c_{ti}dW_t + d_{ti}dV_t, \tag{3}
\]
where \( b_{ti}, c_{ti}, d_{ti} \) are bounded and adapted processes, and \( W_t, V_t \) are independent Brownian motions.

### 2.1 The statistic

In this subsection we define the statistic that will be used in order to test jumps. The construction of the statistic requires the estimation of the spot volatility \( \sigma_{ti} \). The volatility estimator we will use is the truncated two-scales realized volatility (TTSRV) introduced in Brownlees et al. (2016). In this paper, the authors show that the TTSRV consistently
estimates the integrated volatility \( \int_0^1 \sigma_t^2 dt \) in the presence of noise and jumps. The TTSRV is derived from the two-scales realized volatility estimator (TSRV) introduced in Zhang, Mykland, and Aït-Sahalia (2005), by truncating the intervals that are likely to contain a jump. The reason why the TTSRV is immune to jumps is that the intervals that contain jumps are truncated from the estimator. Let us now give a review on the mechanism of the truncation.

In order to detect these intervals, the TTSRV uses the local average return \( \beta_i \) which is defined as

\[
\beta_i(K_0) = \frac{1}{K_0} \sum_{j=i}^{i+K_0-1} (x_j - x_{j-K_0}), i = 1, \ldots, m
\]

where \( K_0 = cm^\gamma \) and \( \gamma \in \left( 0, \frac{1}{3} \right) \). \(|\beta_i|\) is small when there are no jump on the interval, but when there is a jump on \((t_{i-1}, t_i]\), \( \beta_i \) is roughly equal to the jump size. This can be quantified in terms of a threshold \( r(m) \). Accordingly, the TTSRV is defined as

\[
\hat{\sigma}_{\text{TTS}}^2 = \frac{1}{K} \sum_{j=K}^{m} (x_j - x_{j-K})^2 1_{E_j} - \frac{1}{K} \sum_{j=K}^{m} (x_j - x_{j-1})^2 1_{E_j},
\]

where \( K = cm^2 \), and \( E_j = \{ |\beta_i| \leq r(m), \text{ for all } i = j - K + 1, \ldots, j \} \). The threshold is chosen as \( r(m) = m^{-\beta} \) where \( \beta \in \left( 0, \frac{1}{2} \right) \), in order to make it larger than the contribution of the continuous and noise terms in \( \beta_i \).

We are now going to define the test statistic that will detect jumps on each small interval of \((0, 1]\). Specifically, as shown in Figure 1, the period \((0, 1] \) is split into \( \frac{m}{2K_1} \) intervals, and the \( i \)th interval is \( \left( \frac{2(i-1)K_1}{m}, \frac{2iK_1}{m} \right) \), where \( K_1 = m^{\alpha_1} \) and \( \alpha_1 \in (0, 1) \). The value of \( \alpha_1 \) will be choosen later, and for simplicity, we assume \( \frac{m}{2K_1} \) is an integer.

INSERT Figure 1 ABOUT HERE.

In order to construct the statistic for the interval \( \left( \frac{2(i-1)K_1}{m}, \frac{2iK_1}{m} \right) \), we first define the rescaled sum of returns that will smooth away the effect of the noise by:

\[
A_i = M \sum_{j=K_1}^{2K_1} \left( x_{j+2(i-1)K_1} - x_{j-K_1+2(i-1)K_1} \right),
\]
where $M = \sqrt{\frac{3m}{K_1(K_1+1)(2K_1+1)}}$. As shown in Figure 2, $A_i$ is obtained by rescaling the sum of the returns over the intervals that lie inside $\left[\frac{2(i-1)K_1}{m}, \frac{2iK_1}{m}\right]$. Each of these intervals has the form $\left[\frac{2(i-1)K_1+j-1}{m}, \frac{(2i-1)K_1+j-1}{m}\right]$, for $j \in \{K_1, K_1 + 1, \ldots, 2K_1\}$. For large $m$, since the contribution to $A_i$ by the drift $u_i$ and the noise $u_j$ is negligible, $A_i$ is dominated by the dynamics of $(\sigma_i, B_i)$. When there is no jump on $\left[\frac{2(i-1)K_1}{m}, \frac{2iK_1}{m}\right]$, it can be checked that

$$A_i \approx \sigma_{\frac{2(i-1)K_1}{m}} \cdot Z,$$

where $Z$ is a standard normal random variable.

In order to get rid of the spot volatility $\sigma_{\frac{2(i-1)K_1}{m}}$ in the asymptotic distribution of $A_i$, we need to construct an estimator of $\sigma_{\frac{2(i-1)K_1}{m}}$. We will use the TTSRV to define an estimator of $\int_{\frac{2(i-1)K_1}{m}}^{\frac{2iK_1}{m}} \sigma_t^2 dt$. Here $K_2 = cm^{\alpha_2}$ and $\alpha_2 \in (0,1)$ whose value will be specified later. Because of Assumption 1, $\int_{\frac{2(i-1)K_1}{m}}^{\frac{2iK_1}{m}} \sigma_t^2 dt \approx \frac{K_2}{m} \sigma_{\frac{2(i-1)K_1}{m}}^2$. Then we can estimate $\sigma_{\frac{2(i-1)K_1}{m}}$ by taking the square root of the rescaled TTSRV. That is,

$$\hat{\sigma}_{\frac{2(i-1)K_1}{m}} = \sqrt{\frac{m}{K_2} \hat{\sigma}_{\text{TTSRV}}^2}.$$

Observe the TTSRV in the interval $\left[\frac{2(i-1)K_1-K_2}{m}, \frac{2(i-1)K_1}{m}\right]$, we have

$$\hat{\sigma}_{\text{TTSRV}}^2 = \frac{1}{K_3} \sum_{j=2(i-1)K_1-K_2+K_3}^{2(i-1)K_1} (x_j - x_{j-K_3})^2 1_{E_j} - \frac{1}{K_3} \sum_{j=2(i-1)K_1-K_2+K_3}^{2(i-1)K_1} (x_j - x_{j-1})^2 1_{E_j},$$

where $E_j = \{|\hat{\beta}_i(K_4)| \leq r(m), \text{ for all } i = j - K_3 + 1, \ldots, j\}$. Here $K_3 = m^{\alpha_3}$, $K_4 = m^{\alpha_4}$ and $r(m) = m^{-\alpha_2}$, and $\alpha_3, \alpha_4, \alpha_5 \in (0,1)$.

Then the statistic $S_i$ can be obtained by standardizing $A_i$ with $\hat{\sigma}_{\frac{2(i-1)K_1}{m}}$, that is,

$$S_i = \frac{A_i}{\hat{\sigma}_{\frac{2(i-1)K_1}{m}}}.$$

The following lemma shows how to choose the values of $\alpha_2, \alpha_3, \alpha_4, \alpha_5$ in order to make
the denominator a consistent estimator of \( \sigma_{2(i-1)K_1}^2 \).

**Lemma 1.** Assume \( 0 < \alpha_3 < \alpha_2 < 1, \alpha_5 < \frac{\alpha_4}{2} \) and \( 0 < \alpha_4 < \frac{1}{4} \). Then for any \( \epsilon > 0 \), we have

\[
\sup_{i \in \{1, \ldots, \frac{m}{2K_1}\}} \left| \frac{m}{K_2} \hat{\sigma}_{TTSi}^2 - \sigma_{2(i-1)K_1}^2 \right| = O_P \left( m^{\bar{\alpha} + \epsilon} \right),
\]

as \( m \to \infty \), where \( \bar{\alpha} = \max \left( \frac{1}{2}(\alpha_3 - \alpha_2), 1 - \frac{1}{2}\alpha_2 - \alpha_3, \frac{1}{2}(\alpha_2 - 1), \frac{2}{3} + \alpha_4 - \alpha_2 \right) \).

The proof of Lemma 1 is similar to the proof of the consistency of the TTSRV in Brownlees et al. (2016). First we show that since the returns affected by jumps are removed from \( \hat{\sigma}_{TTSi}^2 \), the difference between \( \hat{\sigma}_{TTSi}^2 \) constructed with the jump-involved price data and the TSRV based on the no-jump data is caused only by the information loss due to the truncations. Then we compute the magnitude of the information loss, and the difference between the TSRV and \( \int_{2(i-1)K_1}^{2(i)K_1} \sigma_i^2 dt \).

