Signals and Systems

#14: Inverse Laplace transform

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Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

Outline

- …
- #09 First-order systems
- #10 Second-order systems
- #11 Sampling and reconstruction
- #12 Communication systems
- #13 Laplace transform
- **#14 Inverse Laplace transform**
- #15 Properties of the Laplace transform

Properties of the ROC

Recall:

$$x(t) \;\overset{\mathcal{L}}{\longleftrightarrow}\; X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} \left[x(t)\, e^{-\sigma\, t}\right] e^{-j\omega t}\, dt$$

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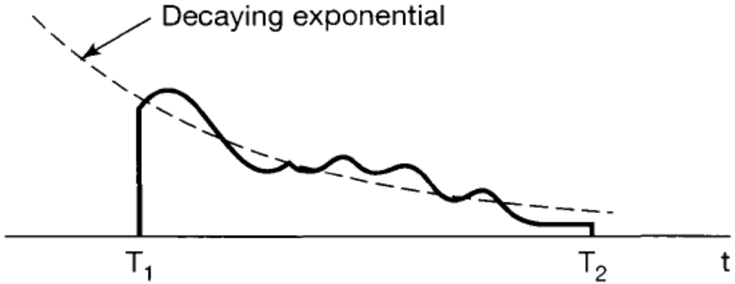
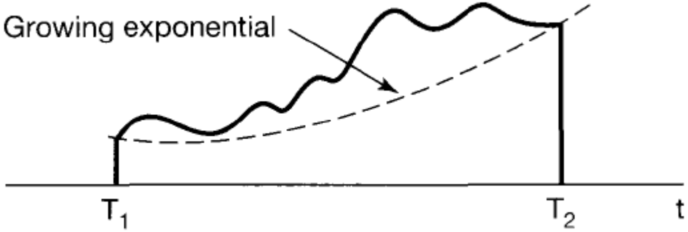
1. ROC consists of strips parallel to the $j\omega-$axis
2. For rationale Laplace transform, the ROC does not contain poles
3. For $x(t)$ of finite duration and absolutely integrable, the ROC is the entire $s-$plane





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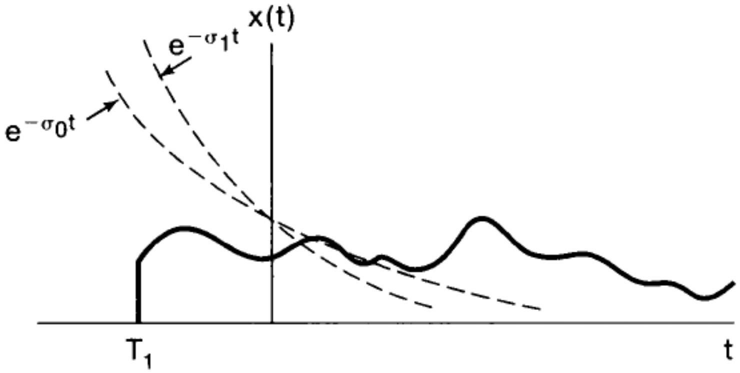
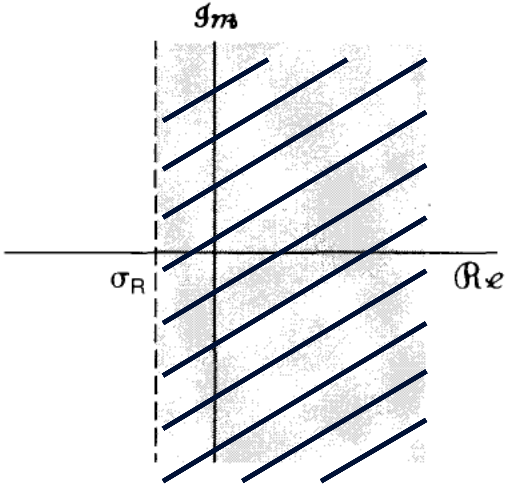
4. For $x(t)$ right sided, the ROC is a right-half plane





