Signals and Systems

#08: Multiplication property

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Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

Outline

- #01 Continuous-time signals
- #02 Continuous-time systems
- #03 Fourier series of periodic signals
- #04 Fourier transform of aperiodic signals
- #05 Fourier transform of periodic signals
- #06 Basic properties of the Fourier transform
- #07 Convolution property
- **#08 Multiplication property**
- #09 First-order systems

Multiplication property

We saw that

$$y(t) = h(t) * x(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

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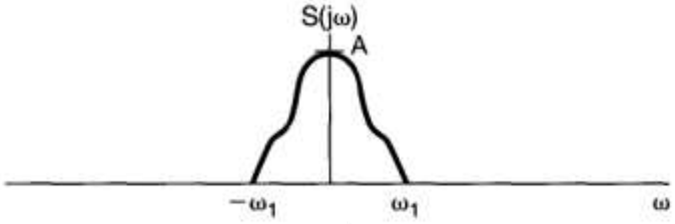
The converse holds

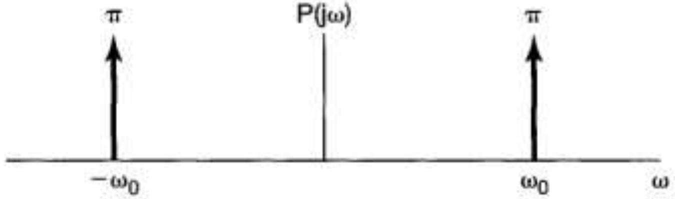
$$r(t) = s(t) \cdot p(t) \overset{\mathcal{F}}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi} S(j\omega) * P(j\omega) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta) P(j(\omega - \theta)) d\theta$$

Examples

Example: modulation



$$p(t) = \cos \omega_0 t$$
$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



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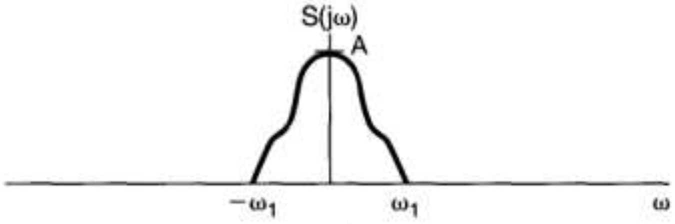

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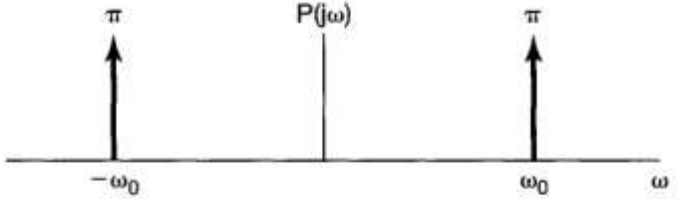
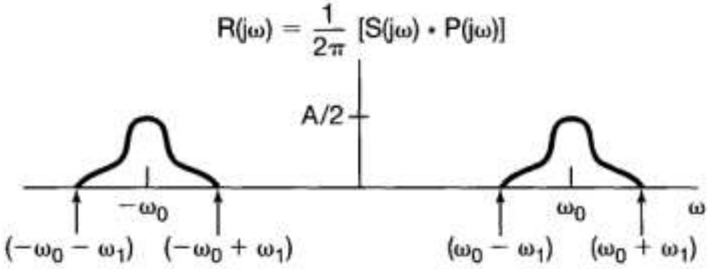


$$r(t) = s(t) \cdot p(t)$$

$$R(j\omega) = \frac{1}{2\pi} S(j\omega) * P(j\omega)$$

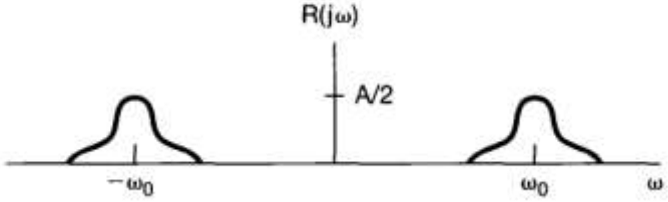
$$= \frac{1}{2} S(j(\omega - \omega_0)) + \frac{1}{2} S(j(\omega + \omega_0))$$



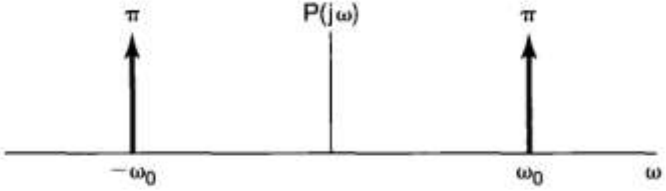
information preserved but shifted at higher frequencies ($\omega_0 > \omega_1$)