Given (11), in order to minimize the difference between \( \frac{m}{K_2} \hat{\sigma}_{TTSi}^2 \) and \( \sigma_{2(i-1)K_1}^2 \), it is natural to minimize \( \bar{\alpha} \) over the values of \( \alpha_2, \alpha_3, \alpha_4 \). It is easy to check that the minimum of \( \bar{\alpha} \) is \( -\frac{1}{12} \), which is achieved when \( \alpha_2 = \frac{5}{6}, \alpha_3 = \frac{2}{3}, 0 < \alpha_4 < \frac{1}{4} \) and \( 0 < \alpha_5 < \frac{\alpha_4}{2} \). In the rest of the paper we will assume this set of values for \( \alpha_2, \alpha_3, \alpha_4, \alpha_5 \) unless stated otherwise.

### 2.2 Under the null hypothesis

In this subsection we compute the asymptotic distribution of \( S_i \) under the null hypothesis that there is no jump on \( \left( \frac{2(i-1)K_1}{m}, \frac{2K_1}{m} \right) \). Since for large \( m \), \( A_i \) is roughly \( \sigma_{2(i-1)K_1} \) times a standard normal random variable, and \( \hat{\sigma}_{2(i-1)K_1} \) is approximately \( \sigma_{2(i-1)K_1} \), \( S_i \) is asymptotically standard normal, as shown in the following theorem where

\[
B_m = \left\{ i \in \left\{ 1, \ldots, \frac{m}{2K_1} \right\}, \text{ such that } N_{i2/K_1} - N_{i2(i-1)K_1} = 0 \right\}. \tag{12}
\]

**Theorem 1.** For any \( \epsilon > 0 \), as \( m \to \infty \),

\[
\sup_{i \in B_m} \left| S_i - \hat{S}_i \right| = O_P \left( m^{\beta + \epsilon} \right), \tag{13}
\]

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where \( \beta = \max \left( \frac{1}{2} - \alpha_1, \frac{1}{2}\alpha_1 - \frac{1}{2}, -\frac{1}{12} \right) \), and

\[
\hat{S}_i = M \sum_{j=0}^{K_1} \left( B_{t(2i-1)K_1+j} - B_{t(2i-1)K_1+j} \right). \tag{14}
\]

Moreover, \( \hat{S}_i \) is a standard normal random variable.

Let \( \alpha_1 \in \left( \frac{1}{2}, 1 \right) \) as a constant on \( \left( \frac{1}{2}, 1 \right) \). Then Theorem 1 asserts that the difference between \( S_i \) and \( \hat{S}_i \) is \( o_P(1) \), which means that asymptotically, \( S_i \) is also standard normal. In addition, the \( \hat{S}_i \)'s are pairwise independent. Thus asymptotically, so are the \( S_i \)'s.

### 2.3 Under the alternative hypothesis

In this subsection we study the property of \( S_i \) under the alternative hypothesis that there is a jump on \( (t_{2(i-1)K_1}, t_{2iK_1}] \). Then the following Theorem 2 can help us check whether a jump can significantly affect the behavior of \( S_i \), which is important for jump detection.

**Theorem 2.** Suppose there is a jump at time \( \tau \) and \( i \) is the integer such that \( \tau \in (t_{2(i-1)K_1}, t_{2iK_1}] \). Then for any \( \epsilon > 0 \) we have

\[
S_i = \hat{S}_i + M \frac{D_m(\tau)}{\sigma_{t_{2(i-1)K_1}}} Y(\tau) + o_P \left( m^{\beta + \epsilon} + MD_m(\tau) \right), \tag{15}
\]

where \( \hat{S}_i \) is defined in Theorem 1, \( Y(\tau) \) is the jump size at time \( \tau \), and

\[
D_m(\tau) = \min \left( \left[ \left( \tau - \frac{2(i - 1)K_1}{m} \right) m \right] + 1, K_1, \left[ \left( \frac{2iK_1}{m} - \tau \right) m \right] + 1 \right),
\]

where \( \lfloor x \rfloor \) means the largest integer not larger than \( x \).

\( D_m(\tau) \) shows how the jump time \( \tau \) affects the contribution by the jump to \( S_i \). In order to know the magnitude of \( D_m(\tau) \) when \( \tau \) is random, we assume the following condition for \( N_i \):

\[
P \left( N_{t_i} - N_{t_{i-1}} > 0 \right) = O \left( \frac{1}{m} \right), \tag{16}
\]

for all \( i \in \{1, \ldots, m\} \). Notice that this condition holds for the Poisson process, and it is also adopted by Mancini (2009). Then we have the following corollary:
Corollary 1. For any $\epsilon > 0$,
\[
\frac{D_m(\tau)}{K_1^{1-\epsilon}} \xrightarrow{p} \infty,
\] (17)
as $m \to \infty$.

Corollary 1 implies that $D_m(\tau)$ is approximately of the same order as $K_1$. The intuition is that when $K_1$ is large, $D_m(\tau)$ will also be large unless $\tau$ is close to $t_{2(i-1)K_1}$ or $t_{2iK_1}$, but the probability that $\tau$ is near $t_{2(i-1)K_1}$ or $t_{2iK_1}$ is small. Given the second component on the right-side of (15), the contribution to $S_i$ by the jump is approximately of order $m^{\beta_1 - \frac{1}{2}}$. As $\alpha_1 \in \left(\frac{1}{2}, 1\right)$, the jump is likely to make the value of $|S_i|$ large for large $m$.

2.4 Selection of rejection region

From Theorems 1 and 2, when there is a jump on $(t_{2(i-1)K_1}, t_{2iK_1})$, $|S_i|$ is likely to be larger than $|S_j|$ for a generic $j$ such that there is no jump on $(t_{2(i-1)K_1}, t_{2iK_1}]$. From the next lemma, we can see that $|S_i|$ is also likely to be larger than the maximum of $|S_j|$ across different values of $j$ such that there is no jump on $(t_{2(i-1)K_1}, t_{2iK_1}]$.

Lemma 2. As $m \to \infty$,
\[
\frac{\max_{i \in B_m} |S_i| - C_m}{L_m} \to X,
\] (18)
where $P(X \leq x) = \exp(-e^{-x})$,
\[
C_m = \left(2 \log \frac{m}{2K_1}\right)^{\frac{1}{2}} - \log \pi + \log \left(\log \frac{m}{2K_1}\right) - \frac{1}{2} \left(2 \log \frac{m}{2K_1}\right)^{\frac{3}{2}}
\]
and $L_m = \frac{1}{\left(2 \log \frac{m}{2K_1}\right)^{\frac{3}{2}}}$.

The principle of our test is that if $|S_i|$ is too large compared with the asymptotic distribution of $\max_{i \in B_m} |S_i|$, we reject the null hypothesis that there is no jump on $(t_{2(i-1)K_1}, t_{2iK_1}]$. Thus the value of the threshold for the test is determined by the significance level and the asymptotic distribution of $\max_{i \in B_m} |S_i|$. For example, if we define the significance level as 5%, because the 95% quantile of the random variable $X$ is 2.97, as $\exp(-e^{-2.97}) = 0.95$, we believe there is a jump on $(t_{2(i-1)K_1}, t_{2iK_1}]$ if $\frac{|S_i| - c_m}{L_m} > 2.97$. 


3 Misclassifications

We consider four types of misclassifications. The first is such that a jump occurs on \((t_{2(i-1)}K_1, t_{2iK_1}]\), but the test does not detect it. We call this a *failure to detect actual jump* on \((t_{2(i-1)}K_1, t_{2iK_1}]\) \((FTD_i)\). The second is such that the test indicates there is a jump on \((t_{2(i-1)}K_1, t_{2iK_1}]\), but actually there is not. We call this a *spurious detection of jump* on \((t_{2(i-1)}K_1, t_{2iK_1}]\) \((SD_i)\). If we commit an \(FTD_i\) for any \(i \in \{1, \ldots, m\}^{2K_1}\), a *global failure to detect actual jump* \((GFTD)\) happens. In other words, we commit a \(GFTD\) unless we detect all the jumps on \((0, 1]\). If we commit an \(SD_i\) for any \(i \in \{1, \ldots, m^{2K_1}\}\), a *global spurious detection of jump* \((GSD)\) happens. Thus we commit a \(GSD\) unless we correctly identify all the intervals \((t_{2(i-1)}K_1, t_{2iK_1}]\) where \(i \in \mathcal{B}_m\) as not containing a jump. We define \(J_i\) as the event that there is a jump on \((t_{2(i-1)}K_1, t_{2iK_1}]\), and \(E_i\) as the event that based on the value of \(|S_i|\), we declare there is a jump on \((t_{2(i-1)}K_1, t_{2iK_1}]\). Then the four types of misclassifications can be expressed as:

\[
\text{failure to detect actual jump on } (t_{2(i-1)}K_1, t_{2iK_1}](\text{local property})(FTD_i) = J_i \bigcap E_i^c,
\]

\[
\text{spurious detection of jump on } (t_{2(i-1)}K_1, t_{2iK_1}](\text{local property})(SD_i) = J_i^c \bigcap E_i,
\]

\[
\text{failure to detect actual jumps (global property)}(GFTD) = \bigcup_{i=1}^{m^{2K_1}} \left( J_i \bigcap E_i^c \right),
\]

\[
\text{spurious detection of jumps (global property)}(GSD) = \bigcup_{i=1}^{m^{2K_1}} \left( J_i^c \bigcap E_i \right).
\]

Define \(\alpha_m\) as the significance level of our test, \(\gamma_m\) as the \((100 - 100\alpha_m)th\) quantile of the distribution of \(X\) in Lemma 2, and \(F(y)\) as the cumulative distribution function of the absolute value of jump size. Then Theorem 3 shows the limiting probability of \(GFTD\) as \(m \to \infty\).

