Signals and Systems

#11: Sampling and reconstruction

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Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

Outline

- …
- #06 Basic properties of the Fourier transform
- #07 Convolution property
- #08 Multiplication property
- #09 First-order systems
- #10 Second-order systems
- **#11 Sampling and reconstruction**
- #12 Communication systems

Sampling

We want to represent a signal $x(t)$ through its samples

T is the sampling interval

$\omega_s = \frac{2\pi}{T}$ is the sampling frequency

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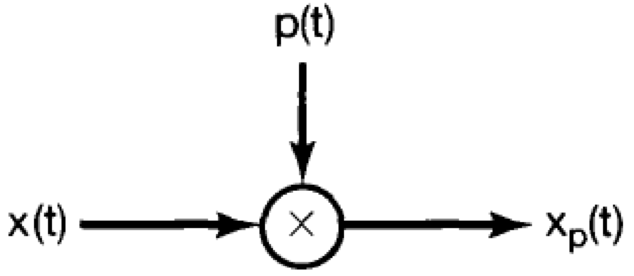

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = x(t) \cdot p(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - nT)$$
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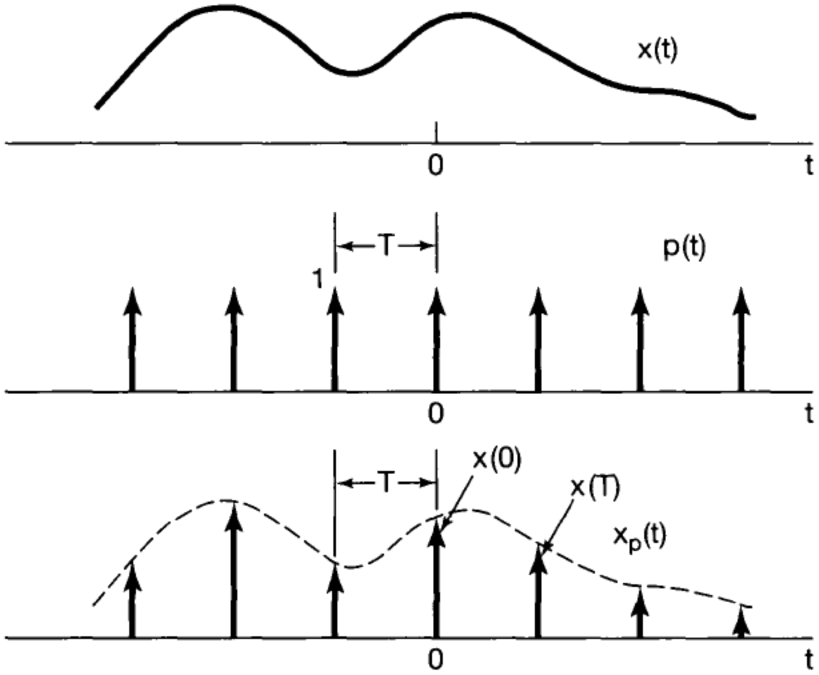

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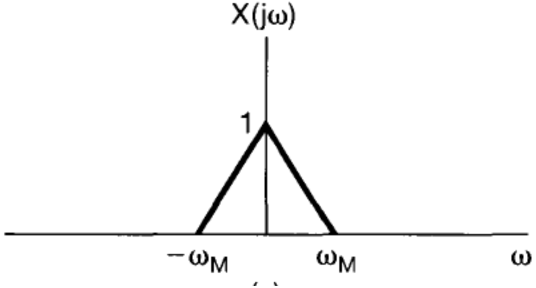
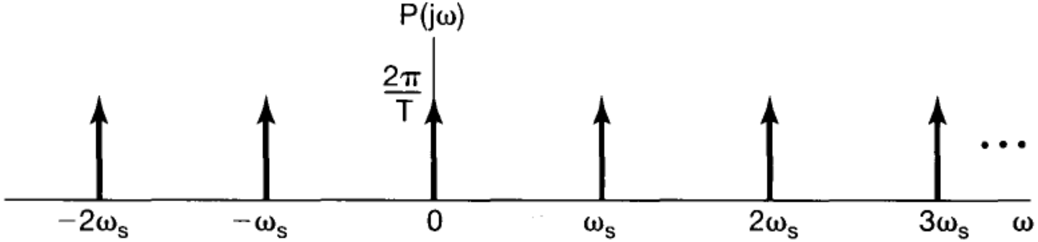
In the frequency domain:

$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta = \ldots$$



recall

$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\,\omega_s)$$



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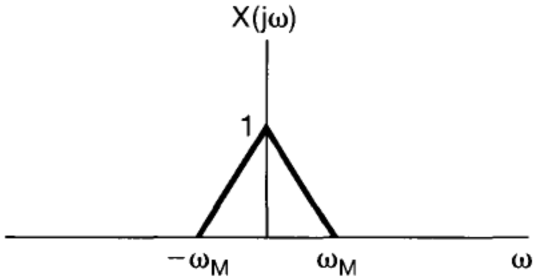

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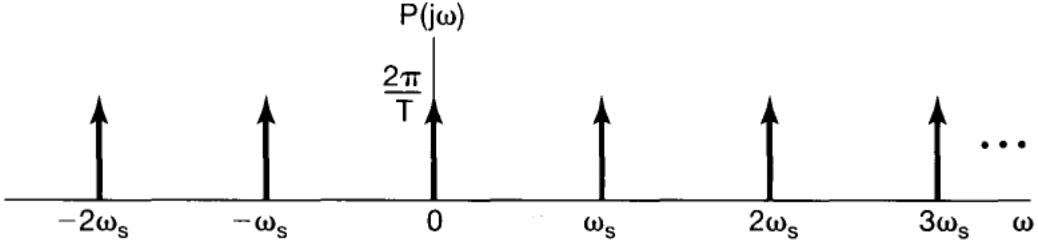
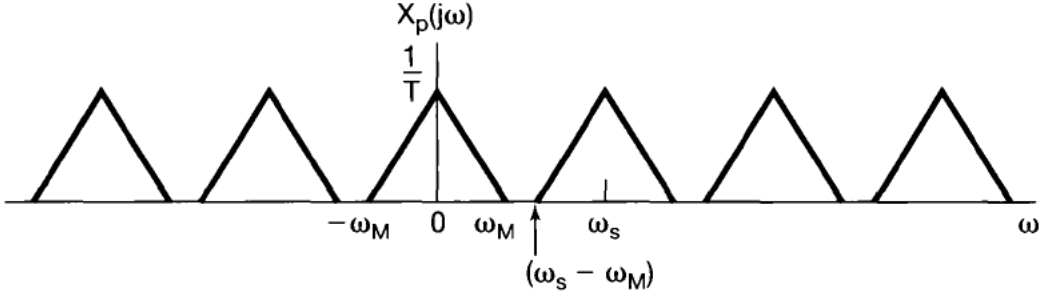


$$\ldots = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\,\omega_s))$$



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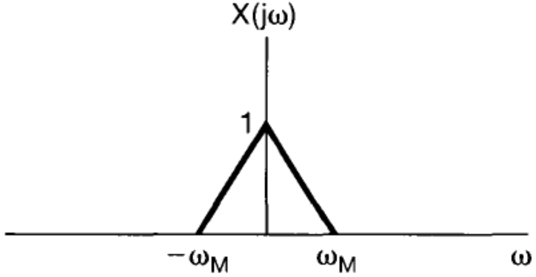

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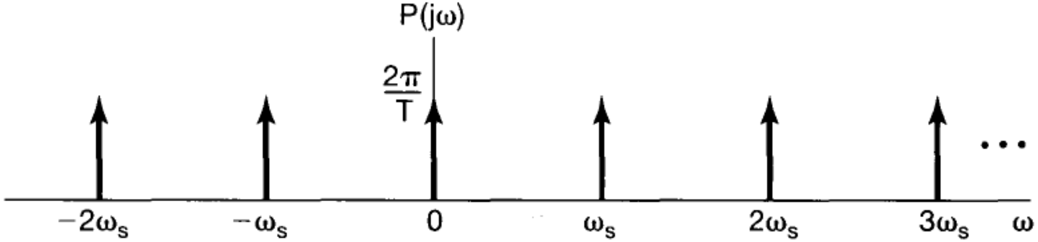

