Signals and Systems

#09: First-order systems

Giovanni Geraci
Universitat Pompeu Fabra, Barcelona
https://www.upf.edu/web/giovanni-geraci
giovanni.geraci@upf.edu

Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

Outline

- #01 Continuous-time signals
- #02 Continuous-time systems
- #03 Fourier series of periodic signals
- #04 Fourier transform of aperiodic signals
- #05 Fourier transform of periodic signals
- #06 Basic properties of the Fourier transform
- #07 Convolution property
- #08 Multiplication property
- **#09 First-order systems**
- #10 Second-order systems

Magnitude-phase representation of the frequency response

The Fourier transform is a complex function of ω

$$X(j\omega) = |X(j\omega)|\, e^{j\sphericalangle X(j\omega)}$$

$|X(j\omega)|$ describes the frequency content of a signal

$\sphericalangle X(j\omega)$ describes the relative phase of the complex exponentials

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For an LTI system: $\quad Y(j\omega) = H(j\omega) \cdot X(j\omega)$

$|H(j\omega)|$ scales the magnitude of $X(j\omega)$

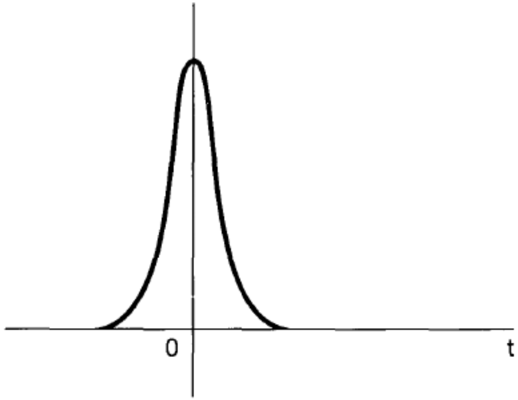
$\sphericalangle H(j\omega)$ introduces a phase shift on $X(j\omega)$

A linear phase shift corresponds to a delay (no distortion)

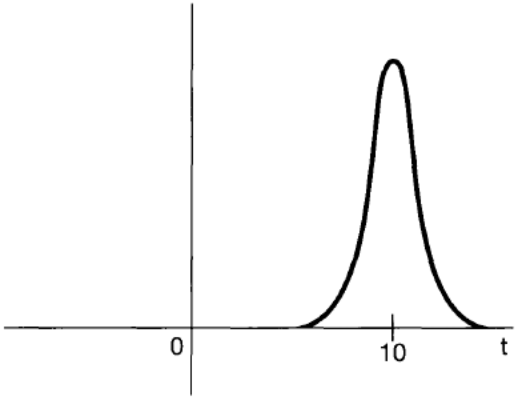
$$H(j\omega) = e^{-j\omega t_0}, \qquad |H(j\omega)| = 1 \qquad \sphericalangle H(j\omega) = -\omega\, t_0$$

$$h(t) = \delta(t - t_0) \quad \Rightarrow \quad y(t) = x(t - t_0)$$

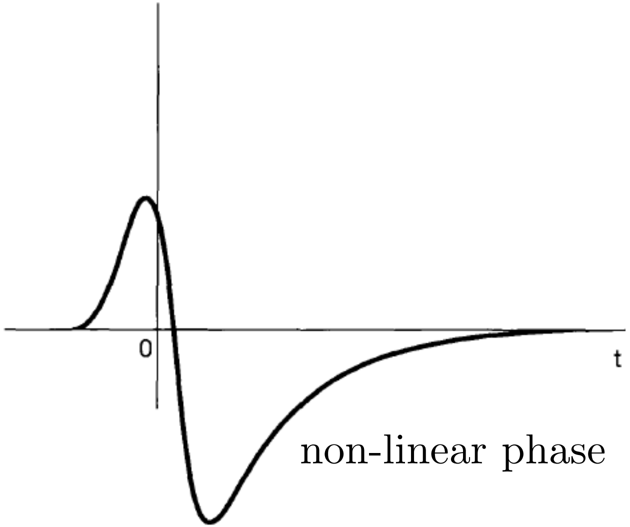
a non-linear phase causes distortion



original



linear phase



non-linear phase

Bode plots

Plotting the magnitude of the Fourier transform in logarithmic scale:

- spans a wider range and highlights value in the stopband
- one can simply obtain $\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$

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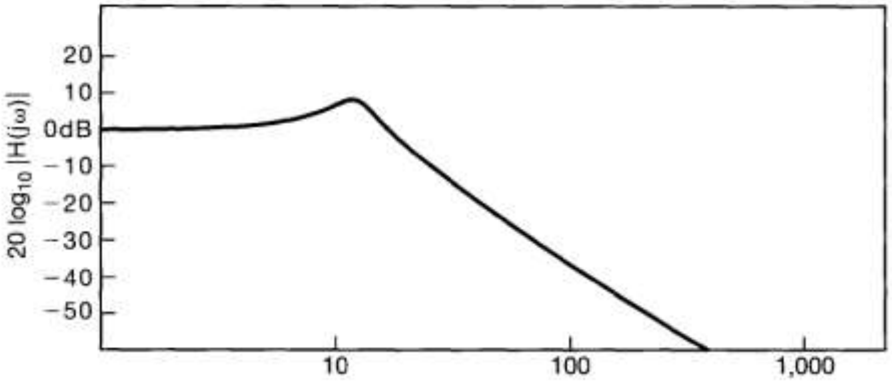
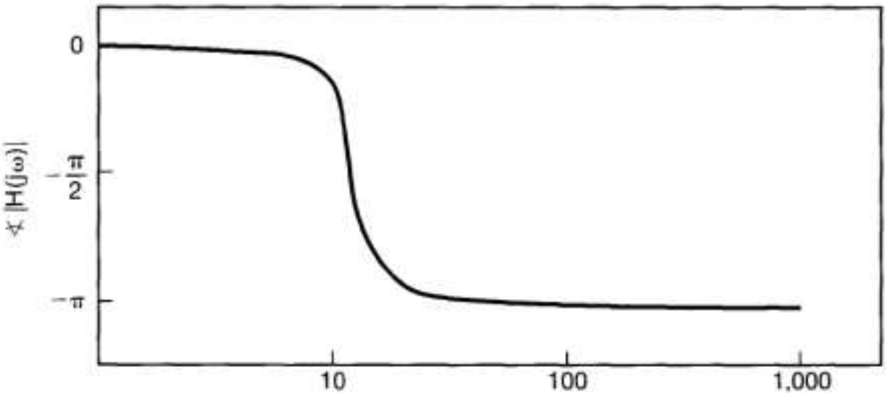
We typically use a $20\log_{10}$ scale (*decibels* or dB) $\Rightarrow \begin{cases} 1\times = 0\text{dB} \\ 10\times = 20\text{dB} \\ 0.1\times = -20\text{dB} \\ 2\times \approx 6\text{dB} \end{cases}$

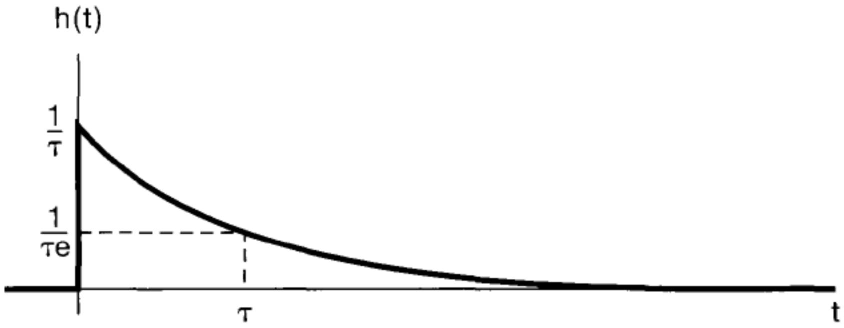




First-order systems

Differential equation: $$\tau \frac{d\,y(t)}{d\,t} + y(t) = x(t) \qquad \tau > 0$$

frequency and impulse response: $$H(j\omega) = \frac{1}{j\omega\tau + 1} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad h(t) = \frac{1}{\tau}\, e^{-t/\tau}\, u(t)$$