**Theorem 3.** Assume \(1 - \alpha_m\) bounded away from zero. Let \\{\(\tau_1, \tau_2, \ldots, \tau_{N_1}\)\} be the time points when jumps occur on \((0, 1]\). Then
\[
P(GFTD|\{\tau_1, \tau_2, \ldots, \tau_{N_1}\}) \to 1 - \prod_{i=1}^{N_1} (1 - F(G_{im})) \tag{19}
\]
as \(m \to \infty\), where
\[
G_{im} = \frac{1}{M} \frac{\sigma_{t_2(-1)K_1}}{D_{im}} (\gamma_m L_m + C_m),
\]
\[
D_{im} = \min \left( \left[ \left( \tau_i - \frac{2(b_{im} - 1)K_1}{m} \right) m \right] + 1, K_1, \left[ \left( \frac{2b_{im}K_1}{m} - \tau_i \right) m \right] + 1 \right),
\]
and \(b_{im} = \left\lceil \frac{m \tau_i}{2K_1} \right\rceil + 1\).

Now we assume for each \(i\), \(Y_i\) is a.s. nonzero. Then the following corollary shows how to set the value of \(\gamma_m\) such that the probability of \(GFTD\) converges to zero as \(m \to \infty\).

**Corollary 2.** Set \(\gamma_m\) such that there exists \(\epsilon > 0\) such that
\[
\frac{\gamma_m}{\sqrt{\log(m)}} m^{-\frac{1}{2} + \frac{1}{2} \alpha_1 + \epsilon} \to 0 \tag{20}
\]
as \(m \to \infty\). Then we have
\[
P(GFTD) \to 0, \tag{21}
\]
as \(m \to \infty\).

The event that \(GFTD\) does not occur means that for the \(S_i's\) corresponding to the intervals that contain a jump, their absolute values are all larger than the threshold \(\gamma_m L_m + C_m\). On the right-side of (15), the main contribution by the jump is the second term, and \(\hat{S}_i\) would be negligible compared to the threshold for large \(m\). Thus the probability that \(|S_i|\) is larger than the threshold approximately equals to the probability that the second term is larger than the threshold \(L_m \gamma_m + C_m\).

From (19) we can see that when \(G_{im} \to 0\) as \(m \to \infty\), the probability of \(GFTD\) converges to 0. Same as \(D_m(\tau)\) in Theorem 2, \(D_{im}\) shows how the position of the jump affects its contribution to \(|S_i|\). From Corollary 1 we can see that the order of \(D_{im}\) is
approximately the same as $K_1$. Thus (21) can be obtained by setting the value of $\gamma_m$ as, for example, any constant or $m^\epsilon$ for any $0 < \epsilon < \frac{1}{2} - \frac{1}{2} \alpha_1$.

By definition $GSD$ is the same as the event that $\max_{i \in B_m}|S_i|$ is larger than the threshold $L_m \gamma_m + C_m$. Then from the asymptotic distribution of $\max_{i \in B_m}|S_i|$, we can obtain the limiting probability of $GSD$ as follows.

**Theorem 4.** As $m \to \infty$,

$$P(GSD) \to 1 - \exp \left( -e^{-\gamma_m} \right) = \alpha_m. \quad (22)$$

Theorems 3 and 4 suggest that if $\lim_{m \to \infty} \gamma_m = \infty$, and there exists $\epsilon > 0$ such that $\lim_{m \to \infty} \frac{\gamma_m}{\sqrt{\log(m)}} m^{-\frac{1}{2} + \frac{1}{2} \alpha_1 + \epsilon} = 0$, then both probabilities of $GFTD$ and $GSD$ converge to zero as $m \to \infty$. For example, we can set $\gamma_m = m^\alpha$ for any $0 < \alpha < \frac{1}{2} - \frac{1}{2} \alpha_1$.

If we find there is a jump on $(t_{2(i-1)K_1}, t_{2iK_1}]$, the jump size can be estimated by taking the average of the returns over the intervals that contain $(t_{2(i-1)K_1}, t_{2iK_1}]$. For example, there are $2K_1 + 1$ intervals in the form of $(t_j, t_{j+4K_1}]$ that contain $(t_{2(i-1)K_1}, t_{2iK_1}]$, which are $(t_{2iK_1-4K_1}, t_{2iK_1}], \ldots, (t_{2(i-1)K_1}, t_{2iK_1+2K_1}]$ (see Figure 3). Then we can consistently estimate the jump size by taking the average of the $2K_1 + 1$ returns over these intervals. It can be checked that the contribution to the average return from the diffusion process is $O_p \left( m^{\frac{1}{2}(\alpha_1-1)} \right)$, and that from the noise is $O_p \left( m^{-\frac{1}{2} \alpha_1} \right)$. Thus for large $m$, the effect from the noise and the diffusion will vanish, and the value of the average return will be approximately equal to the jump size.

### 4 Monte Carlo Simulation

In this section we perform Monte Carlo simulations to explore the efficiency of our jump test, and compare it with the tests by Lee and Mykland (2008) and Jiang and Oomen (2008). We assume there are 8 hours in the whole time horizon. Thus for example, if the sampling frequency is 1 second, we have $\Delta t = t_i - t_{i-1} = \frac{1}{3600 \times 8}$. We assume that there is one price jump in the whole horizon, and its occurrence time follows the uniform distribution on $[0, 1]$. Moreover, we set the drift term $a_t = 0$ and the standard deviation
of the noise as 0.1, unless stated otherwise. We also set \( K_1 = \frac{7}{12} \), \( K_2 = \frac{5}{6} \), \( K_3 = 0.25 \), \( K_4 = 2 \), which is in line with our theory to make the test consistent. We use the Euler scheme to approximate the continuous-time process \( y_t \) which is assumed to satisfy (1), with the discretization being 0.0625 seconds. We assume two conditions for the spot volatility \( \sigma_t \). In the first setting, \( \sigma_t \) is a constant whose value is 0.6. In the second condition, \( \sigma_t \) follows the CIR process:

\[
d\sigma_t = \kappa (\nu - \sigma_t)dt + \theta \sqrt{\sigma_t}dW_t, \tag{23}
\]

where the mean reversion parameter \( \kappa = 0.1 \), the long run mean of the process \( \nu = 0.6 \), and the volatility \( \theta = 0.3 \). The sampling frequency we consider ranges from 0.125 second to 1 second, and for each sampling frequency, we simulate 1000 different series of price observations. Table 1 shows the probability of spurious detection for a generic interval \((t_{2(i-1)K_1}, t_{2iK_1}] \). The significance level is set as \( \alpha = 5\% \). From Table 1 we can see that more frequent observations can reduce the probability of spurious detection.

**INSERT TABLE 1 ABOUT HERE.**

Table 2 shows the probability of correctly detecting a jump on \((t_{2(i-1)K_1}, t_{2iK_1}] \) with respect to different jump sizes and sampling frequencies, with the significance level \( \alpha = 5\% \). We set the jump size relative to the spot volatility \( \sigma_t \), since in the stochastic volatility condition, \( \nu \) is the long-run mean of \( \sigma_t \). From Table 2 we can see that it would be easier to detect large jumps than small ones, and sampling more frequently improves the chance of successful detection.