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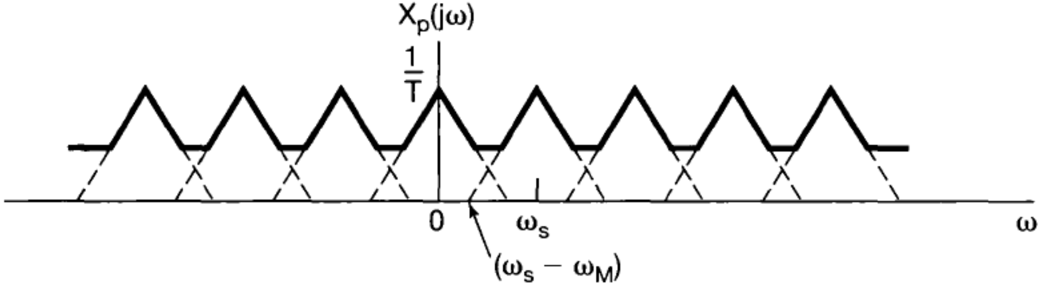
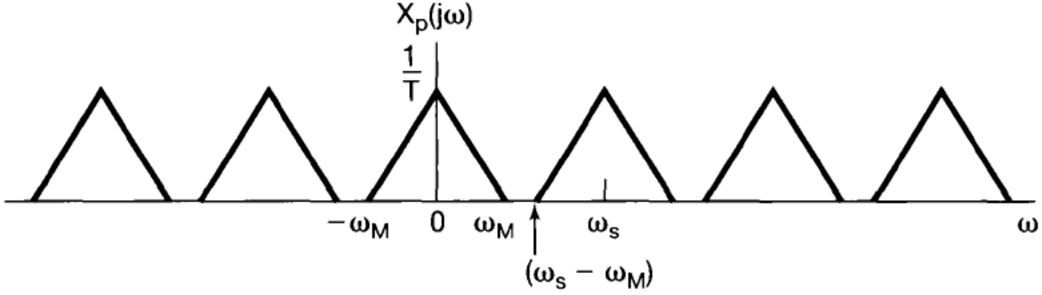


Sampling theorem:



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no overlap
perfect reconstruction possible

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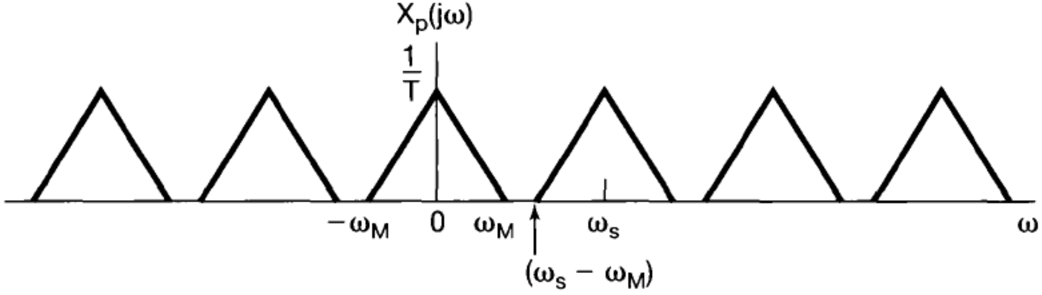
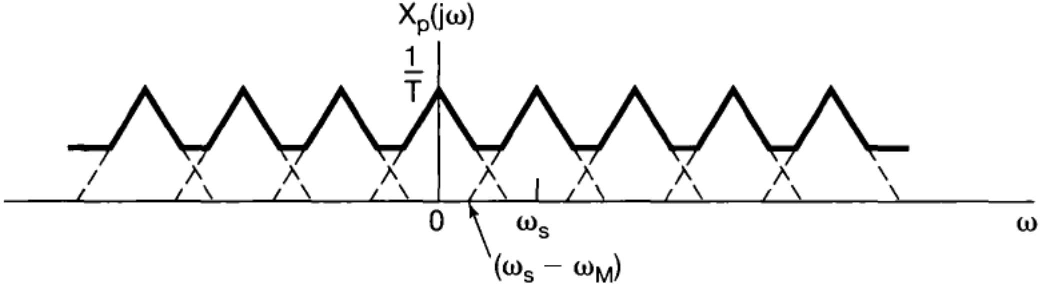


if $\omega_s < 2\,\omega_M$
overlap (*aliasing*)
perfect reconstruction not possible



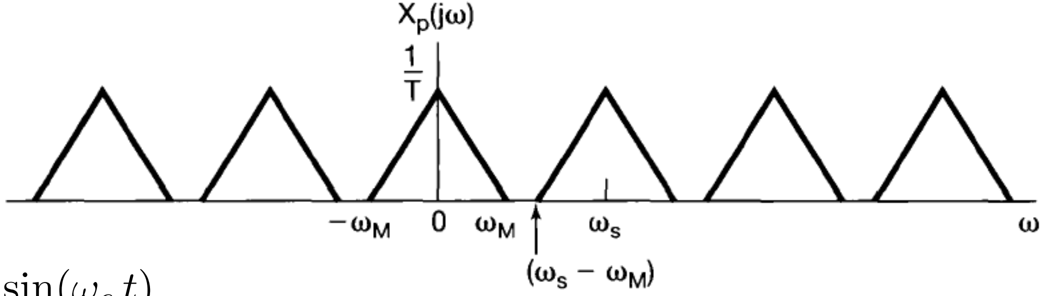
the frequency $2\,\omega_M$ is denoted the **Nyquist rate**