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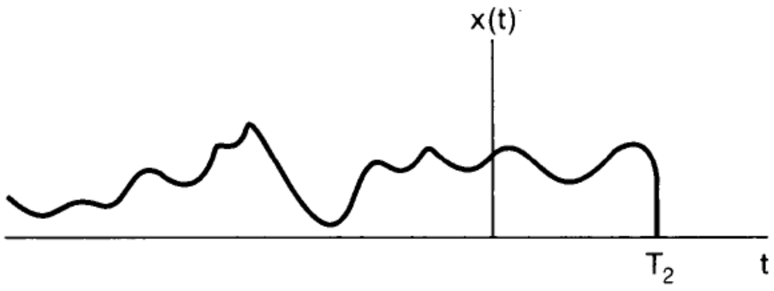
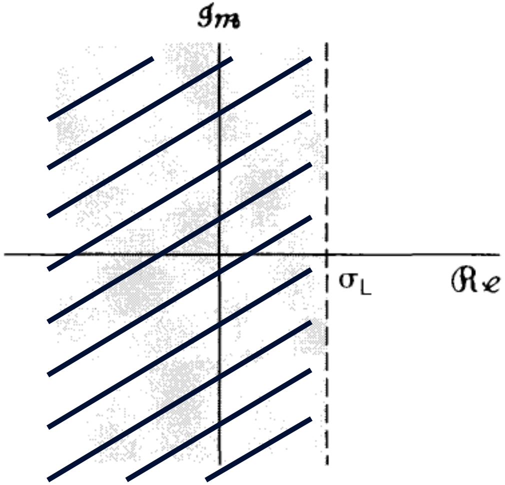
5. For $x(t)$ left sided, the ROC is a left-half plane





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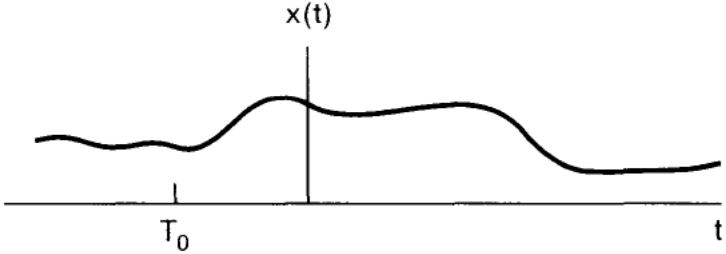
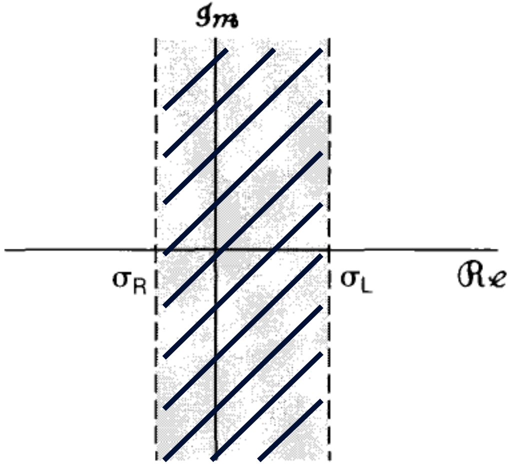
6. For $x(t)$ two sided, either the ROC does not exist, or it is a vertical strip

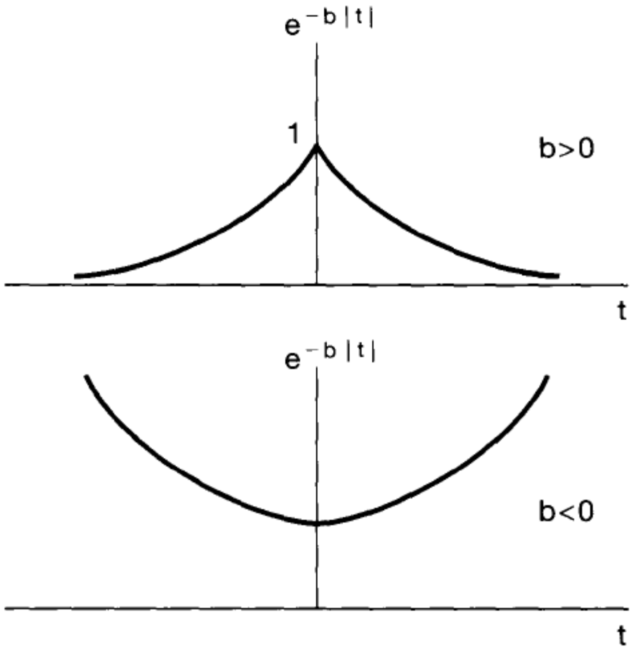




Examples

Example:

$$x(t) = e^{-|b|t} = e^{-bt}\, u(t) + e^{bt}\, u(-t) \qquad \text{(two sided)}$$



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$$e^{-bt}\, u(t) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{s+b} \qquad \Re\{s\} > -b$$

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If $b < 0$, there is no common ROC

$\Rightarrow$ the Laplace transform does not exist

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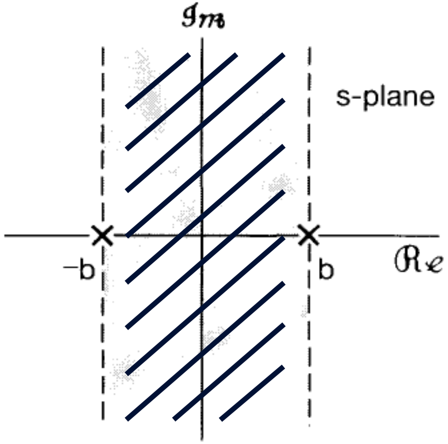
If $b > 0$, then

$$e^{-|b|t} \;\overset{\mathcal{L}}{\longleftrightarrow}\; \tfrac{1}{s+b} - \tfrac{1}{s-b} = \tfrac{-2b}{s^2-b^2} \qquad -b < \Re\{s\} < b$$



Properties of the ROC (cont’d)

Recap — When it exists, the ROC can be:

- The entire $s-$plane, for finite-length signals
- A left-half plane, for left-sided signals
- A right-half plane, for right-sided signals
- A vertical strip, for two-sided signals

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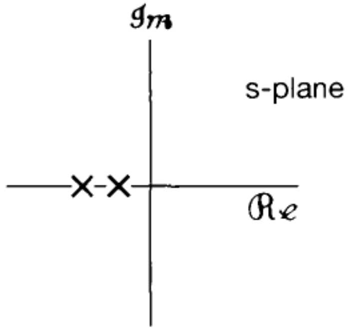
More properties — If the Laplace transform is rational, then:

7. The ROC is either bounded by poles or it extends to infinity. No poles in the ROC.
8. The ROC is to the right of the rightmost pole (for $x(t)$ right sided) or to the left of the leftmost pole (for $x(t)$ left sided).

Example:

$$X(s) = \frac{1}{(s+1)(s+2)}$$



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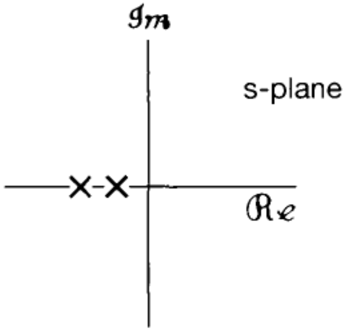
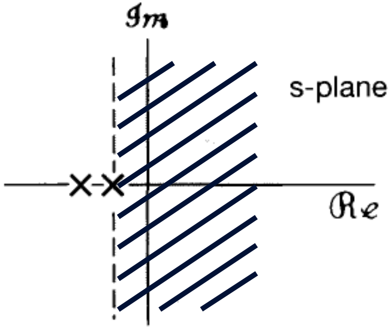
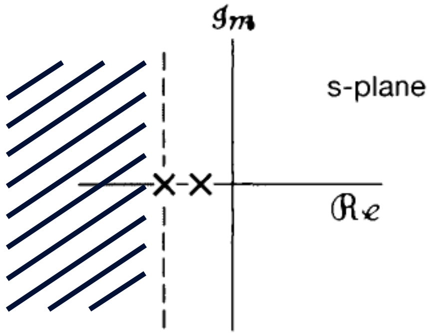