**INSERT TABLE 2 ABOUT HERE.**

We define the event global misclassification \((GM)\) such that it happens if and only if either \(GSD\) or \(GFTD\) occurs. Table 3 shows the probability of \(GM\) under the condition where \( \sigma_t = 0.6 \). Like the results of Table 2, from Table 3 we can also see that more frequent sampling makes the test more efficient, and larger jumps are easier to detect, which reduces the probability of \(GM\) by decreasing the likelihood of \(GFTD\).
We also compare the probabilities of $GFTD$, $GSD$ and $GM$ for our test and the tests by Lee and Mykland (2008) (LM test) and Jiang and Oomen (2008) (JO test). Now let us introduce LM and JO tests. The statistic proposed by Lee and Mykland (2008) to detect whether there is a jump on $(t_{i-1}, t_i]$ is defined as follows:

\[ L_i = \frac{x_i - x_{i-1}}{\hat{\sigma}_i}, \]  

(24)

where

\[ \hat{\sigma}^2_i = \frac{1}{K_5 - 2} \sum_{j=i-K+2}^{i-1} (x_j - x_{j-1})(x_{j-1} - x_{j-2}). \]  

(25)

Lee and Mykland (2008) set $K_5 = m^{\delta}$ and $\delta$ is a constant on $(0, \frac{1}{2})$. They do not show how to choose the optimal value of $\delta$. We have tried many values on $\delta$, and we will show the results when $\delta = \frac{1}{2}$, because the results for other values of $\delta$ are not better. Lee and Mykland (2008) justify the test in the absence of the noise. In this case, under the null hypothesis that there is no jump on $(t_{i-1}, t_i]$,

\[ \sqrt{m}(x_i - x_{i-1}) \approx \sigma_t U_i, \]  

(26)

where $U_i$ is a standard normal random variable generated by the Brownian motion on $(t_{i-1}, t_i]$. Moreover, in the noiseless setting, $\sqrt{m}\hat{\sigma}_i$ is approximately equal to the local volatility $\sigma_t$. Thus without the impact of the noise and jump, $L_i$ follows asymptotically a standard normal distribution and they are independent of each other. Still in the noiseless setting, a jump on $(t_{i-1}, t_i]$ can significantly increase the value of $|L_i|$. Thus the principle that Lee and Mykland (2008) adopt to determine the threshold based on the significance level is similar to ours. However, neither the nominator nor the denominator on the right-side of (24) are immune to the noise, so the efficiency of LM test decreases with the presence of noise.
The statistic proposed by Jiang and Oomen (2008) to detect jumps on \((0,1]\) is as follows:

\[
S_m = \frac{S_{wV_m} - RV_m}{\sqrt{\hat{\Omega}}},
\]

where \(S_{wV_m} = 2 \sum_{i=1}^{m} (e^{x_i-x_{i-1}} - 1 - (x_i-x_{i-1}))\), \(RV_m = \sum_{i=1}^{m} (x_i-x_{i-1})^2\), and \(\hat{\Omega}\) is some approximation of

\[
\Omega = 4m\eta^6 + 12\eta^4 \int_0^1 \sigma_t^2 dt + 8\eta^2 \frac{1}{m} \int_0^1 \sigma_t^4 dt + \frac{5}{3} \frac{1}{m^2} \int_0^1 \sigma_t^6 dt.
\]

We use the procedure described by Jiang and Oomen (2008) to construct \(\hat{\Omega}\). Under the null hypothesis that there is no jump on \((0,1]\), \(S_m\) has approximately zero expectation and variance 1. In order to obtain acceptable results from the test, as shown in Jiang and Oomen (2008), we need to impose several conditions. For example, \(\sigma_t\) is a constant, \(\eta^2 << \sigma_t^2\) and \(m\sigma_t^4\) is small. Here recall that \(\eta\) is the standard deviation of the noise. The first two conditions are basically true in our simulations, since we define \(\sigma_t = 0.6\) and \(\eta^2 = 0.01\) or 0.0144, much smaller than \(\sigma_t^2 = 0.36\). The last condition implies that \(m\) cannot be too large, so it may not be proper to sample frequently for JO test with noisy data, and we will see that the simulation results also illustrate it. Thus we sample much less frequently in performing JO test than the other two tests. Moreover, unlike our and LM tests, for a fixed period \((0,T]\), JO test can only indicate whether there is a jump on that period, without further indication on the jump arrival time, like locating the jump arrival within a small interval. Thus in order to make the results comparable, we use the JO test to detect jump respectively on \((0,\frac{1}{2}]\) and \((\frac{1}{2},1]\) in one simulation.

For all the tests, the significance level is set as 5%, and the jump size is subject to normal distribution with mean 0 and standard deviation 1.2. Table 4 reports the results. We can see that across different levels of sampling frequency and \(\eta\), our test is less likely to commit misspecifications than LM and JO test, since the probability of \(GM\) is uniformly lower for our test. Further investigation reveals that this is because the probability of \(GFTD\) of our test is always smaller than LM and JO tests. It is obvious that relative to our and LM tests, we use much less frequent price observations to construct the JO
test. This is because for JD test, when \( m \) is large, the absolute value of the statistic will be significantly increased by the noise, so it is likely to generate a misleading result when there is no jump on the interval. For example, when the sampling frequency is 10 seconds, over 90 percent of the JO tests will commit a spurious detection. Unreported results suggest that for JO test, the optimal sampling frequency is around 120 seconds. In addition, when \( \eta = 0.12 \), more frequent sampling does not reduce the likelihood of misspecification for LM test. This implies that the consistency of LM test, which is established in the noiseless setting, is not immune to the noise. However, sampling more frequently can always improve the efficiency of our test.

Insert Table 4 about here.

Table 5 displays the values of the mean squared errors (MSE) on estimating the jump size by considering the local average return as the estimator. The procedure to construct the estimator is the same as we have explained in the last paragraph of Section 3. That is, for any jump on \((0, 1]\), it is contained in an interval in the form of \((t_{2(i-1)K_1}, t_{2iK_1}]\) for some \( i \in \{1, \ldots, \frac{m}{2K_1}\} \). Then we take the average of the returns over the intervals \((t_{(2i-4)K_1}, t_{2iK_1}], (t_{(2i-4)K_1+1}, t_{2iK_1+1}], \ldots, (t_{(2i-2)K_1}, t_{(2i+2)K_1}]\) as the estimator on the jump size. We consider 4 combinations in terms of the spot volatility \( \sigma_t \) and jump size \( Y_i \). \( \sigma_t \) can be either a constant 0.6 or follows the stochastic process as specified in (23). \( Y_i \) can be either a constant 1.2 or subject to the distribution \( \text{N}(0,1.44) \). From Table 5 we can see that sampling more frequently increases the precision of estimating the jump size.

Insert Table 5 about here.

5 Conclusions

In this work we propose a novel test on detecting jumps in the process of efficient asset price. This test is based on high-frequency price data that are affected by market microstructure noise. We perform multiple tests over the whole period, and each single test checks whether there is a jump in a small interval. The statistic is the rescaled local
average return standardized by an estimator of spot volatility. The volatility estimator is derived from the truncated two-scales realized variance (TTSRV), because the TTSRV consistently estimates the integrated volatility in the presence of noise and jumps. Since a jump can greatly increase the absolute value of the statistic, we will reject the null hypothesis that there is no jump in the interval, if the corresponding statistic is suspiciously large in its absolute value. We show that when the threshold is defined properly, the probability of testing errors will converge to zero as the number of price observations increases. Simulations also illustrate the consistency of our test, because more frequent sampling reduces the likelihood of errors. Simulations also indicate that compared to some other prevalent jump tests which are not immune to the noise, our test has the advantage of dealing with noisy data.
References


6 Appendix

Proof of Lemma 1. For simplicity we assume \( a_t = 0 \). We define

\[
\bar{x}_i = \int_0^{t_i} \sigma_s dB_s + u_{t_i},
\]

\[
\hat{\sigma}_{TTS_i}^2 = \frac{1}{K_3} \sum_{j=2(i-1)K_1-K_2+1}^{2(i-1)K_1} (\bar{x}_j - \bar{x}_j-K_3)^2 1_{E_j} - \frac{1}{K_3} \sum_{j=2(i-1)K_1-K_2+1}^{2(i-1)K_1} (\bar{x}_j - \bar{x}_j-1)^2 1_{E_j},
\]

and

\[
\hat{\sigma}_{TTS_i}^2 = \frac{1}{K_3} \sum_{j=2(i-1)K_1-K_2+1}^{2(i-1)K_1} (\bar{x}_j - \bar{x}_j-K_3)^2 - \frac{1}{K_3} \sum_{j=2(i-1)K_1-K_2+1}^{2(i-1)K_1} (\bar{x}_j - \bar{x}_j-1)^2.
\]

Following similar steps as those justifying equations (11) and (12) in Brownlees et al. (2016), it can be checked that a.s.,

\[
\hat{\sigma}_{TTS_i}^2 = \sigma_{TTS_i}^2,
\]

(28)

for all \( i \in \{1, \ldots, \frac{m}{2K_1}\} \), as \( m \to \infty \), and for any \( \epsilon > 0 \),

\[
\sup_{i \in \{1, \ldots, \frac{m}{2K_1}\}} \left| \hat{\sigma}_{TTS_i}^2 - \sigma_{TTS_i}^2 \right| = O_P \left( m^{-\frac{1}{3}+\alpha_4+\epsilon} \right).
\]

(29)

Therefore,

\[
\sup_{i \in \{1, \ldots, \frac{m}{2K_1}\}} \left| \hat{\sigma}_{TTS_i}^2 - \sigma_{TTS_i}^2 \right| = O_P \left( m^{-\frac{1}{3}+\alpha_4+\epsilon} \right).
\]

(30)

\( \sigma_{TTS_i}^2 \) can be seen as a two-scales realized volatility estimator on \([t_2(i-1)K_1-K_2, t_2(i-1)K_1] \).

