Signals and Systems

#10: Second-order systems

Giovanni Geraci
Universitat Pompeu Fabra, Barcelona
https://www.upf.edu/web/giovanni-geraci
giovanni.geraci@upf.edu

Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

Outline

Second-order systems

Differential equation: $$\frac{d^2\, y(t)}{d\, t^2} + 2\,\zeta\,\omega_n\,\frac{d\, y(t)}{d\, t} + \omega_n^2\, y(t) = \omega_n^2\, x(t) \qquad \zeta, \omega_n^2 > 0$$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\,\zeta\,\omega_n\,(j\omega) + \omega_n^2} = \frac{\omega_n^2}{(j\omega - c_1)\,(j\omega - c_2)} \qquad c_1, c_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Differential equation: $$\frac{d^2\,y(t)}{d\,t^2} + 2\,\zeta\,\omega_n\,\frac{d\,y(t)}{d\,t} + \omega_n^2\,y(t) = \omega_n^2\,x(t) \qquad \zeta, \omega_n^2 > 0$$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\,\zeta\,\omega_n\,(j\omega) + \omega_n^2} = \frac{\omega_n^2}{(j\omega - c_1)\,(j\omega - c_2)} \qquad c_1, c_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- For $0 < \zeta < 1$, c_1 and c_2 are complex conjugate

$$h(t) = \ldots = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]u(t) \qquad \textit{underdamped}$$

Differential equation: $$\frac{d^2\, y(t)}{d\,t^2} + 2\,\zeta\,\omega_n\,\frac{d\,y(t)}{d\,t} + \omega_n^2\,y(t) = \omega_n^2\,x(t) \qquad \zeta, \omega_n^2 > 0$$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\,\zeta\,\omega_n\,(j\omega) + \omega_n^2} = \frac{\omega_n^2}{(j\omega - c_1)\,(j\omega - c_2)} \qquad c_1, c_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- For $0 < \zeta < 1$, c_1 and c_2 are complex conjugate

$$h(t) = \ldots = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]u(t) \qquad \textit{underdamped}$$

- For $\zeta > 1$, c_1 and c_2 are real negative

$$h(t) = \ldots = \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[e^{c_1 t} - e^{c_2 t}\right]u(t) \qquad \textit{overdamped}$$

Differential equation: $$\frac{d^2\,y(t)}{d\,t^2} + 2\,\zeta\,\omega_n\,\frac{d\,y(t)}{d\,t} + \omega_n^2\,y(t) = \omega_n^2\,x(t) \qquad \zeta, \omega_n^2 > 0$$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\,\zeta\,\omega_n\,(j\omega) + \omega_n^2} = \frac{\omega_n^2}{(j\omega - c_1)\,(j\omega - c_2)} \qquad c_1, c_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- For $0 < \zeta < 1$, c_1 and c_2 are complex conjugate

$$h(t) = \ldots = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]u(t) \qquad \textit{underdamped}$$

- For $\zeta > 1$, c_1 and c_2 are real negative

$$h(t) = \ldots = \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[e^{c_1 t} - e^{c_2 t}\right]u(t) \qquad \textit{overdamped}$$