Reconstruction

Band-limited interpolation

Interpolate the sampled signal

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$



through lowpass filtering $\quad h(t) = \dfrac{\omega_c T \, \sin(\omega_c t)}{\pi\, \omega_c t}$

$$x_r(t) = x_p(t) * h(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\,\frac{\omega_c T}{\pi}\,\frac{\sin(\omega_c(t - nT))}{\omega_c(t - nT)}$$

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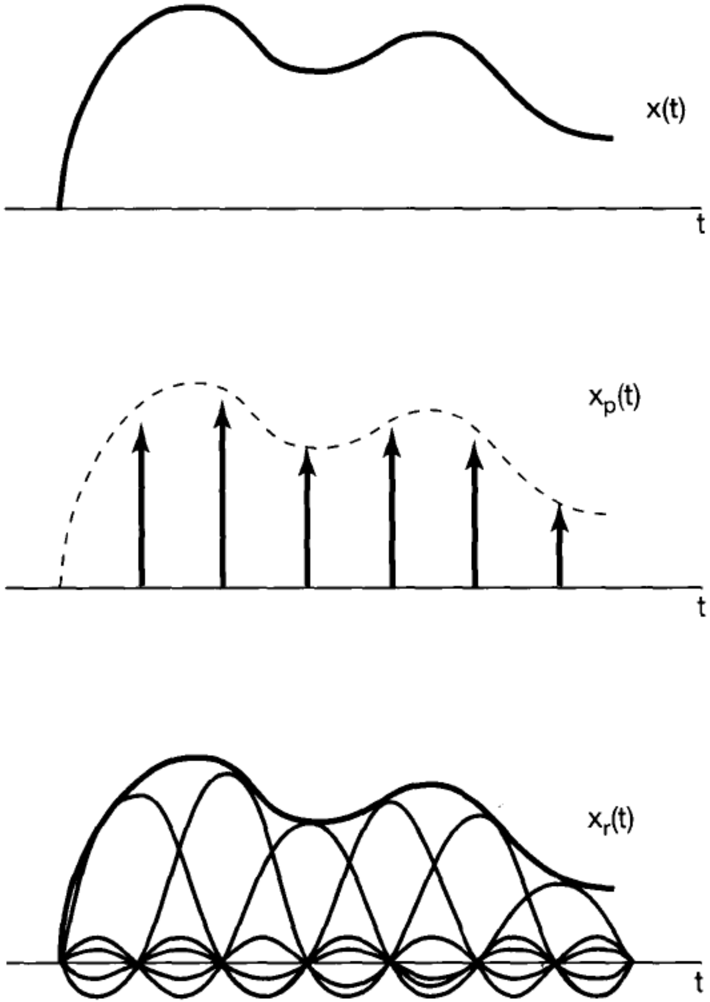
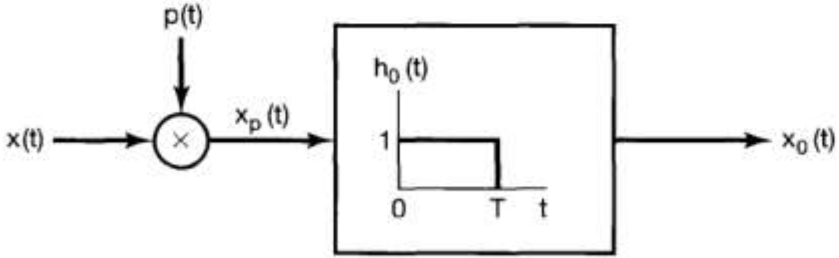
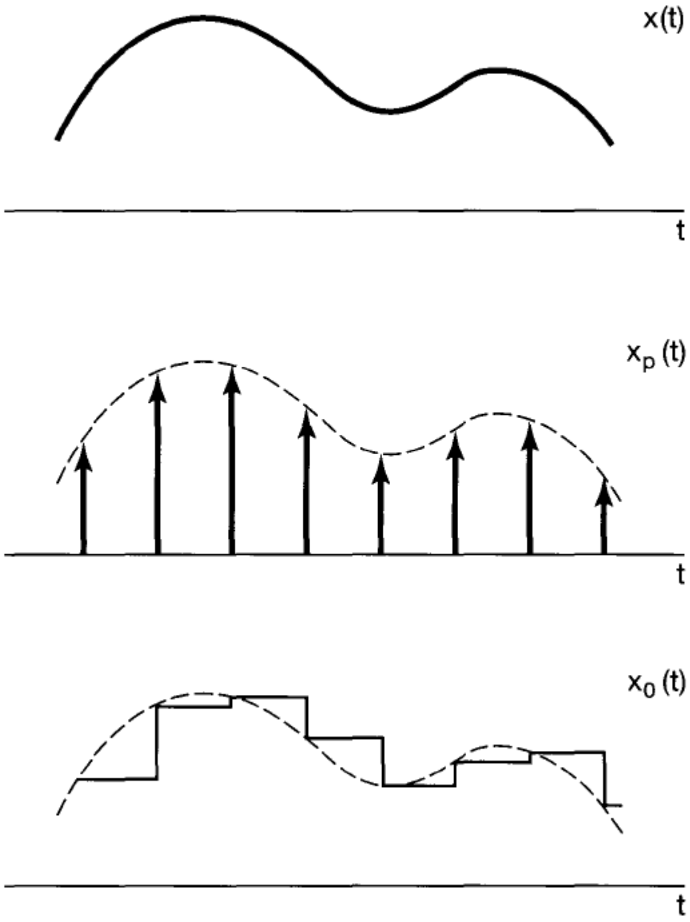


