Signals and Systems

#15: Properties of the Laplace transform

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Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

Outline

- …
- #13 Laplace transform
- #14 Inverse Laplace transform
- **#15 Properties of the Laplace transform**
- #16 Analysis of LTI systems via Laplace transform

Properties of the Laplace transform

Linearity:

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \quad \text{with ROC} = R_1$$

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s) \quad \text{with ROC} = R_2$$

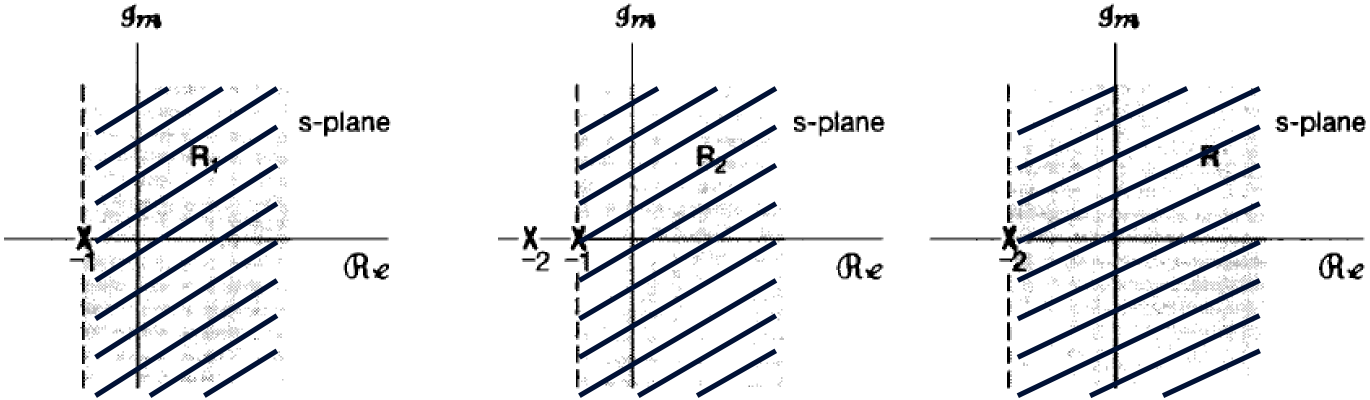
$$\Rightarrow a\,x_1(t) + b\,x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} a\,X_1(s) + b\,X_2(s) \quad \text{with ROC containing } R_1 \cap R_2$$

(the ROC can be larger than $R_1 \cap R_2$)

Example:

$$x(t) = x_1(t) - x_2(t), \quad \text{where} \quad \begin{cases} X_1(s) = \frac{1}{s+1} & \Re\{s\} > -1 \\ X_2(s) = \frac{1}{(s+1)(s+2)} & \Re\{s\} > -1 \end{cases}$$

$$X(s) = X_1(s) - X_2(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2} \quad \Re\{s\} > -2$$



Time shifting:

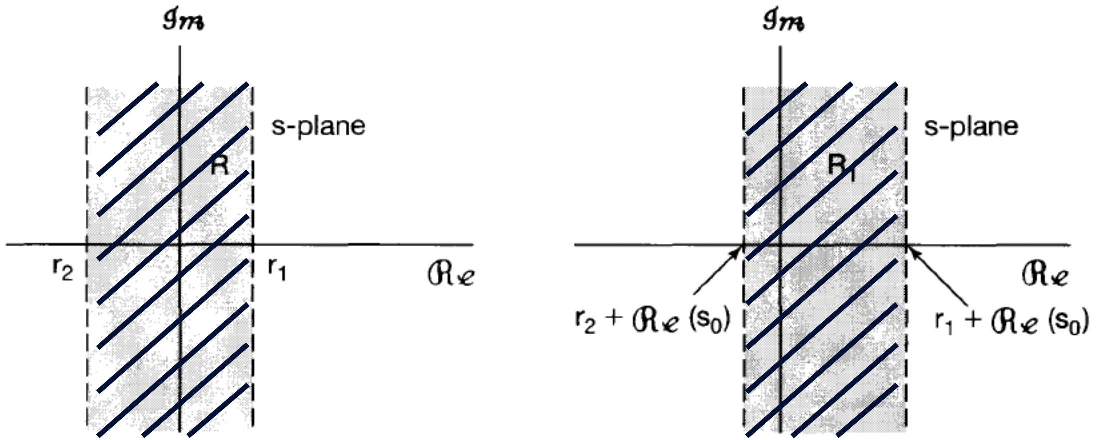
$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \quad \text{with ROC} = R$$

$$\Rightarrow \; x(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} X(s)\, e^{-st_0} \quad \text{with ROC} = R$$

Shifting in the $s-$domain:

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s) \quad \text{with ROC} = R$$

$$\Rightarrow \quad e^{s_0 t}x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s-s_0) \quad \text{with ROC} = R + \Re\{s_0\}$$



poles/zeroes at $s = a$ will be moved to $s = a + s_0$

Time scaling:

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \quad \text{with ROC} = R$$

$$\Rightarrow \; x(a\,t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad \text{with ROC} = aR$$