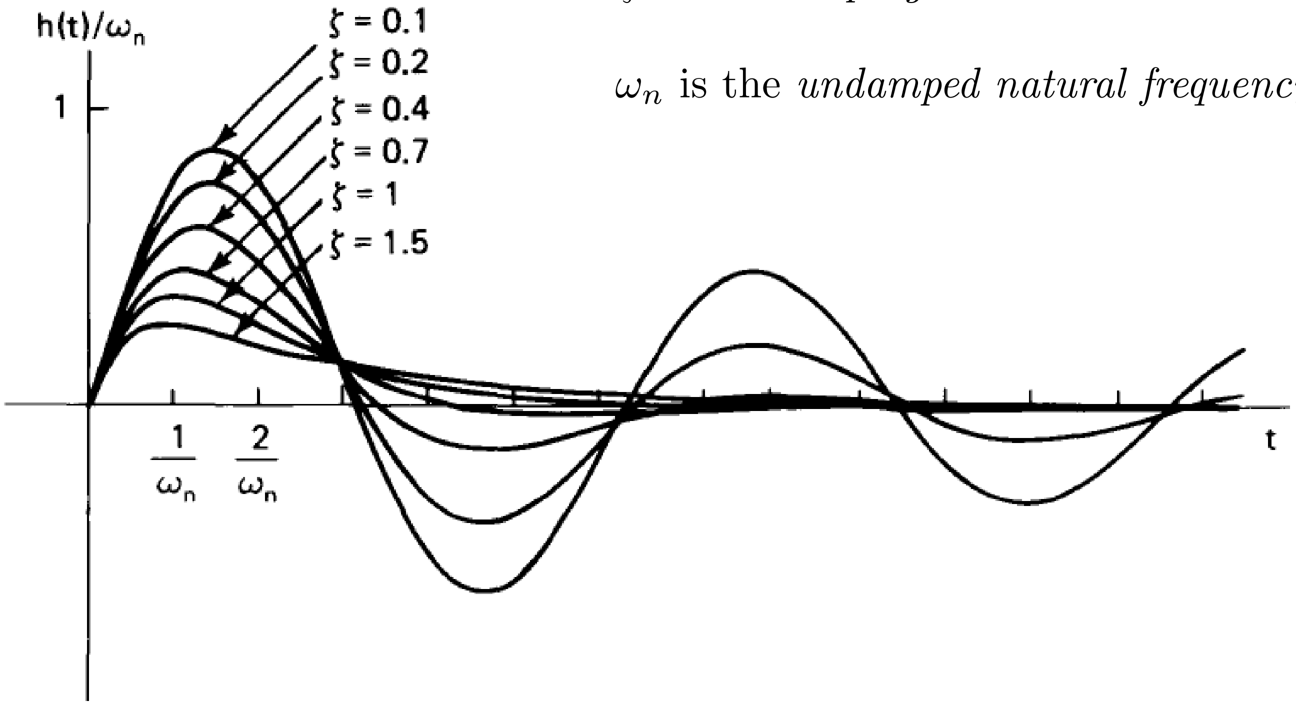
- For $\zeta = 1$, $c_1 = c_2 = -\omega_n$ real negative

$$h(t) = \ldots = \omega_n^2\,t\,e^{-\omega_n t}\,u(t) \qquad \textit{critically damped}$$