Based on the proof of Theorem 1 in Fan, Li, and Yu (2012), we have that there exists positive constants \( c \) and \( c' \) such that for all \( x \in [0, c] \) and large \( m \),

\[
P \left( \left| \sigma_{TTS_i}^2 - \int_{t_2(i-1)K_1-K_2}^{t_2(i-1)K_1} \sigma_s^2 ds \right| \geq x \right) \leq c' \exp \left( -cx^2 m^{\min(2-\alpha_2-\alpha_3,2\alpha_3-\alpha_2)} \right),
\]

(31)
from which we can see that for any \( \epsilon > 0 \) we have

\[
\sup_{i \in \{1, \ldots, \frac{m}{2K_1} + 1\}} \left| \sigma_{TS_i}^2 - \int_{t_{2(i-1)K_1 - K_2}}^{t_{2(i-1)K_1}} \sigma_s^2 ds \right| = O_P \left( m^{\max\left( \frac{1}{2} (\alpha_2 + \alpha_3 - 1), \frac{1}{2} \alpha_2 - 1, \frac{1}{2} (\alpha_1 - 1) \right) + \epsilon} \right). \tag{32}
\]

From Assumption 2 for all \( i \in \{1, \ldots, \frac{m}{2K_1}\} \) we have

\[
\int_{t_{2(i-1)K_1 - K_2}}^{t_{2(i-1)K_1}} \sigma_s^2 ds = \sigma_{t_{2(i-1)K_1}}^2 (t_{2(i-1)K_1} - t_{2(i-1)K_1 - K_2}) + \int_{t_{2(i-1)K_1 - K_2}}^{t_{2(i-1)K_1}} (\sigma_s^2 - \sigma_{t_{2(i-1)K_1}}^2) ds
\]

\[
= \frac{K_2}{m} \sigma_{t_{2(i-1)K_1}}^2 + O_P \left( m^{\frac{1}{2} (\alpha_2 - 1)} \right), \tag{33}
\]

for \( \epsilon > 0 \). Therefore,

\[
\sup_{i \in \{1, \ldots, \frac{m}{2K_1}\}} \left| \frac{m}{K_2} \int_{t_{2(i-1)K_1 - K_2}}^{t_{2(i-1)K_1}} \sigma_s^2 ds - \sigma_{t_{2(i-1)K_1}}^2 \right| = O_P \left( m^{\frac{1}{2} (\alpha_2 - 1) + \epsilon} \right). \tag{34}
\]

From (29) (32) (34), we get that

\[
\sup_{i \in \{1, \ldots, \frac{m}{2K_1}\}} \left| \frac{m}{K_2} \tilde{\sigma}_{TS_i}^2 - \sigma_{t_{2(i-1)K_1}}^2 \right| = O_P \left( m^{\max\left( \frac{1}{2} (\alpha_4 - \alpha_2 + \frac{1}{2} \alpha_3 - \frac{1}{2} \alpha_2 - \frac{1}{2} (\alpha_2 - \alpha_3 + 1), \frac{1}{2} (\alpha_1 - 1) \right) + \epsilon} \right), \tag{35}
\]

for \( \epsilon > 0 \).

**Proof of Theorem 1.** For simplicity we assume \( a_t = 0 \). Define

\[
A_{i1} = \sum_{j=0}^{K_1} \int_{t_{2(i-1)K_1 + j}}^{t_{2(i-1)K_1 + j+1}} \sigma_t dB_t, \tag{36}
\]

and

\[
A_{i2} = \sum_{j=0}^{K_1} (u_{(2i-1)K_1 + j} - u_{2(i-1)K_1 + j}). \tag{37}
\]

Then when \( i \in B_m \),

\[
A_i = \sqrt{\frac{3m}{K_1(K_1 + 1)(2K_1 + 1)}} (A_{i1} + A_{i2}).
\]
For $A_{i1}$, we have

$$A_{i1} = \sum_{j=2(i-1)K_1}^{(2i-1)K_1} \int_{t_j}^{t_j+K_1} (\sigma_t - \sigma_{t_{2(i-1)K_1}}) dB_t + \sigma_{t_{2(i-1)K_1}} \sum_{j=2(i-1)K_1}^{(2i-1)K_1} (B_{t_j+K_1} - B_{t_j}).$$

By Assumption 2, across $i \in \{1, \ldots, \frac{m}{2K_1}\}$, for any $\epsilon > 0$ we have

$$\sup_i \sup_{t \in [t_{2(i-1)K_1}, t_{2iK_1}]} (\sigma_t - \sigma_{t_{2(i-1)K_1}}) = O_P \left( \sigma \left( \frac{1}{2} \alpha_{1-\epsilon} \right) \right).$$

Then it can be checked that

$$\sup_i \int_{t_j}^{t_j+K_1} (\sigma_t - \sigma_{t_{2(i-1)K_1}}) dB_t = O_P \left( \sigma \left( \frac{1}{2} \alpha_{1-\epsilon} \right) \right),$$

for $\epsilon > 0$. For $A_{i2}$, we have $\sup_i |A_{i2}| = O_P \left( \sigma \left( \frac{1}{2} \alpha_{1-\epsilon} \right) \right)$. Therefore,

$$\sup_i \left| M \left( A_{i1} + A_{i2} - \sigma_{t_{2(i-1)K_1}} \sum_{j=2(i-1)K_1}^{(2i-1)K_1} (B_{t_j+K_1} - B_{t_j}) \right) \right|$$

is $O_P \left( \sigma \left( \max(1-2\alpha_1, \frac{1}{2} - \alpha_1) + \epsilon \right) \right)$, as $M = \sqrt{\frac{3m}{K_1(K_1+1)(2K_1+1)}}$. Then we have

$$\sup_i \left| S_i \sqrt{\frac{m}{K_2}} \tilde{\sigma}_{TTS_i}^2 - \sigma_{t_{2(i-1)K_1}} \tilde{S}_i \right| = O_P \left( \sigma \left( \max(1-2\alpha_1, \frac{1}{2} - \alpha_1) + \epsilon \right) \right),$$

for any $\epsilon > 0$. Since we have set $\alpha_2 = \frac{5}{6}$, $\alpha_3 = \frac{2}{3}$, based on Lemma 1, it can be obtained that

$$\sup_i \left| \sqrt{\frac{m}{K_2}} \tilde{\sigma}_{TTS_i}^2 - \sigma_{t_{2(i-1)K_1}} \tilde{S}_i \right| = O_P \left( \sigma \left( \frac{1}{2} \alpha_1 + \epsilon \right) \right),$$

for any $\epsilon > 0$. Then we have

$$\sup_i \left| \left( \sigma_{t_{2(i-1)K_1}} + \epsilon \right) S_i - \sigma_{t_{2(i-1)K_1}} \tilde{S}_i \right| = O_P \left( \sigma \left( \max(1-2\alpha_1, \frac{1}{2} - \alpha_1) + \epsilon \right) \right),$$

where $\sup_i \epsilon_i$ is $O_P \left( \sigma \left( \frac{1}{2} \alpha_1 + \epsilon \right) \right)$. As we will prove later, $\tilde{S}_i$ is subject to the standard normal
distribution. Then \( \sup_i |\hat{S}_i| = O_P(\log m) \). Thus given that \( \sigma_{t_{2(i-1)K_1}} > \sigma^- \), according to (41), we can see that \( \sup_i |S_i| = O_P \left( \log m + m^{\max(1-2\alpha_1, \frac{1}{2}-\alpha_1)+\epsilon} \right) \), because

\[
\sup_i |S_i| \leq \sup_i \left| \frac{\sigma_{t_{2(i-1)K_1}}}{\sigma_{t_{2(i-1)K_1}} + \epsilon} \hat{S}_i \right| + O_P \left( m^{\max(\frac{1}{2}-\alpha_1, \frac{1}{2}-\alpha_1)+\epsilon} \right). \tag{42}
\]