Zero-order hold

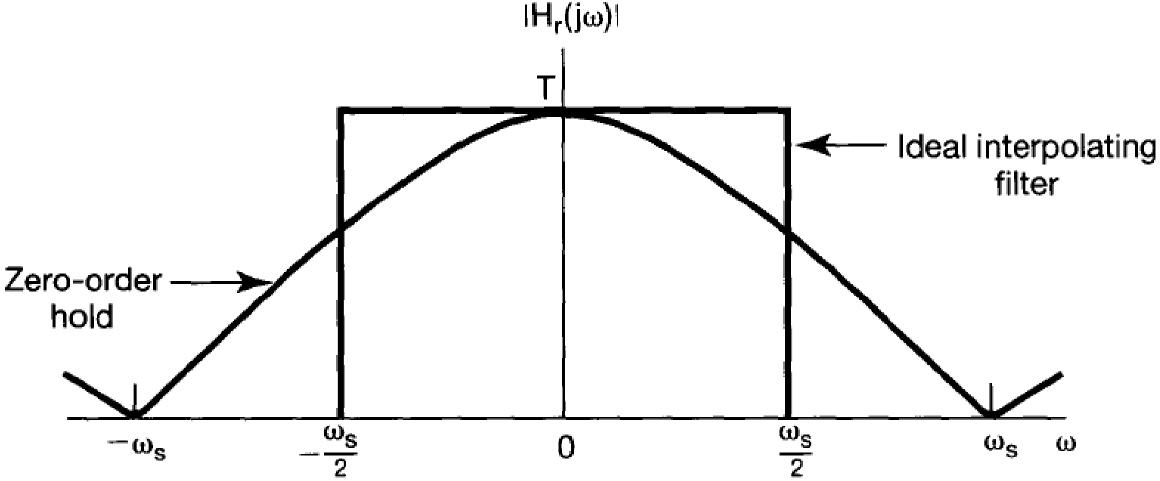
in time domain:





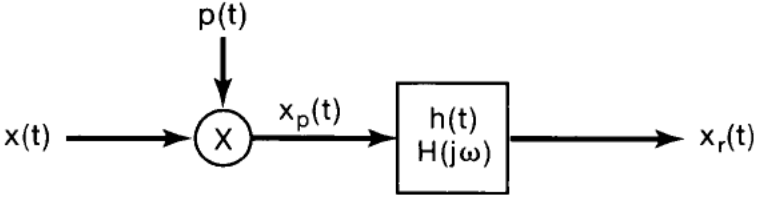
Zero-order hold

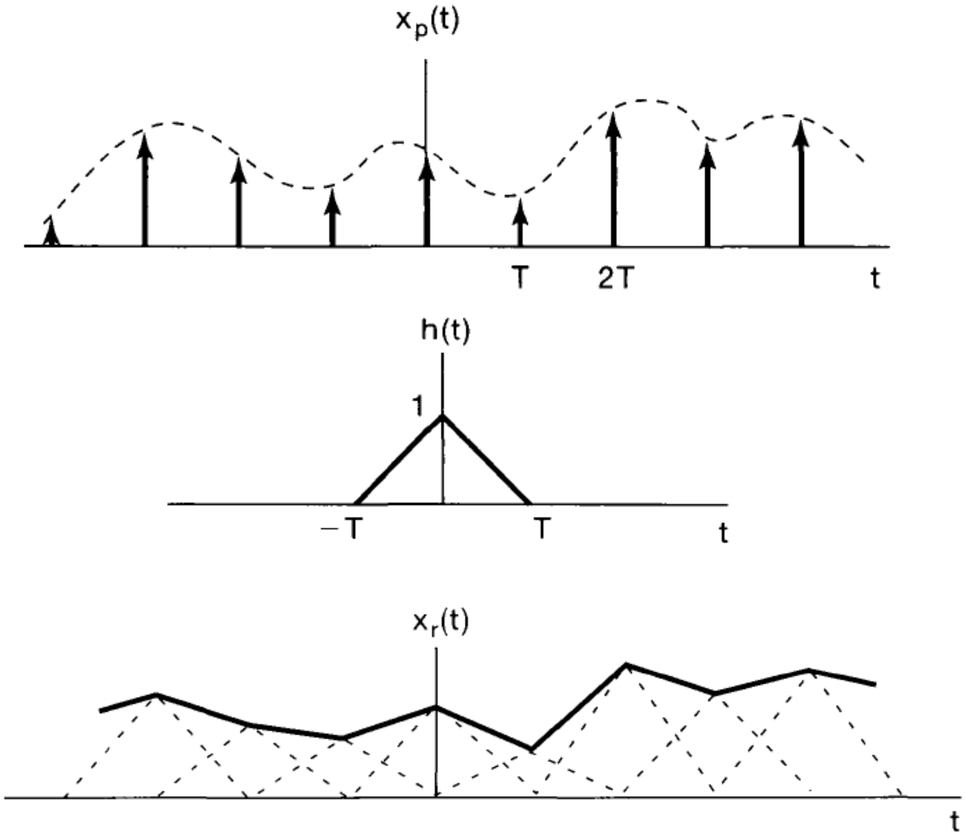
in frequency domain: $$H_0(j\omega) = e^{-j\omega T/2} \left[\frac{2\,\sin(\omega\,T/2)}{\omega} \right]$$



First-order hold (linear interpolation)

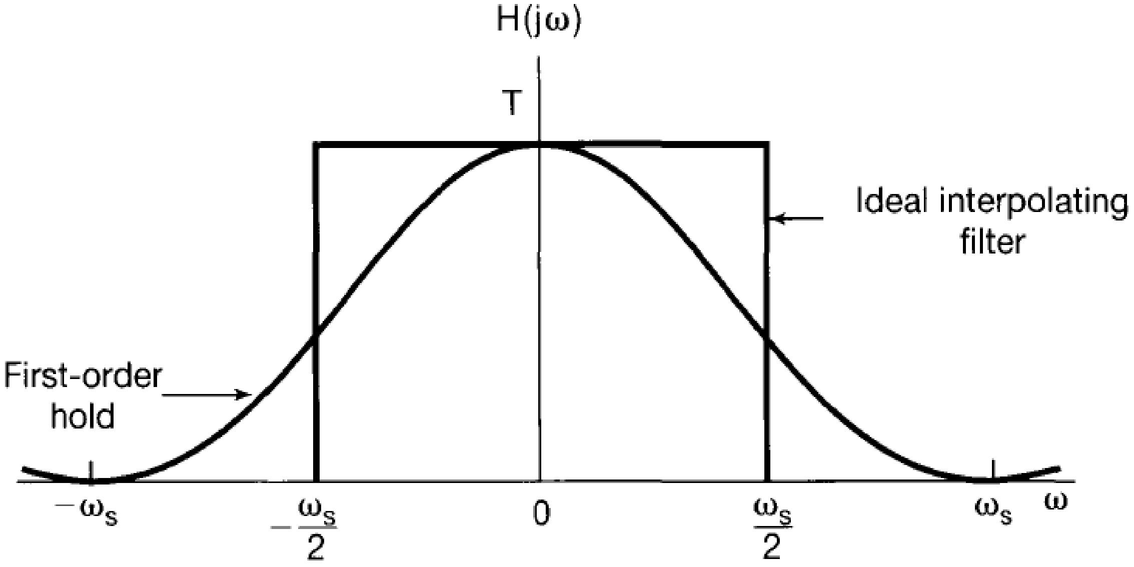
in time domain:





First-order hold (linear interpolation)

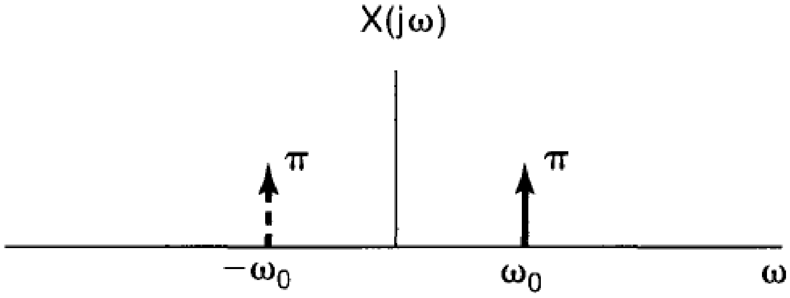
in frequency domain: $$H(j\omega) = \frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2$$

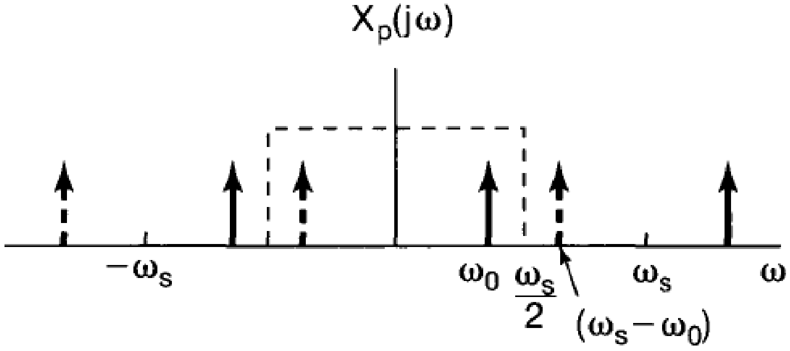


Example

Example: $x(t) = \cos\omega_0 t \;\stackrel{\mathcal{F}}{\longleftrightarrow}\; X(j\omega) = \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$

$\omega_s = 3\,\omega_0$ (frequency domain)





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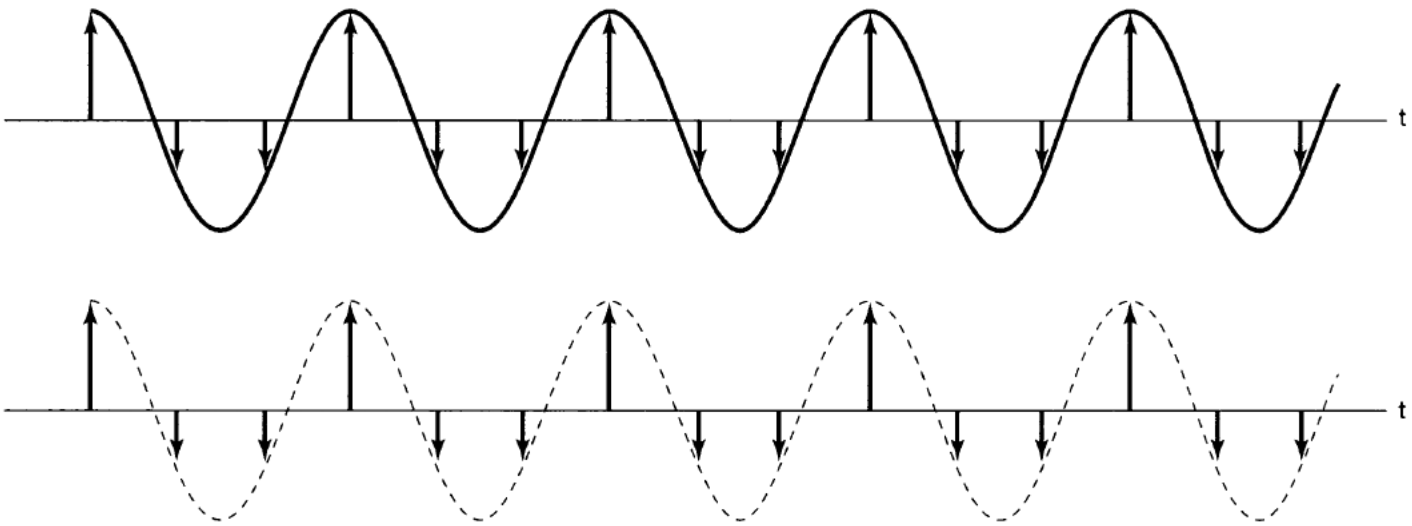