Impulse response

ζ is the *damping ratio*

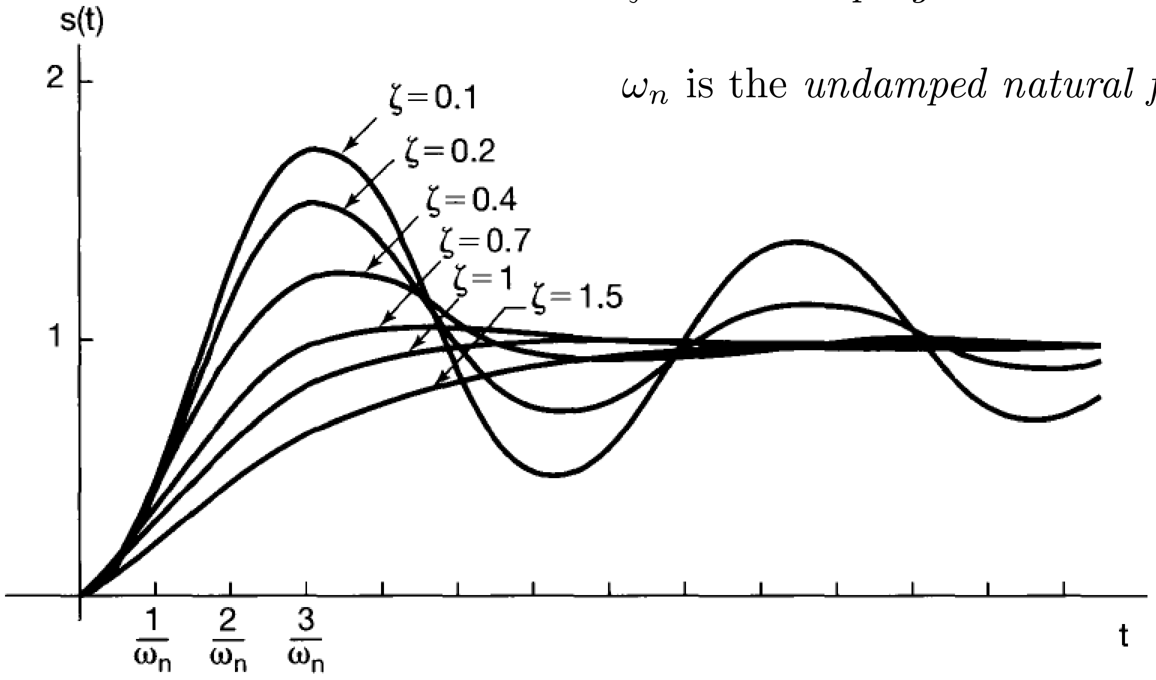
ω_n is the *undamped natural frequency*



Step response

ζ is the *damping ratio*

ω_n is the *undamped natural frequency*



Magnitude of the frequency response:

$$20\log_{10}|H(j\omega)| = -10\log_{10}\left\{\left[1-(\omega/\omega_n)^2\right]^2 + 4\zeta\,(\omega/\omega_n)^2\right\}$$

- For $\omega \ll \omega_n$ $\quad 20\log_{10}|H(j\omega)| \approx 0$
- For $\omega \gg \omega_n$ $\quad 20\log_{10}|H(j\omega)| \approx -40\log_{10}\omega + 40\log_{10}\omega_n$

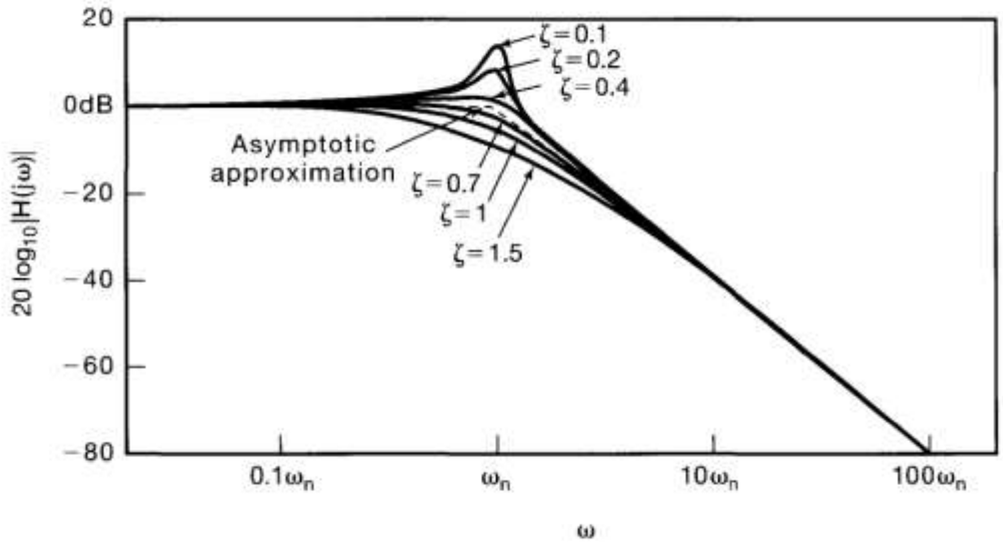
Magnitude of the frequency response:

$$20\log_{10}|H(j\omega)| = -10\log_{10}\left\{\left[1-(\omega/\omega_n)^2\right]^2 + 4\zeta\,(\omega/\omega_n)^2\right\}$$

- For $\omega \ll \omega_n$ $\quad 20\log_{10}|H(j\omega)| \approx 0$
- For $\omega \gg \omega_n$ $\quad 20\log_{10}|H(j\omega)| \approx -40\log_{10}\omega + 40\log_{10}\omega_n$