Also from (41), we have that

\[
\sup_i \left| \sigma_{t_{2(i-1)K_1}} S_i - \sigma_{t_{2(i-1)K_1}} \hat{S}_i \right| \leq O_P \left( m^{\max(1-2\alpha_1, \frac{1}{2}-\alpha_1)+\epsilon} \right) + \sup_i \epsilon_i S_i. \tag{43}
\]

As \( \sup_i \epsilon_i S_i = O_P \left( m^{\max(\frac{1}{2}-\alpha_1, \frac{1}{2}-\alpha_1)+\epsilon} \right) = O_P \left( m^{\beta+\epsilon} \right) \) and \( \sigma_{t_{2(i-1)K_1}} > \sigma^- \), proof for (13) is completed. Now we are going to show that \( \hat{S}_i = M \sum_{j=2(i-1)K_1}^{(2i-1)K_1} (B_{t_j + K_1} - B_{t_j}) \) is subject to the standard normal distribution.

\[
\sum_{j=2(i-1)K_1}^{(2i-1)K_1} (B_{t_j + K_1} - B_{t_j}) = \sum_{j=0}^{K_1-1} (K_1 - j)(B_{t_{(2i-1)K_1-j}} - B_{t_{(2i-1)K_1-j-1}}) + \sum_{j=0}^{K_1-1} (K_1 - j)(B_{t_{(2i-1)K_1+j+1}} - B_{t_{(2i-1)K_1+j}}). \tag{44}
\]

As \( 1^2 + 2^2 + \cdots + K_1^2 = \frac{K_1(K_1+1)(2K_1+1)}{6} \),

\[
\sum_{j=2(i-1)K_1}^{(2i-1)K_1} (B_{t_j + K_1} - B_{t_j}) \sim N \left( 0, \frac{K_1(K_1+1)(2K_1+1)}{3m} \right). \tag{45}
\]

As \( \frac{K_1(K_1+1)(2K_1+1)}{3m} = \frac{1}{M^2}, \hat{S}_i \sim N(0,1). \)

\( \square \)

**Proof of Theorem 2.** For simplicity we assume \( \alpha_t = 0 \). Recall that we have defined \( \bar{\pi}_i = \int_0^{\tau_i} \sigma_s dB_s + u_{t_i} \). It can be checked that

\[
S_i = \frac{M \sum_{j=0}^{K_1} (\bar{\pi}_{(2i-1)K_1+j} - \bar{\pi}_{(2i-2)K_1+j})}{\sqrt{\frac{m}{K_2} TTS_i} \sqrt{\frac{m}{K_2} TTS_i}} + \frac{MD_m(\tau)Y_{\tau}}{\sqrt{\frac{m}{K_2} TTS_i}}. \tag{46}
\]
From the proof of Theorem 1, we can see that

\[
M \sum_{j=0}^{K_1} \left( \bar{x}_{(2i-1)K_1+j} - \bar{x}_{(2i-2)K_1+j} \right) \sqrt{\frac{m \Hat{\sigma}^2}{K_2 \sigma_{TTS}}} = \Hat{S}_i + o_p \left( m^{\max(\frac{1}{2} - \alpha_1, \frac{1}{2} \alpha_1 - \frac{1}{2} - \epsilon)} \right),
\]

where \( \Hat{S}_i \) is standard normal. From Lemma 1, we have

\[
\sqrt{\frac{m \Hat{\sigma}^2}{K_2 \sigma_{TTS}}} = \sigma_{t_2(i-1)K_1} + o_p(1).
\]

Therefore,

\[
\frac{MD_{m}(\tau)Y_{\tau}}{\sqrt{\frac{m \Hat{\sigma}^2}{K_2 \sigma_{TTS}}} \sigma_{t_2(i-1)K_1}} = \frac{MD_{m}(\tau)Y_{\tau}}{\sigma_{t_2(i-1)K_1}} + o_p \left( \frac{MD_{m}(\tau)}{\sigma_{t_2(i-1)K_1}} \right).
\]

(49)

Now the proof is completed given (46) (47) (49).

Proof of Corollary 1. When \( D_{m}(\tau) \leq K_1^{1-\epsilon} \), we have

\[
\tau \in \bigcup_{j=0}^{\frac{m}{2K_1} - 1} \left( t_{2K_1j}, t_{2K_1j+K_1^{1-\epsilon}} \right) \quad \text{or} \quad \tau \in \bigcup_{j=0}^{\frac{m}{2K_1} - 1} \left( t_{2K_1(j+1)-K_1^{1-\epsilon}}, t_{2K_1(j+1)} \right).
\]

Therefore,

\[
P (D_{m}(\tau) \leq K_1^{1-\epsilon}) \leq \sum_{j=0}^{\frac{m}{2K_1} - 1} P \left( \tau \in \left( t_{2K_1j}, t_{2K_1j+K_1^{1-\epsilon}} \right) \right) + \sum_{j=0}^{\frac{m}{2K_1} - 1} P \left( \tau \in \left( t_{2K_1(j+1)-K_1^{1-\epsilon}}, t_{2K_1(j+1)} \right) \right)
\]

\[
\leq \sum_{j=0}^{\frac{m}{2K_1} - 1} P \left( N_{t_{2K_1j},K_1^{1-\epsilon}} - N_{t_{2K_1j}} > 0 \right) + \sum_{j=0}^{\frac{m}{2K_1} - 1} P \left( N_{t_{2K_1(j+1)}}, N_{t_{2K_1(j+1)-K_1^{1-\epsilon}}} > 0 \right)
\]

\[
= O \left( \frac{m}{K_1} \cdot \frac{K_1^{1-\epsilon}}{m} \right) = O \left( K_1^{-\epsilon} \right),
\]

(50)

where the second to the last equation is obtained from (16). Thus for any \( \epsilon \in (0, 1) \) it is
a.s. that \( D_m(\tau) > K_1^{1-\epsilon} \) for large \( m \) and any \( \epsilon > 0 \), so we have

\[
\frac{D_m(\tau)}{K_1^{1-\epsilon}} > \frac{K_1^{1-\epsilon}}{K_1^{1-\epsilon}} = K_1^{\frac{\epsilon}{1-\epsilon}},
\]

for large \( m \). Thus the proof is completed given that \( K_1^{\frac{\epsilon}{1-\epsilon}} \to \infty \) as \( m \to \infty \).

**Proof of Lemma 2.** From Theorem 1 we can see that as \( m \to \infty \), the limiting distribution of \( \max_{i \in \mathcal{B}_m} |S_i| \) is the same as the distribution of the maximum absolute value of \( \frac{m}{2K_1} - N_1 \) standard normal variables. Lemma 1 in Lee and Mykland (2008) shows the distribution of the maximum absolute value of \( m - N_1 \) standard normal variables, as \( m \to \infty \). Thus given that \( \frac{m}{2K_1} \to \infty \) as \( m \to \infty \), Lemma 1 in Lee and Mykland (2008) directly leads to the results of Lemma 2.

**Proof of Theorem 3.** For simplicity we assume \( a_t = 0 \). For a generic \( i \), consider the \( i \)th jump which happens at time \( \tau_i \) with size \( Y_i \). Without loss of generality, we assume \( \tau_i \notin \{t_{2K_1}, t_{4K_1}, \ldots, t_{m-2K_1}, t_m\} \). Then the \( i \)th jump happens on the interval \( (t_{2(b_{im}-1)K_1}, t_{2b_{im}K_1}] \) where \( b_{im} = \left\lfloor \frac{m\tau_i}{2K_1} \right\rfloor + 1 \). Now we study the probability that \( |S_{b_{im}}| \) is larger than the threshold \( L_m \gamma_m + C_m \), conditional on the value of \( \tau_i \). Similarly as (46) we have

\[
S_{b_{im}} = \frac{M \sum_{j=0}^{K_1} (\tau_{(2b_{im}-1)K_1+j} - \tau_{(2b_{im}-2)K_1+j})}{\sqrt{\frac{m}{K_2} \widehat{\sigma}^2 \text{TTS}_{b_{im}}}} + \frac{MD_{im}Y_i}{\sqrt{\frac{m}{K_2} \widehat{\sigma}^2 \text{TTS}_{b_{im}}}}.
\]

Define

\[
S^1_{b_{im}} = \frac{M \sum_{j=0}^{K_1} (\tau_{(2b_{im}-1)K_1+j} - \tau_{(2b_{im}-2)K_1+j})}{\sqrt{\frac{m}{K_2} \widehat{\sigma}^2 \text{TTS}_{b_{im}}}},
\]

\[
S^2_{b_{im}} = \frac{MD_{im}Y_i}{\sigma_{t(2b_{im}-1)K_1}},
\]

and

\[
S^3_{b_{im}} = \frac{MD_{im}Y_i}{\sqrt{\frac{m}{K_2} \widehat{\sigma}^2 \text{TTS}_{b_{im}}}} - S^2_{b_{im}}.
\]

