Signals and Systems

#13: Laplace transform

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Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

Outline

- …
- #09 First-order systems
- #10 Second-order systems
- #11 Sampling and reconstruction
- #12 Communication systems
- **#13 Laplace transform**
- #14 Inverse Laplace transform

Laplace transform

Fourier transform: signals as linear combinations of $e^{j\omega t}$

Laplace transform: signals as linear combinations of e^{st}

Generalization allows to study unstable systems,

for which $\mathcal{F}\{h(t)\}$ is not defined

Fourier transform: signals as linear combinations of $e^{j\omega t}$

Laplace transform: signals as linear combinations of e^{st}

$$x(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad X(s) \triangleq \int_{-\infty}^{\infty} x(t)\, e^{-s\,t}\, dt \qquad\qquad s = \sigma + j\omega \in \mathbb{C}$$

We can see it as:

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)\, t}\, dt = \int_{-\infty}^{\infty} \left[x(t)\, e^{-\sigma\, t}\right] e^{-j\omega t}\, dt$$

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The Laplace transform of $x(t)$ is the Fourier transform of $x(t)\, e^{-\sigma t}$,

whose convergence depends on the value of σ

Examples

Example: $x(t) = e^{-at}\, u(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = \frac{1}{a + j\omega} \qquad a > 0$$

Example: $x(t) = e^{-at}\, u(t)$

$$X(s) = \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+\sigma)\, t}\, e^{-j\omega t}\, dt = \frac{1}{a + \sigma + j\omega} \qquad \sigma + a > 0$$

$$= \frac{1}{a + s} \qquad \Re\{s\} > -a$$

Example: $x(t) = -e^{-at}\,u(-t)$

$$X(s) = -\int_{-\infty}^{\infty} e^{-at}\,u(-t)\,e^{-st}\,dt = -\int_{-\infty}^{\infty} e^{-at}\,u(-t)\,e^{-(\sigma+j\omega)t}\,dt$$

$$= \int_{-\infty}^{0} e^{-(a+\sigma)\,t}\,e^{-j\omega t}\,dt = \frac{1}{a+\sigma+j\omega} \qquad \sigma + a < 0$$

$$= \frac{1}{a+s} \qquad \Re\{s\} < -a$$

Note: $X(s)$ is identical, but the region of convergence (ROC) changes

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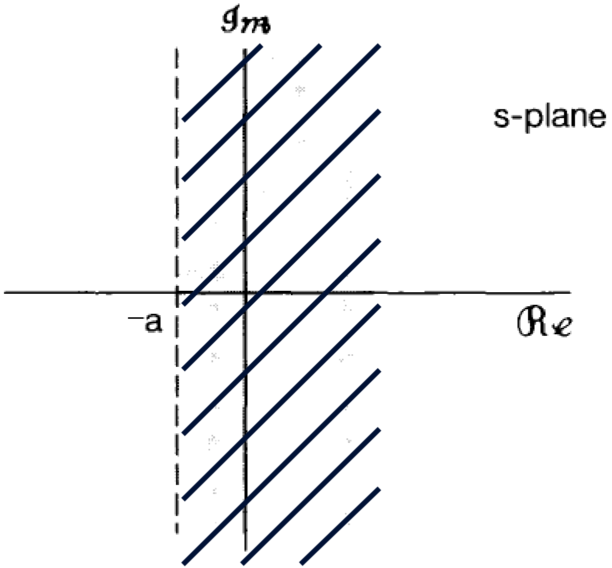
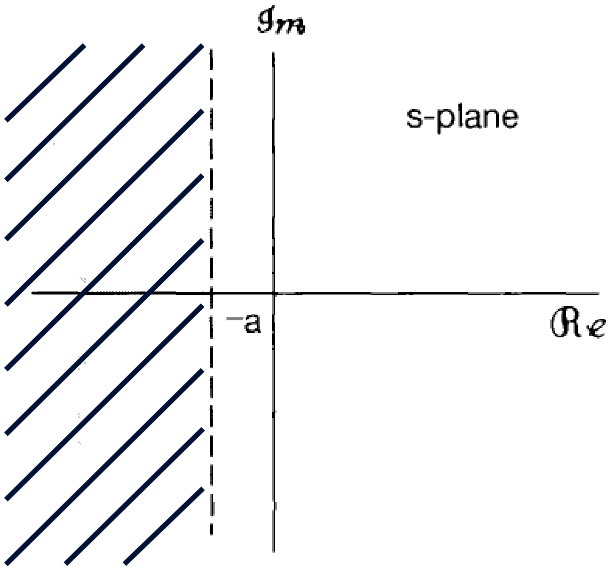