$\omega = \omega_n$ is the *break frequency*

approximation does not depend on ζ
actual plot does



Phase of the frequency response:

$$\sphericalangle H(j\omega) = -\tan^{-1}\left[\frac{2\,\zeta\,(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right] \approx \begin{cases} 0 & \omega \ll \omega_n \ \ (\text{e.g., } \omega < 0.1\,\omega_n) \\ -\pi & \omega \gg \omega_n \ \ (\text{e.g., } \omega > 10\,\omega_n) \end{cases}$$

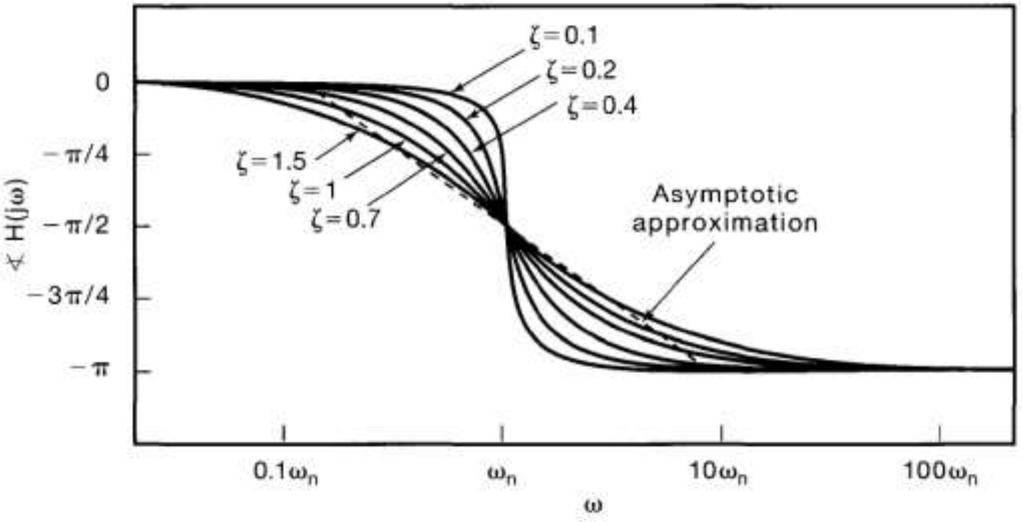
Phase of the frequency response:

$$\sphericalangle H(j\omega) = -\tan^{-1}\left[\frac{2\,\zeta\,(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right] \approx \begin{cases} 0 & \omega \ll \omega_n \;\; (\text{e.g., } \omega < 0.1\,\omega_n) \\ -\pi & \omega \gg \omega_n \;\; (\text{e.g., } \omega > 10\,\omega_n) \end{cases}$$

between $0.1\,\omega_n < \omega < 10\,\omega_n$
we can use a linear approximation

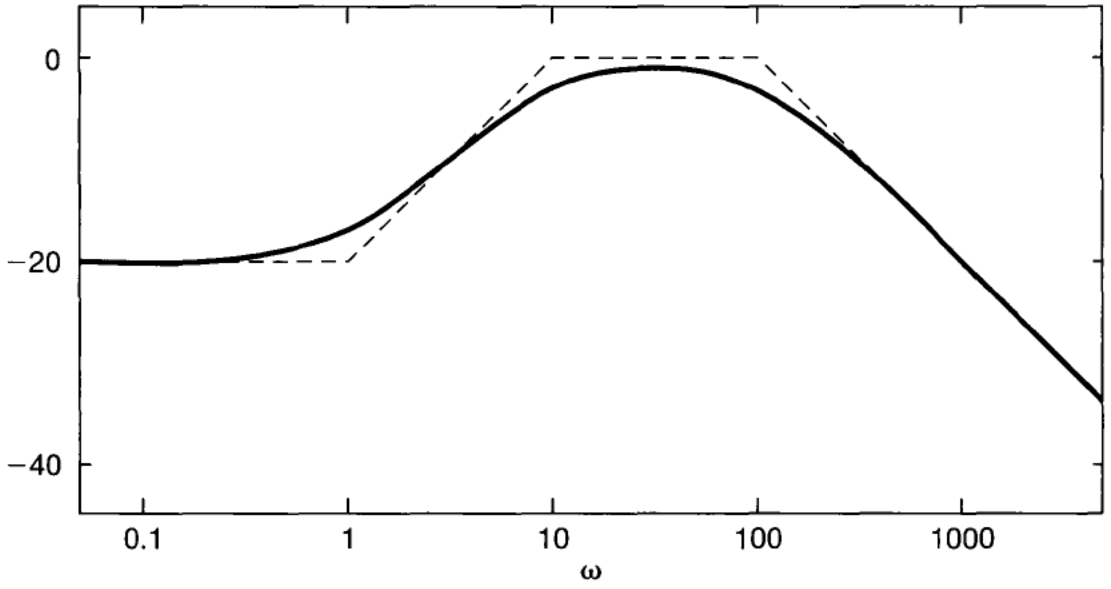
the exact value at $\omega = \omega_n$ is:
$\sphericalangle H(j\omega) = -\pi/2$