From the proof of Theorem 1, we can see that \( S^1_{b_{im}} \) is asymptotically standard normal as
\( m \to \infty \). Since \( L_m \gamma_m + C_m \xrightarrow{m \to \infty} \infty \), we have as \( m \to \infty \),

\[
\lim_{m \to \infty} \frac{S_{b,m}^1}{L_m \gamma_m + C_m} \xrightarrow{P} 0.
\] (53)

Given (48), \( S_{b,m}^3 \) is \( o_P \left( S_{b,m}^2 \right) \). As

\[
P \left( |S_{b,m}^1| > L_m \gamma_m + C_m \right) = P \left( |S_{b,m}^1 + S_{b,m}^2 + S_{b,m}^3| > L_m \gamma_m + C_m \right),
\] (54)

for any positive constants \( c_1, c_2 \) we have

\[
P \left( |S_{b,m}^1 + S_{b,m}^2 + S_{b,m}^3| > L_m \gamma_m + C_m \right) \\
\geq P \left( |S_{b,m}^1| < c_1(L_m \gamma_m + C_m), (1 - c_2)|S_{b,m}^2| > (1 + c_1)(L_m \gamma_m + C_m), |S_{b,m}^3| < c_2|S_{b,m}^2| \right) \\
\geq 1 - P \left( |S_{b,m}^1| \geq c_1(L_m \gamma_m + C_m) \right) - P \left( (1 - c_2)|S_{b,m}^2| \leq (1 + c_1)(L_m \gamma_m + C_m) \right) \\
- P \left( |S_{b,m}^3| \geq c_2|S_{b,m}^2| \right). \] (55)

Since \( S_{b,m}^1 \) is \( o_P \left( L_m \gamma_m + C_m \right) \) and \( S_{b,m}^3 \) is \( o_P \left( S_{b,m}^2 \right) \),

\[
P \left( |S_{b,m}^1| \geq c_1(L_m \gamma_m + C_m) \right) \to 0 \text{ and } P \left( |S_{b,m}^3| \geq c_2|S_{b,m}^2| \right) \to 0,
\] (56)

as \( m \to \infty \). Since \( c_1, c_2 \) can be any positive constants, when \( c_1 \to 0, c_2 \to 0 \), the right-side of the second inequality of (55) converges to \( 1 - P \left( |S_{b,m}^2| \leq L_m \gamma_m + C_m \right) = P \left( |S_{b,m}^2| > L_m \gamma_m + C_m \right) \), as \( m \to \infty \). In addition,

\[
P \left( |S_{b,m}^1 + S_{b,m}^2 + S_{b,m}^3| > L_m \gamma_m + C_m \right) \\
\leq P \left( (1 + c_2)|S_{b,m}^2| > (1 - c_1)(L_m \gamma_m + C_m) \right) + P \left( |S_{b,m}^3| \geq c_2|S_{b,m}^2| \right) + \\
P \left( |S_{b,m}^1| > c_1(L_m \gamma_m + C_m) \right). \] (57)

Following a similar analysis, it can be checked that when \( c_1 \to 0, c_2 \to 0 \), the right-side of (57) also converges to \( P \left( |S_{b,m}^2| > L_m \gamma_m + C_m \right) \) as \( m \to \infty \). Therefore, we have that

\[
P \left( |S_{b,m}^1 + S_{b,m}^2 + S_{b,m}^3| > L_m \gamma_m + C_m \right) \text{ converges to } P \left( |S_{b,m}^2| > L_m \gamma_m + C_m \right) \text{ as } m \to \infty,
\]
and
\[
P \left( |S_{b_1 m}^2 | > L_m \gamma_m + C_m \right) \\
= P \left( |Y_i| > \frac{\sigma_{(i-1)K_1} (L_m \gamma_m + C_m)}{MD_{1 m}} \right) \\
= 1 - F(G_{1 m}). \tag{58}
\]

Similarly, conditional on the occurrence times of all the jumps on \((0, 1]\) we have
\[
P \left( |S_{b_1 m}^2 | > L_m \gamma_m + C_m, |S_{b_2 m}^2 | > L_m \gamma_m + C_m, \ldots, |S_{b_{N_1 m}^2} | > L_m \gamma_m + C_m \right) \\
\rightarrow P \left( |S_{b_1 m}^2 | > L_m \gamma_m + C_m, |S_{b_2 m}^2 | > L_m \gamma_m + C_m, \ldots, |S_{b_{N_1 m}^2} | > L_m \gamma_m + C_m \right) \\
= \prod_{i=1}^{N_1} P \left( |S_{b_i m}^2 | > L_m \gamma_m + C_m \right) = \prod_{i=1}^{N_1} \left( 1 - F(G_{i m}) \right), \tag{59}
\]

as \(m \to \infty\). Therefore, by the definition of \(GFTD\),
\[
P \left( GFTD | \{\tau_1, \tau_2, \ldots, \tau_{N_1} \} \right) \\
= 1 - P \left( |S_{b_1 m} | > L_m \gamma_m + C_m, |S_{b_2 m} | > L_m \gamma_m + C_m, \ldots, |S_{b_{N_1 m} | > L_m \gamma_m + C_m \right) \\
\rightarrow 1 - \prod_{i=1}^{N_1} \left( 1 - F(G_{i m}) \right), \tag{60}
\]

as \(m \to \infty\).

**Proof of Corollary 2.** As \(Y_i\) is a.s. nonzero, we have \(F(0) = 0\). Thus given (60), (21) is proved if we show that \(G_{1 m} \xrightarrow{P} 0\) as \(m \to \infty\). We have
\[
G_{1 m} = \sqrt{\frac{K_1 (K_1 + 1) (2K_1 + 1)}{3m}} \frac{\sigma_{(i-1)K_1} }{D_{1 m}} \gamma_m L_m + \sqrt{\frac{K_1 (K_1 + 1) (2K_1 + 1)}{3m}} \frac{\sigma_{(i-1)K_1} }{D_{1 m}} C_m. \tag{61}
\]

For the first part on the right-side of (61), recall that there exists \(\epsilon > 0\) such that
\[
\frac{\gamma_m}{\sqrt{\log(m)}} m^{-\frac{1}{2} + \frac{1}{2} \alpha_1 + \epsilon} \to 0, \text{ as } m \to \infty.
\]

We have
\[
\sqrt{\frac{K_1 (K_1 + 1) (2K_1 + 1)}{3m}} \frac{\sigma_{(i-1)K_1} }{D_{1 m}} \gamma_m L_m
\]
\[
\sigma_{t_2(\gamma_{i1})}^{(i)} \frac{\gamma m(m^{1/2} + m^{1/2})}{\log(m)} \left( L_m \sqrt{\log(m)} \right) \left( \frac{\sqrt{K_1(K_1+1)/2K_1+1}}{3m^{3/2} \alpha_{1}^{1/2}} \right) \frac{m^{\alpha_{1}-\epsilon}}{D_i m} \to 0, \quad (62)
\]
as \( m \to \infty \), given that \( L_m \sqrt{\log(m)} \), \( \frac{\sqrt{K_1(K_1+1)/2K_1+1}}{3m^{3/2} \alpha_{1}^{1/2}} \) are \( O(1) \) and based on Corollary 1, \( \frac{m^{\alpha_{1}-\epsilon}}{D_i m} \) is \( o_P(1) \). It can be also checked that the second part on the right-side of (61) converges to zero in probability as \( m \to \infty \). Thus \( G_{i m} \to 0 \) as \( m \to \infty \).

\textbf{Proof of Theorem 4.} Recall the definition of \( X \) in Lemma 2. We have

\[
P(GSD) = 1 - P \left( \max_{i \in B_m} |S_i| - C_m \leq \gamma_m \right) = 1 - P(X \leq \gamma_m)
\]

\[
= 1 - \exp \left( -e^{-\gamma_m} \right) = \alpha_m.
\]

(63)
Figure 1. The figure shows how we divide the whole horizon \((0, 1]\) into \(\frac{m-2K_1}{2K_1}\) intervals. Then test can be performed on each interval to detect jumps.

\[
\begin{array}{cccccc}
0 & \frac{2K_1}{m} & \frac{4K_1}{m} & \cdots & \frac{m-2K_1}{m} & 1 \\
\end{array}
\]

Figure 2. The figure provides a schematic representation of how we compute the nominator of \(S_i\)

Figure 3. The figure shows there are \(2K_1 + 1\) intervals in the form of \((t_j, t_{j+4K_1}]\) that contain the interval \((t_{2i-2K_1}, t_{2iK_1}]\) in red color.