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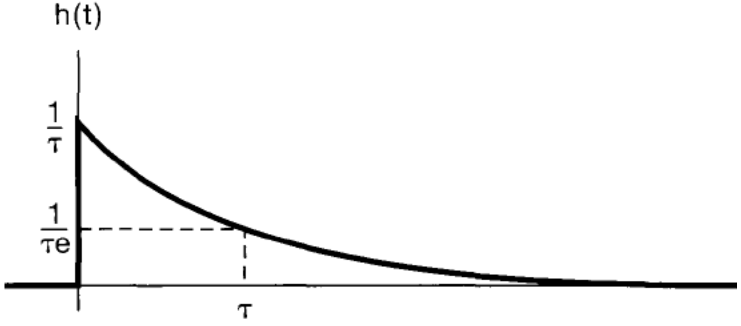
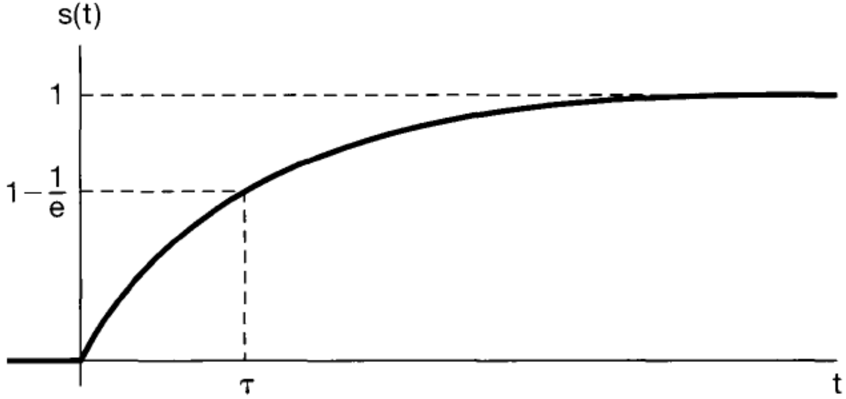
step response: $s(t) = h(t) * u(t) = \left(1 - e^{-t/\tau}\right)\, u(t)$





smaller $\tau \quad \Rightarrow \quad$ faster response

Magnitude of the frequency response:

$$20\log_{10}|H(j\omega)| = 20\log_{10}\left|\frac{1}{j\omega\tau+1}\right| = 20\log_{10}\left[\frac{1}{\sqrt{1+(\omega\tau)^2}}\right] = -10\log_{10}\left[1+(\omega\tau)^2\right]$$

- For $\omega \ll \frac{1}{\tau}$ $\quad 20\log_{10}|H(j\omega)| \approx 0$
- For $\omega \gg \frac{1}{\tau}$ $\quad 20\log_{10}|H(j\omega)| \approx -20\log_{10}\omega\tau = -20\log_{10}\omega - 20\log_{10}\tau$