approximation does not depend on ζ
actual plot does

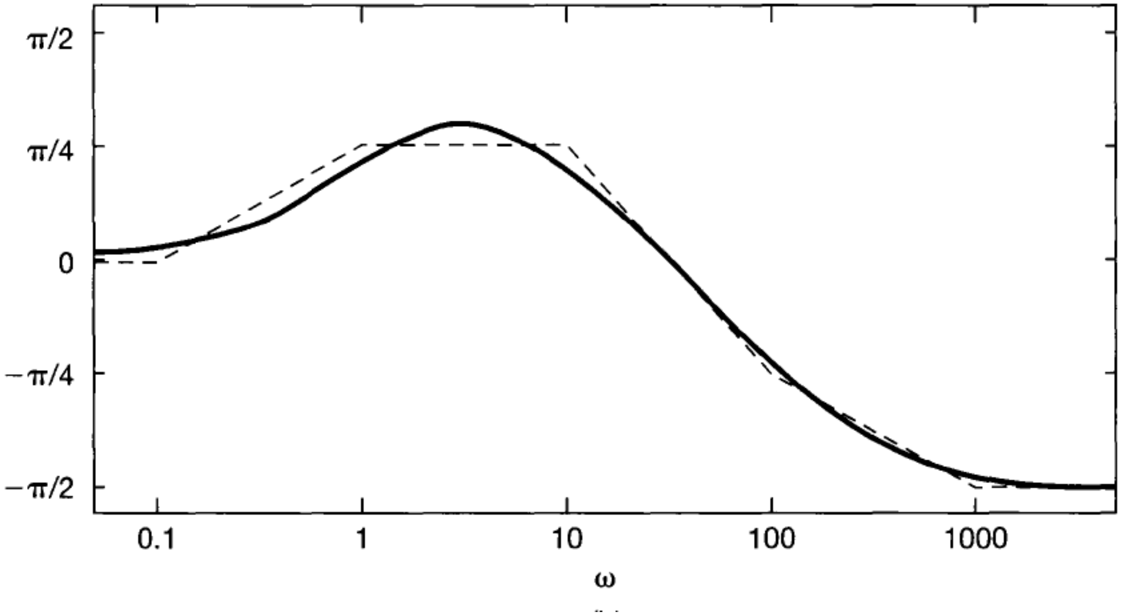


Example

$$H(j\omega) = \frac{100\,(1+j\omega)}{(10+j\omega)\,(100+j\omega)} = \frac{1}{10} \cdot \frac{1}{1+j\omega/10} \cdot \frac{1}{1+j\omega/100} \cdot (1+j\omega)$$



$$H(j\omega) = \frac{100\,(1+j\omega)}{(10+j\omega)\,(100+j\omega)} = \frac{1}{10}\cdot\frac{1}{1+j\omega/10}\cdot\frac{1}{1+j\omega/100}\cdot(1+j\omega)$$



Outline

- …
- #06 Basic properties of the Fourier transform
- #07 Convolution property
- #08 Multiplication property
- #09 First-order systems
- **#10 Second-order systems**
- #11 Sampling and reconstruction