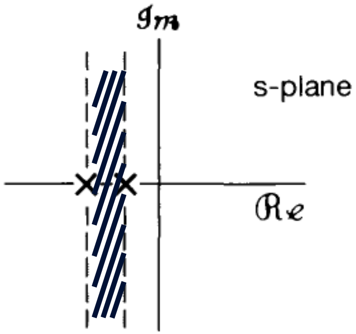
right-sided $x(t)$

$\mathcal{F}\{x(t)\}$ exists



left-sided $x(t)$

$\mathcal{F}\{x(t)\}$ does not exist



two-sided $x(t)$

$\mathcal{F}\{x(t)\}$ does not exist

Inverse Laplace transform

Note that:

$$x(t)\,e^{-\sigma t} = \mathcal{F}^{-1}\left\{X(\sigma + j\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}\,d\omega \qquad \text{(multiply by } e^{\sigma t}\text{)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}\,d\omega \qquad (\omega \to s,\;\; ds = j\,d\omega)$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\,e^{st}\,ds \qquad \text{Laplace anti-transform}$$

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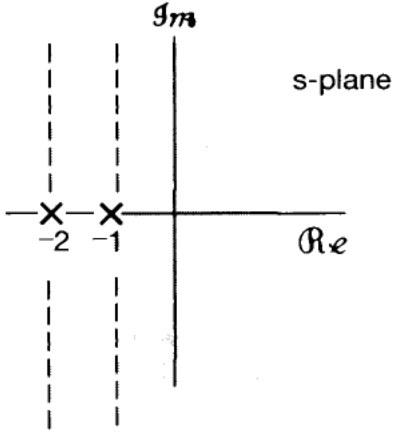
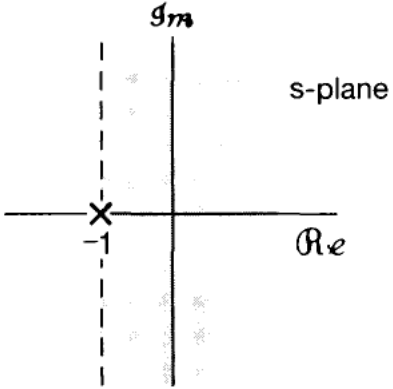
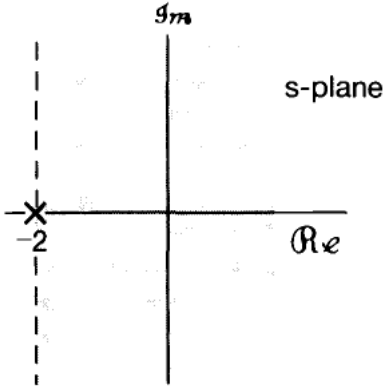
For ratios of polynomials, this formula is not necessary

Example: $X(s) = \dfrac{1}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2} = \dfrac{1}{s+1} - \dfrac{1}{s+2}$







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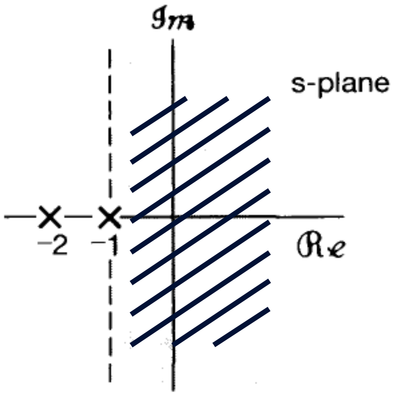

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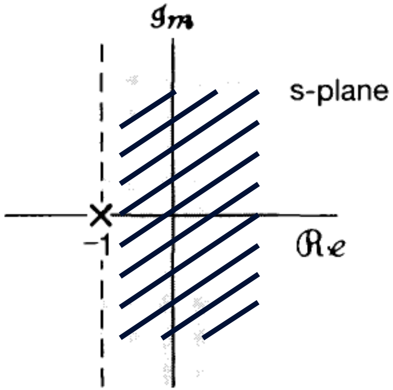