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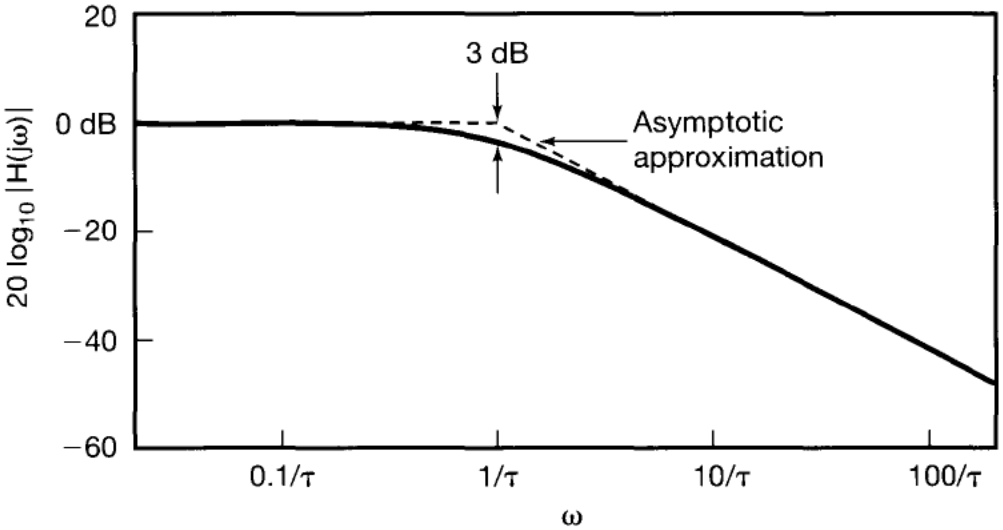
$\omega = \frac{1}{\tau}$ is the *break frequency*

the exact value at $\omega = \frac{1}{\tau}$ is:

$$-10\log_{10}[1+1] \approx -3\text{dB}$$

smaller $\tau \Rightarrow$ larger break frequency



Phase of the frequency response:

$$\sphericalangle H(j\omega) = -\tan^{-1}\omega\tau \approx \begin{cases} 0 & \omega \ll \frac{1}{\tau} \;\; (\text{e.g., } \omega < \frac{0.1}{\tau}) \\ -\pi/2 & \omega \gg \frac{1}{\tau} \;\; (\text{e.g., } \omega > \frac{10}{\tau}) \end{cases}$$

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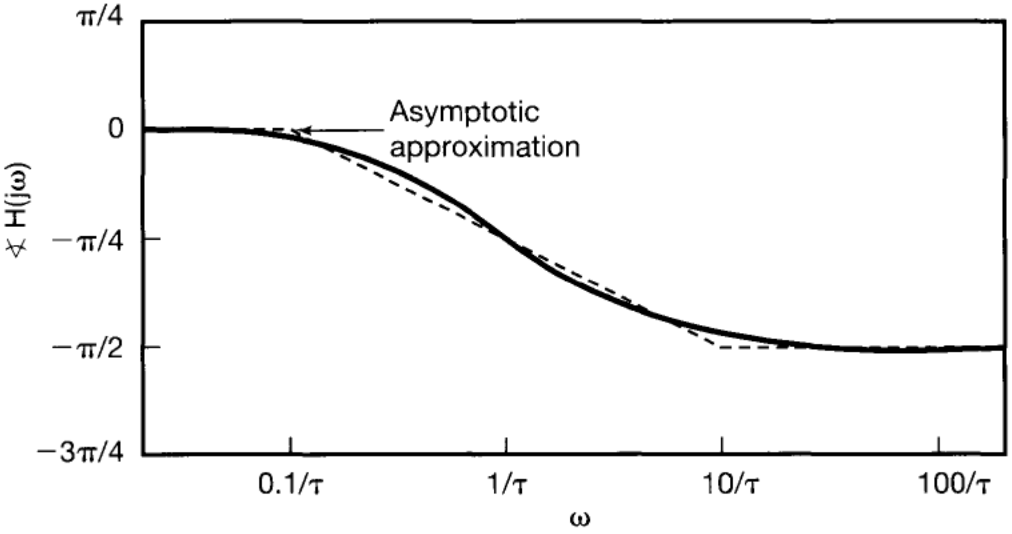
between $\frac{0.1}{\tau} < \omega < \frac{10}{\tau}$
we can use a linear approximation

the exact value at $\omega = \frac{1}{\tau}$ is:
$\sphericalangle H(j\omega) = -\pi/4$



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