Example: demodulation

take the same $r(t)$



multiply again by $p(t)$



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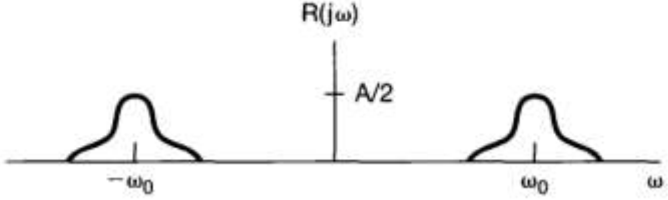

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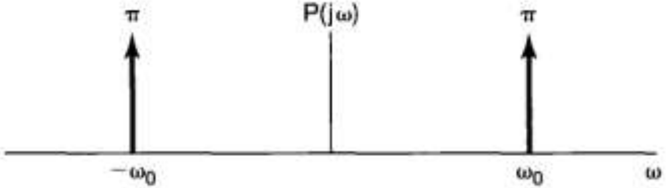
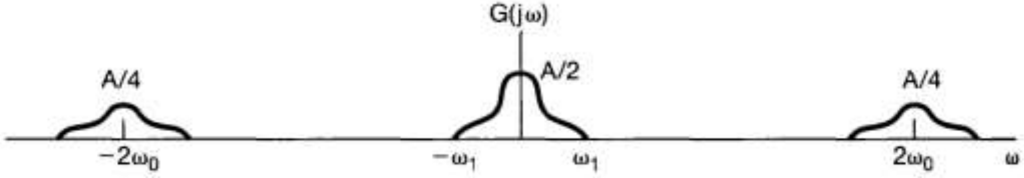


$g(t) = r(t) \cdot p(t)$

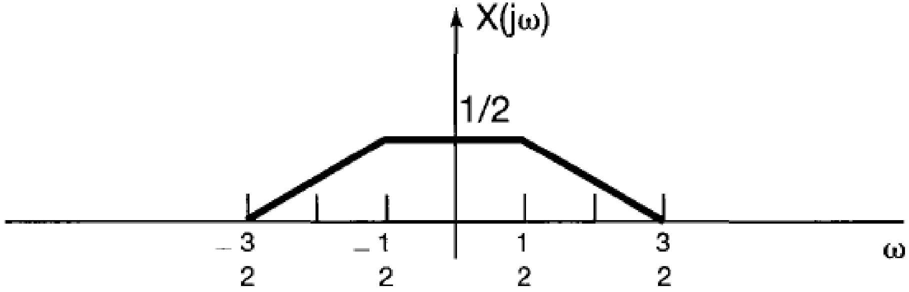


we can recover $s(t)$ with a lowpass filter

Example: multiplication property

$$x(t) = \frac{\sin(t)\,\sin(t/2)}{\pi\, t^2} = \pi \left(\frac{\sin(t)}{\pi\, t}\right) \left(\frac{\sin(t/2)}{\pi\, t}\right)$$

$$X(j\omega) = \frac{1}{2}\,\mathcal{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathcal{F}\left\{\frac{\sin(t/2)}{\pi t}\right\}$$



Systems characterized by linear constant-coefficient differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{d t^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{d t^k}$$

To determine the frequency response $H(j\omega)$ we Fourier transform both sides

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$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

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$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

ratio of polynomials

Examples

Example: first-order system

$$\frac{d\,y(t)}{d\,t} + a\,y(t) = x(t) \qquad a > 0$$

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$$j\omega\,Y(j\omega) + a\,Y(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{j\omega + a} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad h(t) = e^{-at}\,u(t)$$

Example: second-order system

$$\frac{d^2\,y(t)}{d\,t^2} + 4\,\frac{d\,y(t)}{d\,t} + 3\,y(t) = \frac{d\,x(t)}{d\,t} + 2\,x(t)$$

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$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3} = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} = \frac{1/2}{j\omega + 1} + \frac{1/2}{j\omega + 3}$$

$$h(t) = \frac{1}{2}e^{-t}\,u(t) + \frac{1}{2}e^{-3t}\,u(t)$$

now assume $\quad x(t) = e^{-t}u(t), \quad$ what is $y(t)$?

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$$Y(j\omega) = H(j\omega) \cdot X(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} \cdot \frac{1}{j\omega + 1}$$

$$= \frac{j\omega + 2}{(j\omega + 1)^2\,(j\omega + 3)} = \ldots = \frac{1/4}{j\omega + 1} + \frac{1/2}{(j\omega + 1)^2} - \frac{1/4}{j\omega + 3}$$

$$y(t) = \frac{1}{4}\, e^{-t}\, u(t) + \frac{1}{2}\, t\, e^{-t}\, u(t) - \frac{1}{4}\, e^{-3t}\, u(t)$$

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