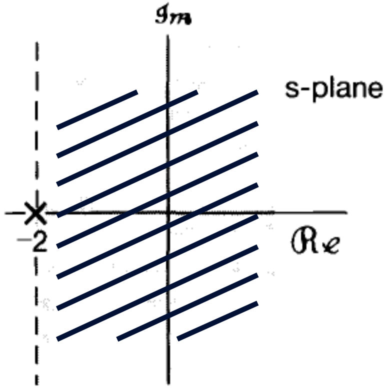

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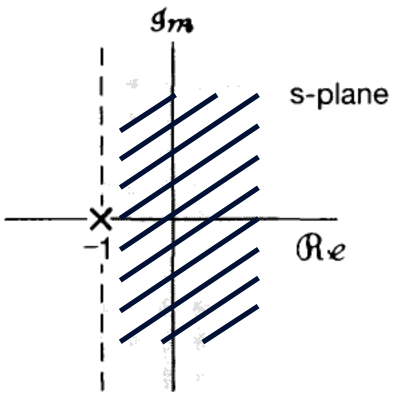

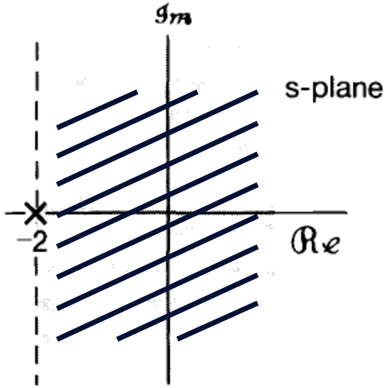

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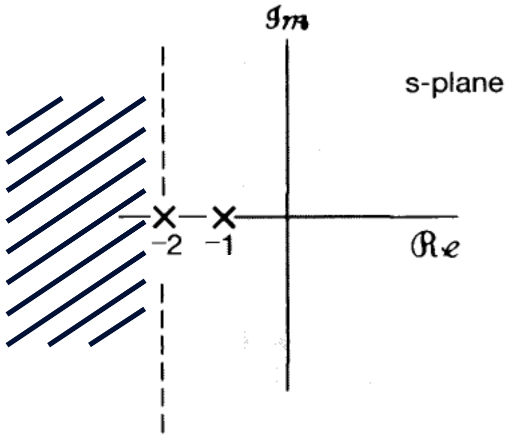
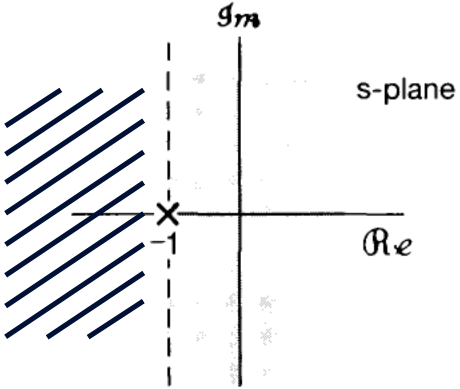
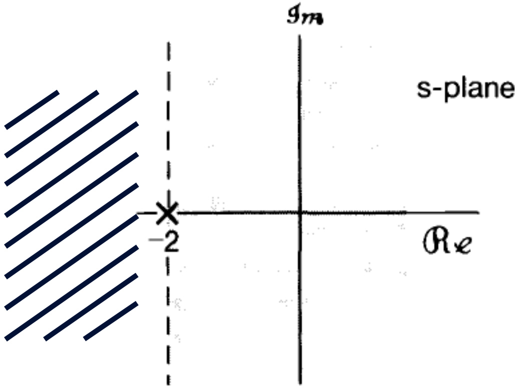