$\Re\{s\} > -a$



$\Re\{s\} < -a$

Example:

$$x(t) = e^{-2t}\,u(t) + e^{-t}\,(\cos 3t)\,u(t) = \left[e^{-2t} + \frac{1}{2}e^{-(1-3j)t} + \frac{1}{2}e^{-(1+3j)t}\right]\,u(t)$$

$$X(s) = \int_{-\infty}^{\infty} e^{-2t}\,u(t)\,e^{-st}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-(1-3j)t}\,u(t)\,e^{-st}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-(1+3j)t}\,u(t)\,e^{-st}\,dt$$

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$$e^{-2t}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2} \qquad \Re\{s\} > -2$$

$$e^{-(1-3j)t}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1-3j)} \qquad \Re\{s\} > -1$$

$$e^{-(1+3j)t}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1+3j)} \qquad \Re\{s\} > -1$$

They all converge

only if $\Re\{s\} > -1$

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$$X(s) = \frac{1}{s+2} + \frac{1}{2}\,\frac{1}{s+(1-3j)} + \frac{1}{2}\,\frac{1}{s+(1+3j)} \qquad \Re\{s\} > -1$$

$$= \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)} \qquad \Re\{s\} > -1$$

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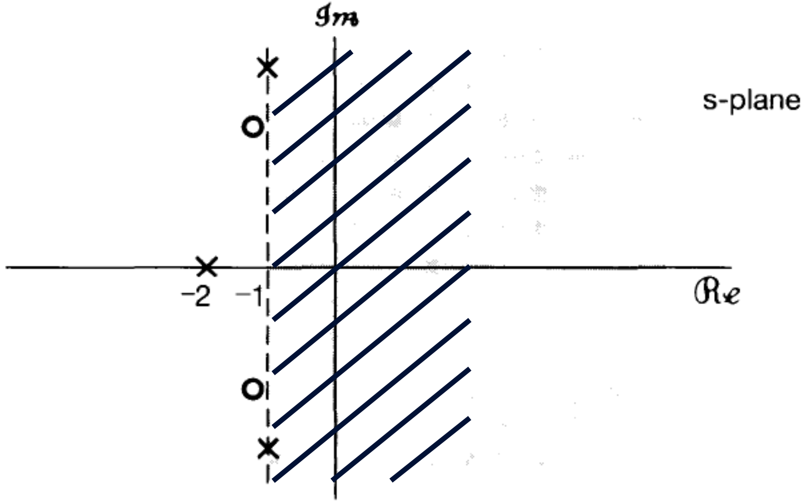


$x(t)$ made of complex exponentials
$\Rightarrow X(s)$ ratio of polynomials

Example:

$$x(t) = \delta(t) - \frac{4}{3}\, e^{-t}\, u(t) + \frac{1}{3}\, e^{2t}\, u(t)$$

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} \delta(t) e^{-st}\, dt = 1 \qquad\qquad \forall s$$

$$e^{-t}\, u(t) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{s+1} \qquad\qquad \Re\{s\} > -1$$

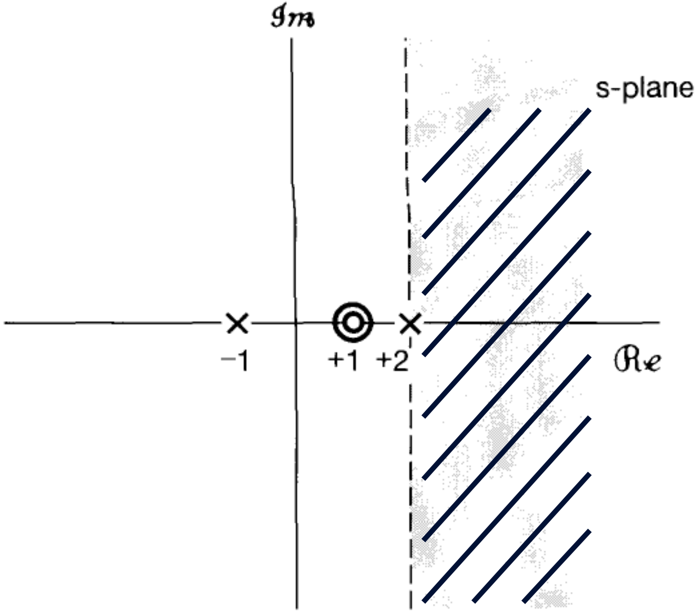
$$e^{2t}\, u(t) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{s-2} \qquad\qquad \Re\{s\} > 2$$

Example:

$$x(t) = \delta(t) - \frac{4}{3}\, e^{-t}\, u(t) + \frac{1}{3}\, e^{2t}\, u(t)$$

$$X(s) = 1 - \frac{4}{3}\,\frac{1}{s+1} + \frac{1}{3}\,\frac{1}{s-2} \qquad \Re\{s\} > 2$$

$$= \frac{(s-1)^2}{(s+1)(s-2)} \qquad \Re\{s\} > 2$$



ROC does not include $s = j\omega$,

Fourier transform does not converge

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