For $0 < a < 1 \;\Rightarrow\;$ ROC compression

For $a > 1 \;\Rightarrow\;$ ROC expansion

For $a < 0 \;\Rightarrow\;$ ROC reversal and scaling

$$x(-t) \overset{\mathcal{L}}{\longleftrightarrow} X(-s) \quad \text{with ROC} = -R$$

Conjugation:

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \quad \text{with ROC} = R$$

$$\Rightarrow \; x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*) \quad \text{with ROC} = R$$

If $x(t)$ is real $\Rightarrow \; X(s) = X^*(s^*)$

Convolution:

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \quad \text{with ROC} = R_1$$

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s) \quad \text{with ROC} = R_2$$

$$\Rightarrow x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \cdot X_2(s) \quad \text{with ROC containing } R_1 \cap R_2 \text{ (can be larger)}$$

Example: $\begin{cases} X_1(s) = \frac{s+1}{s+2} \quad \Re\{s\} > -2 \\ X_2(s) = \frac{s+2}{s+1} \quad \Re\{s\} > -1 \end{cases}$ $\qquad X_1(s) \cdot X_2(s) = 1 \quad \forall s$

Time-domain differentiation:

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \quad \text{with ROC} = R$$

$$\Rightarrow \quad \frac{d\,x(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} s\,X(s) \quad \text{with ROC containing } R \text{ (can be larger)}$$

Example:

$$x(t) = u(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) = \frac{1}{s} \quad \Re\{s\} > 0$$

$$\frac{d\,x(t)}{dt} = \delta(t) \overset{\mathcal{L}}{\longleftrightarrow} s\,X(s) = 1 \quad \forall\, s$$

$s-$domain differentiation:

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad \text{with ROC} = R$$

$$\Rightarrow \quad -t\,x(t) \xleftrightarrow{\mathcal{L}} \frac{d\,X(s)}{ds} \quad \text{with ROC} = R$$

$s-$domain differentiation:

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \quad \text{with ROC} = R$$

$$\Rightarrow \quad -t\,x(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{d\,X(s)}{ds} \quad \text{with ROC} = R$$

It follows that:

$$e^{-at}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \quad \Re\{s\} > -a$$

$$t\,e^{-at}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds}\,\frac{1}{s+a} = \frac{1}{(s+a)^2} \quad \Re\{s\} > -a$$

$$\frac{t^2}{2}\,e^{-at}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds}\,\frac{1}{(s+a)^2} = \frac{1}{(s+a)^3} \quad \Re\{s\} > -a$$

$$\frac{t^{n-1}}{(n-1)!}\,e^{-at}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^n} \quad \Re\{s\} > -a$$

Example:

$$X(s) = \frac{2s^2 + 5s + 5}{(s+1)^2\,(s+2)} = \frac{2}{(s+1)^2} - \frac{1}{s+1} + \frac{3}{s+2} \quad \Re\{s\} > -1$$

$$\Rightarrow\ x(t) = \left[2te^{-t} - e^{-t} + 3e^{-2t}\right]\,u(t)$$

It follows that:

$$e^{-at}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \quad \Re\{s\} > -a$$

$$t\,e^{-at}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds}\,\frac{1}{s+a} = \frac{1}{(s+a)^2} \quad \Re\{s\} > -a$$

$$\frac{t^2}{2}\,e^{-at}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds}\,\frac{1}{(s+a)^2} = \frac{1}{(s+a)^3} \quad \Re\{s\} > -a$$

$$\frac{t^{n-1}}{(n-1)!}\,e^{-at}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^n} \quad \Re\{s\} > -a$$

Time-domain integration:

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \quad \text{with ROC} = R$$

$$\Rightarrow \int_{-\infty}^{t} x(\tau)\,d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s) \quad \text{with ROC containing } R \cap \{\Re\{s\} > 0\}$$

Initial-value theorem:

If $x(t) = 0$ for $t < 0$, and

$x(t)$ has no impulses or higher-order singularities at $t = 0$

$$\Rightarrow \quad x(0^+) = \lim_{s \to \infty} s \cdot X(s)$$

Initial-value theorem:

If $x(t) = 0$ for $t < 0$, and

$x(t)$ has no impulses or higher-order singularities at $t = 0$

$$\Rightarrow \quad x(0^+) = \lim_{s \to \infty} s \cdot X(s)$$

Final-value theorem:

If $x(t) = 0$ for $t < 0$, and

$x(t)$ has finite limit as $t \to \infty$

$$\Rightarrow \quad \lim_{t \to \infty} x(t) = \lim_{s \to 0} s \cdot X(s)$$

Initial-value theorem:

If $x(t) = 0$ for $t < 0$, and

$x(t)$ has no impulses or higher-order singularities at $t = 0$

$$\Rightarrow \quad x(0^+) = \lim_{s \to \infty} s \cdot X(s)$$

Example:

$$x(t) = e^{-2t}\, u(t) + e^{-t}\,(\cos 3t)\, u(t)$$

$$X(s) = \frac{2s^2 + 5s + 12}{(s^2 + 2s + 10)\,(s + 2)}$$

$$x(0^+) = 2$$

$$\lim_{s \to \infty} s \cdot X(s) = \lim_{s \to \infty} \frac{2s^3 + 5s^2 + 12s}{s^3 + 4s^2 + 14s + 20} = 2$$

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