$\Re\{s\} < -1$





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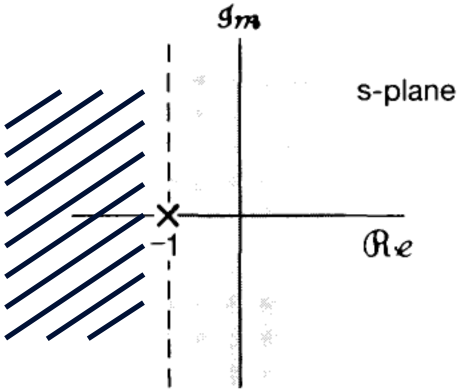

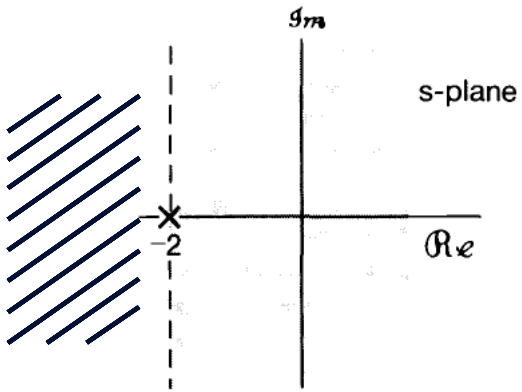

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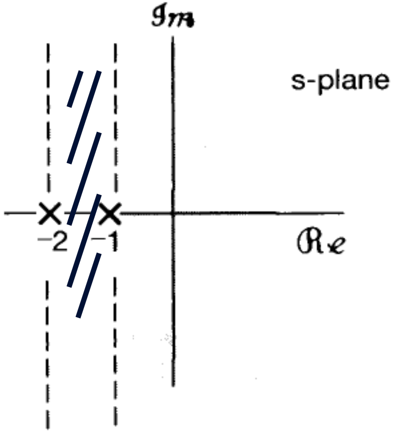

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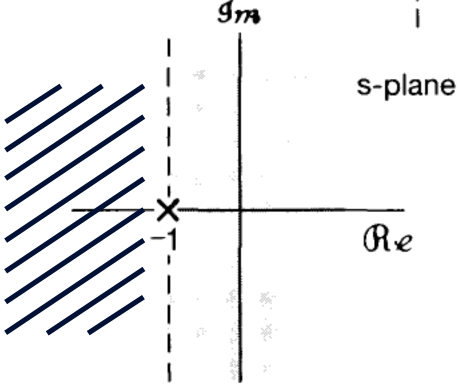

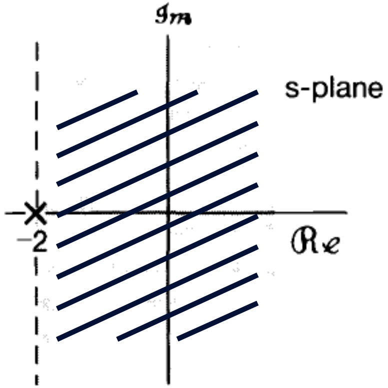

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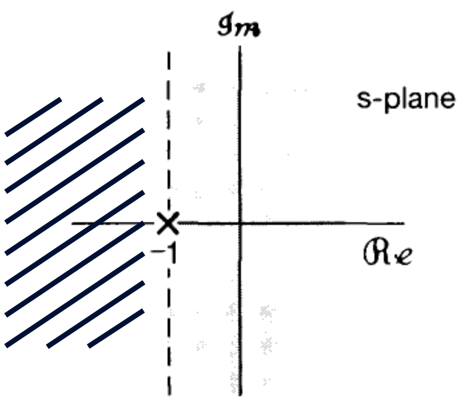

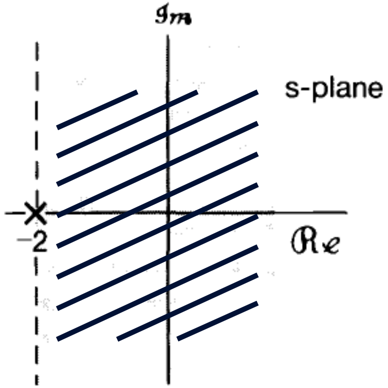

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$$-e^{-t}u(-t) - e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1} - \frac{1}{s+2} \qquad -2 < \Re\{s\} < -1$$

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