Signals and Systems

#16: Analysis of LTI systems via Laplace transform

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Some of the images in this presentation are from “Signals and Systems”, A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

Outline

- …
- #13 Laplace transform
- #14 Inverse Laplace transform
- #15 Properties of the Laplace transform
- **#16 Analysis of LTI systems via Laplace transform**

Causality

For an LTI system:

$$y(t) = h(t) * x(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad Y(s) = H(s)\,X(s)$$

$H(s) = \mathcal{L}\{h(t)\}$ is the transfer function

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Causality:

$h(t) = 0 \;\; \forall t < 0$, i.e., right sided $\;\Rightarrow\;$ the ROC of $H(s)$ is a right-half plane

(the converse is true when $H(s)$ is rational)

Example:

$$h(t) = e^{-t}\, u(t) \quad \Rightarrow \quad \text{causal}$$

$$H(s) = \frac{1}{s+1}, \quad \Re\{s\} > -1 \quad \text{(right-half plane)}$$

Causality:

$$h(t) = 0 \;\; \forall t < 0, \;\; \text{i.e., right sided} \;\; \Rightarrow \;\; \text{the ROC of } H(s) \text{ is a right-half plane}$$

(the converse is true when $H(s)$ is rational)

Example:

$h(t) = e^{-|t|} \quad \Rightarrow \quad \text{not causal}$

$$H(s) = \frac{-2}{s^2 - 1}, \quad -1 < \Re\{s\} < 1 \quad (\textit{not} \text{ to the right of the rightmost pole})$$

Causality:

$h(t) = 0 \ \forall t < 0$, i.e., right sided $\Rightarrow$ the ROC of $H(s)$ is a right-half plane

(the converse is true when $H(s)$ is rational)

Example:

$$H(s) = \frac{e^s}{s+1}, \ \ \Re\{s\} > -1 \quad \text{(right of the rightmost pole} \ \ \Rightarrow \ \ h(t) \text{ right sided)}$$

$$e^{-t}\, u(t) \ \ \overset{\mathcal{L}}{\longleftrightarrow} \ \ \frac{1}{s+1}, \ \ \Re\{s\} > -1$$

$$e^{-(t+1)}\, u(t+1) \ \ \overset{\mathcal{L}}{\longleftrightarrow} \ \ e^s\, \frac{1}{s+1}, \ \ \Re\{s\} > -1$$

thus $h(t) = e^{-(t+1)}\, u(t+1) \ \ \Rightarrow$ not causal

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(the converse is true when $H(s)$ is rational)

Stability

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$H(s) = \mathcal{L}\{h(t)\}$ is the transfer function

Stability:

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty \quad \Rightarrow \quad \text{the Fourier transform } H(j\omega) \text{ exists}$$

$\Rightarrow$ the ROC of $H(s)$ must include the $j\omega-$axis, i.e., $\Re\{s\} = 0$

Example:

$$H(s) = \frac{s-1}{(s+1)(s-2)} \quad \text{ROC not specified}$$

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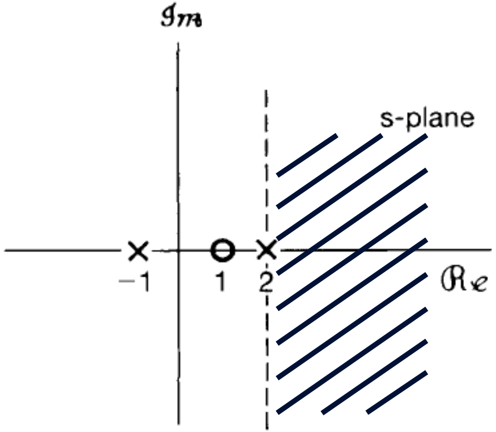


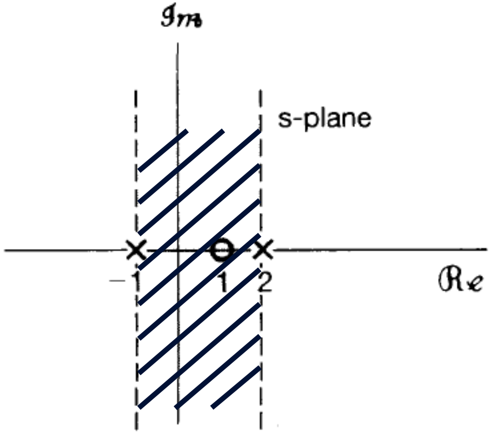
If the system is causal, then:

$$h(t) = \left[\frac{2}{3}\,e^{-t} + \frac{1}{3}\,e^{2t}\right]\,u(t)$$

ROC does not include the $j\omega-$axis $\Rightarrow$ unstable

Example:

$$H(s) = \frac{s-1}{(s+1)(s-2)} \quad \text{ROC not specified}$$



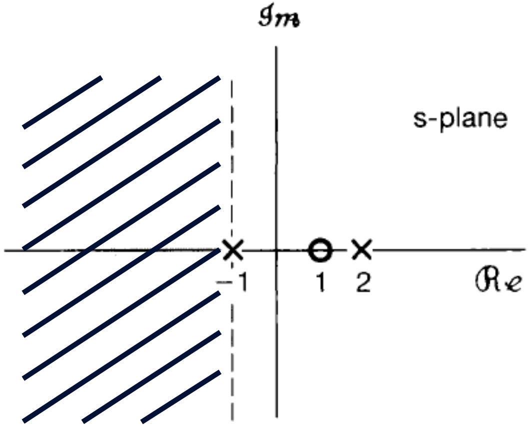
If the system is stable, then:

$$h(t) = \frac{2}{3} e^{-t} u(t) - \frac{1}{3} e^{2t} u(-t)$$

ROC not a right-half plane $\Rightarrow$ not causal

Example:

$$H(s) = \frac{s-1}{(s+1)\,(s-2)} \quad \text{ROC not specified}$$



The last option is:

$$h(t) = -\left[\frac{2}{3}\,e^{-t} + \frac{1}{3}\,e^{2t}\right] u(-t)$$

anticausal and unstable

If LTI system is causal $\Rightarrow$ ROC to the right of rightmost pole

If LTI system is also stable $\Rightarrow$ rightmost pole must be to the left of $j\omega$−axis

Causal LTI system with rational $H(s)$ is stable
if and only if all its poles have negative real part

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Example:

$h(t) = e^{-t}\,u(t)$ (causal) $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$

$H(s) = \frac{1}{s+1}, \quad \Re\{s\} > -1$ all poles have negative real part $\Rightarrow$ stable

If LTI system is causal $\Rightarrow$ ROC to the right of rightmost pole

If LTI system is also stable $\Rightarrow$ rightmost pole must be to the left of $j\omega-$axis

Causal LTI system with rational $H(s)$ is stable
if and only if all its poles have negative real part

Example:

$h(t) = e^{2t}\, u(t)$ (causal) $\quad\quad \displaystyle\int_{-\infty}^{\infty} |h(t)|\, dt = \infty$

$H(s) = \dfrac{1}{s-2}, \quad \Re\{s\} > 2$ pole w/ positive real part $\Rightarrow$ unstable

Systems characterized by differential equations

Example:

$$\frac{dy(t)}{dt} + 3\,y(t) = x(t) \qquad \text{apply } \mathcal{L}\{\cdot\} \text{ to both sides}$$

$$s\,Y(s) + 3\,Y(s) = X(s) \;\;\Rightarrow\;\; H(s) = \frac{1}{s+3} \quad \text{(ROC unknown)}$$

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If the system is causal, then:

$$\Re\{s\} > -3 \;\;\Rightarrow\;\; h(t) = e^{-3t}\,u(t)$$

If the system is anticausal, then:

$$\Re\{s\} < -3 \;\;\Rightarrow\;\; h(t) = -e^{-3t}\,u(-t)$$

Example:

we know input and output $\begin{cases} x(t) = e^{-3t}\, u(t) \\ \\ y(t) = \left[e^{-t} - e^{-2t}\right]\, u(t) \end{cases}$

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we know input and output
$$\begin{cases} x(t) = e^{-3t}\, u(t) \\ \\ y(t) = \left[e^{-t} - e^{-2t}\right]\, u(t) \end{cases}$$

we calculate
$$\begin{cases} X(s) = \frac{1}{s+3} & \Re\{s\} > -3 \\ \\ Y(s) = \frac{1}{(s+1)\,(s+2)} & \Re\{s\} > -1 \end{cases}$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)\,(s+2)} = \frac{s+3}{s^2+3s+2}, \quad \Re\{s\} > -1$$

Therefore:

$H(s)$ rational and ROC right-half plane $\quad\Rightarrow\quad$ causal system

all poles in $H(s)$ have negative real part $\quad\Rightarrow\quad$ stable system

$$\Rightarrow\quad H(s)=\frac{Y(s)}{X(s)}=\frac{s+3}{(s+1)(s+2)}=\frac{s+3}{s^2+3s+2},\qquad \Re\{s\}>-1$$

Therefore:

$H(s)$ rational and ROC right-half plane $\Rightarrow$ causal system

all poles in $H(s)$ have negative real part $\Rightarrow$ stable system

the system equation is: $$\frac{d^2y(t)}{dt^2} + 3\,\frac{dy(t)}{dt} + 2\,y(t) = \frac{dx(t)}{dt} + 3\,x(t)$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)\,(s+2)} = \frac{s+3}{s^2+3s+2}, \quad \Re\{s\} > -1$$

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