Table 1. The table shows means and standard errors (in parentheses) of probability of spurious detection \(P(\text{SD}_i)\). The significance level \(\alpha\) is 5%. SV means the spot volatility follows the process of \((23)\), and \(freq\) denotes the sampling frequency.

<table>
<thead>
<tr>
<th>(freq)</th>
<th>(\sigma_t = 0.6)</th>
<th>(SE)</th>
<th>(SV)</th>
<th>(SE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-second</td>
<td>(3.89 \times 10^{-3})</td>
<td>((3.54 \times 10^{-4}))</td>
<td>(4.44 \times 10^{-3})</td>
<td>((3.89 \times 10^{-4}))</td>
</tr>
<tr>
<td>0.5-second</td>
<td>(3.13 \times 10^{-3})</td>
<td>((2.71 \times 10^{-4}))</td>
<td>(3.08 \times 10^{-3})</td>
<td>((2.36 \times 10^{-4}))</td>
</tr>
<tr>
<td>0.25-second</td>
<td>(2.19 \times 10^{-3})</td>
<td>((1.99 \times 10^{-4}))</td>
<td>(1.56 \times 10^{-3})</td>
<td>((1.65 \times 10^{-4}))</td>
</tr>
<tr>
<td>0.125-second</td>
<td>(1.53 \times 10^{-3})</td>
<td>((1.36 \times 10^{-4}))</td>
<td>(1.41 \times 10^{-3})</td>
<td>((1.26 \times 10^{-4}))</td>
</tr>
</tbody>
</table>
Table 2. This table shows means and standard errors (in parentheses) of probability of detecting actual jump \(1 - P(FTD_i)\). The significance level \(\alpha\) is 5%. Stochastic volatility means the spot volatility follows the process of (23), and \(freq\) denotes the sampling frequency.

<table>
<thead>
<tr>
<th>Jump Size</th>
<th>(2\sigma_t)</th>
<th>(\sigma_t)</th>
<th>(0.5\sigma_t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(freq=1)-second</td>
<td>0.78</td>
<td>0.67</td>
<td>0.34</td>
</tr>
<tr>
<td></td>
<td>((1.32 \times 10^{-2}))</td>
<td>((1.49 \times 10^{-2}))</td>
<td>((1.51 \times 10^{-2}))</td>
</tr>
<tr>
<td>(freq=0.5)-second</td>
<td>0.85</td>
<td>0.72</td>
<td>0.48</td>
</tr>
<tr>
<td></td>
<td>((1.13 \times 10^{-2}))</td>
<td>((1.43 \times 10^{-2}))</td>
<td>((1.59 \times 10^{-2}))</td>
</tr>
<tr>
<td>(freq=0.25)-second</td>
<td>0.89</td>
<td>0.75</td>
<td>0.55</td>
</tr>
<tr>
<td></td>
<td>((9.94 \times 10^{-3}))</td>
<td>((1.38 \times 10^{-2}))</td>
<td>((1.58 \times 10^{-2}))</td>
</tr>
<tr>
<td>(freq=0.125)-second</td>
<td>0.94</td>
<td>0.77</td>
<td>0.64</td>
</tr>
<tr>
<td></td>
<td>((7.55 \times 10^{-3}))</td>
<td>((1.34 \times 10^{-2}))</td>
<td>((1.53 \times 10^{-2}))</td>
</tr>
</tbody>
</table>

Table 3. This table shows means and standard errors (in parentheses) of probability of global misclassifications by either global spurious detection of jumps \((GSD)\) or global failure to detect actual jumps\((GFTD)\). The significance level \(\alpha\) is 5% and \(\sigma_t = 0.6\). \(freq\) denotes the sampling frequency.

<table>
<thead>
<tr>
<th>Jump size</th>
<th>(2\sigma_t)</th>
<th>(\sigma_t)</th>
<th>(0.5\sigma_t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(freq=1)-second</td>
<td>0.32</td>
<td>0.40</td>
<td>0.70</td>
</tr>
<tr>
<td></td>
<td>((1.48 \times 10^{-2}))</td>
<td>((1.56 \times 10^{-2}))</td>
<td>((1.46 \times 10^{-2}))</td>
</tr>
<tr>
<td>(freq=0.5)-second</td>
<td>0.26</td>
<td>0.33</td>
<td>0.61</td>
</tr>
<tr>
<td></td>
<td>((1.39 \times 10^{-2}))</td>
<td>((1.49 \times 10^{-2}))</td>
<td>((1.55 \times 10^{-2}))</td>
</tr>
<tr>
<td>(freq=0.25)-second</td>
<td>0.20</td>
<td>0.34</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>((1.27 \times 10^{-2}))</td>
<td>((1.51 \times 10^{-2}))</td>
<td>((1.59 \times 10^{-2}))</td>
</tr>
<tr>
<td>(freq=0.125)-second</td>
<td>0.17</td>
<td>0.30</td>
<td>0.43</td>
</tr>
<tr>
<td></td>
<td>((1.19 \times 10^{-2}))</td>
<td>((1.46 \times 10^{-2}))</td>
<td>((1.57 \times 10^{-2}))</td>
</tr>
<tr>
<td>( \eta = 0.1 )</td>
<td>Our test</td>
<td>( \eta = 0.1 )</td>
<td>LM test</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>( freq )</td>
<td>( P(GM) )</td>
<td>(SE)</td>
<td>( P(GFTD) )</td>
</tr>
<tr>
<td>0.5-second</td>
<td>0.38</td>
<td>(1.54 ( \times 10^{-2} ))</td>
<td>0.30</td>
</tr>
<tr>
<td>0.25-second</td>
<td>0.36</td>
<td>(1.53 ( \times 10^{-2} ))</td>
<td>0.28</td>
</tr>
<tr>
<td>0.125-second</td>
<td>0.27</td>
<td>(1.41 ( \times 10^{-2} ))</td>
<td>0.20</td>
</tr>
<tr>
<td>120-second</td>
<td>0.41</td>
<td>(1.56 ( \times 10^{-2} ))</td>
<td>0.34</td>
</tr>
<tr>
<td>60-second</td>
<td>0.43</td>
<td>(1.57 ( \times 10^{-2} ))</td>
<td>0.35</td>
</tr>
<tr>
<td>10-second</td>
<td>0.53</td>
<td>(1.59 ( \times 10^{-2} ))</td>
<td>0.51</td>
</tr>
<tr>
<td>( \eta = 0.1 )</td>
<td>( \text{LGD} )</td>
<td>( \text{GM} )</td>
<td>( \text{GFTD} )</td>
</tr>
</tbody>
</table>

**Table 4.** This table shows means and standard errors of probabilities of \( \text{GFTD} \), \( \text{GSD} \) and \( \text{GM} \) for our test and the tests by Lee and Mykland (2008) (LM test) and Jiang and Oomen (2008) (JO test). The significance level \( \alpha \) is 5% and \( \sigma_t = 0.6 \). The volatility of the jump size is set as 1.2. \( \text{freq} \) denotes the sampling frequency, and \( \eta \) is the standard deviation of the noise.
Table 5. The table shows the MSE values of estimating the jump size for different conditions on the spot volatility $\sigma_t$ and the jump size $Y$. SV means $\sigma_t$ follows the process of (23), and $freq$ denotes the sampling frequency.

<table>
<thead>
<tr>
<th>$freq$</th>
<th>$Y = 1.2, \sigma_t = 0.6$</th>
<th>$Y = 1.2, SV$</th>
<th>$Y \sim N(0, 1.44), \sigma_t = 0.6$</th>
<th>$Y \sim N(0, 1.44), SV$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-second</td>
<td>$1.46 \times 10^{-2}$</td>
<td>$2.53 \times 10^{-2}$</td>
<td>$1.68 \times 10^{-2}$</td>
<td>$2.40 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.5-second</td>
<td>$1.34 \times 10^{-2}$</td>
<td>$2.09 \times 10^{-2}$</td>
<td>$1.42 \times 10^{-2}$</td>
<td>$1.72 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.25-second</td>
<td>$8.16 \times 10^{-3}$</td>
<td>$1.94 \times 10^{-2}$</td>
<td>$9.40 \times 10^{-3}$</td>
<td>$1.32 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.125-second</td>
<td>$7.38 \times 10^{-3}$</td>
<td>$1.11 \times 10^{-2}$</td>
<td>$7.16 \times 10^{-3}$</td>
<td>$1.00 \times 10^{-2}$</td>
</tr>
</tbody>
</table>