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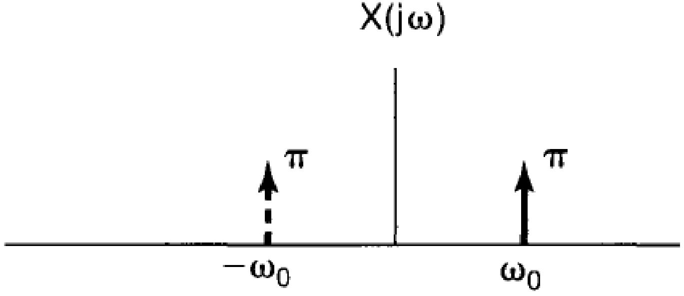

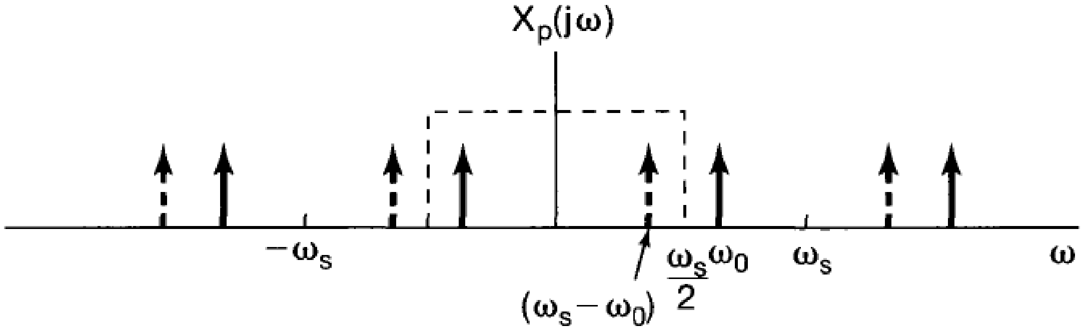

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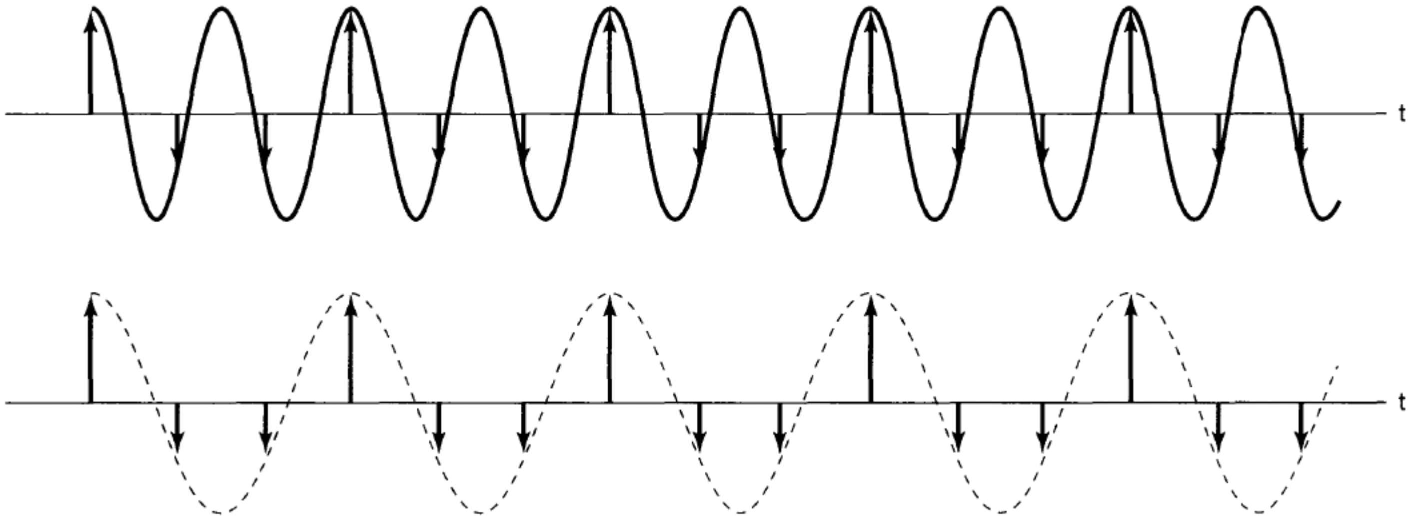

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