# Signals and Systems

## #07: Convolution property

**Giovanni Geraci**

Universitat Pompeu Fabra, Barcelona

https://www.upf.edu/web/giovanni-geraci

giovanni.geraci@upf.edu

*Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.*

# Outline

- #01 Continuous-time signals
- #02 Continuous-time systems
- #03 Fourier series of periodic signals
- #04 Fourier transform of aperiodic signals
- #05 Fourier transform of periodic signals
- #06 Basic properties of the Fourier transform
- **#07 Convolution property**
- #08 Multiplication property

# Convolution property

Recall $\quad h(t) * x(t) \triangleq \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

We have

$$y(t) = h(t) * x(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

$H(j\omega) = \mathcal{F}\left\{h(t)\right\}$ is the **frequency response**

Recall $\quad h(t) * x(t) \triangleq \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

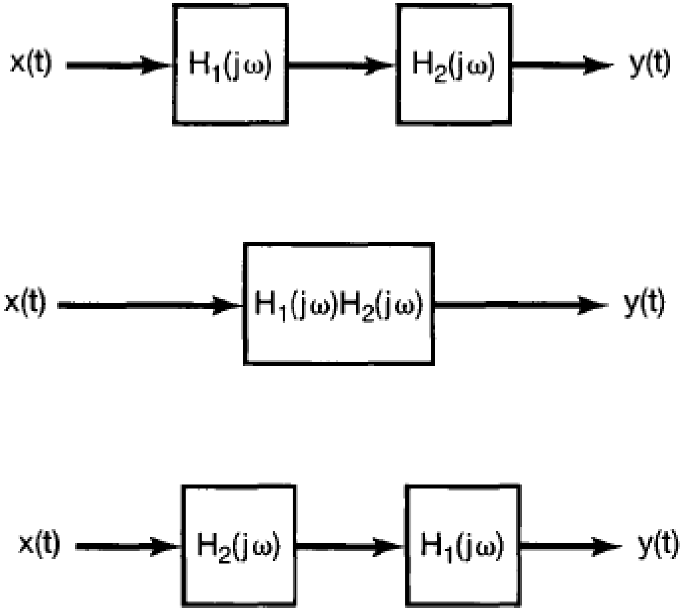
We have

$$y(t) = h(t) * x(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

[Proof:]

$$\begin{aligned} Y(j\omega) = \mathcal{F}\left\{y(t)\right\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right] e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right] d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\, H(j\omega)\, e^{-j\omega\tau} d\tau = H(j\omega) \cdot X(j\omega) \end{aligned}$$

# Cascade of systems



# Examples

Example: time-shift system

$$h(t) = \delta(t - t_0) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H(j\omega) = e^{-j\omega t_0}$$

For an input $x(t)$ with $\mathcal{F}\{x(t)\} = X(j\omega)$

the output is

$$y(t) = x(t) * \delta(t - t_0) = x(t - t_0) \qquad \text{with}$$

$$\mathcal{F}\{y(t)\} = Y(j\omega) = H(j\omega) \cdot X(j\omega) = X(j\omega)\, e^{-j\omega t_0}$$

Example: differentiator

$$y(t) = \frac{dx(t)}{dt}$$

therefore

$$Y(j\omega) = j\omega\, X(j\omega) \qquad \text{and}$$

$$H(j\omega) = j\omega$$

Example: integrator

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$
$$= x(t) * u(t)$$

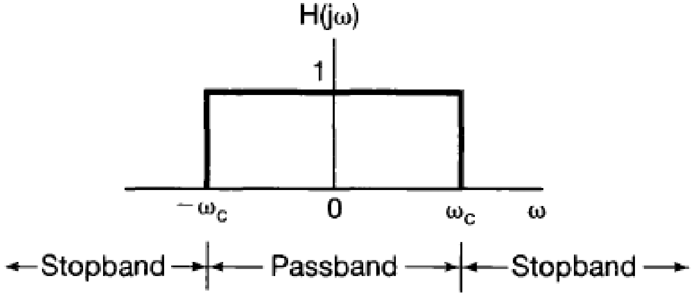
therefore

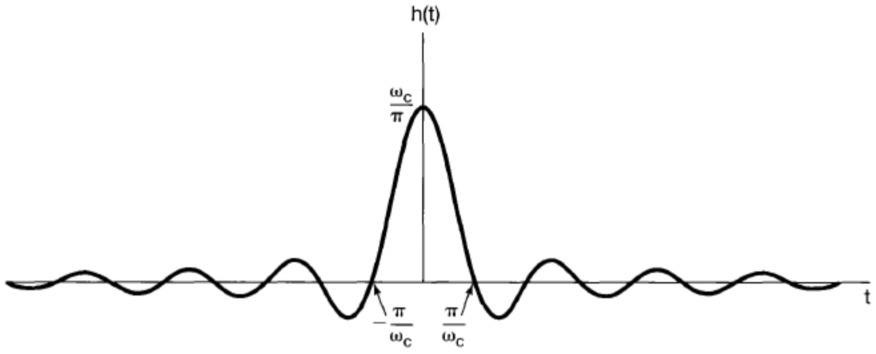
$$H(j\omega) = \mathcal{F}\{u(t)\} = \frac{1}{j\omega} + \pi\,\delta(\omega) \qquad \text{and}$$
$$Y(j\omega) = H(j\omega) \cdot X(j\omega) = \frac{1}{j\omega} X(j\omega) + \pi\, X(0)\, \delta(\omega)$$

Example: frequency-selective filter (ideal)

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad h(t) = \frac{\sin \omega_c t}{\pi t}$$

perfect frequency selectivity $\Rightarrow$ non-causal filter with oscillations

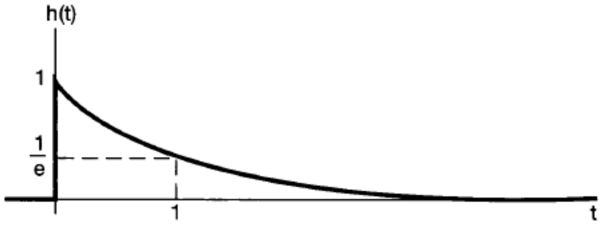


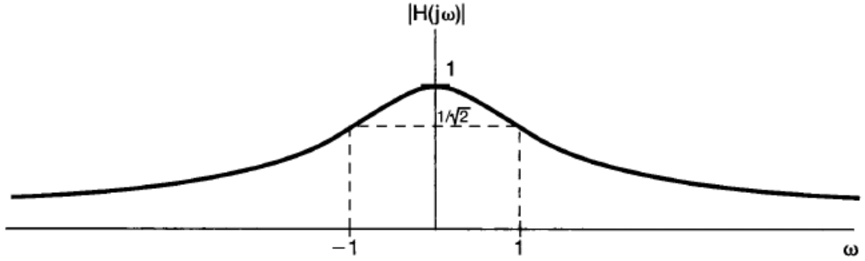



Example: frequency-selective filter (smooth)

$$h(t) = e^{-t}\, u(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H(j\omega) = \frac{1}{j\omega + 1}$$

causal without oscillations $\Rightarrow$ weaker frequency selectivity



Example: evaluating convolution integral

$$h(t) = e^{-at}\,u(t), \quad a > 0$$
$$x(t) = e^{-bt}\,u(t), \quad b > 0$$

$$y(t) = h(t) * x(t) \;\; ?$$

$$H(j\omega) = \frac{1}{a + j\omega}$$
$$X(j\omega) = \frac{1}{b + j\omega}$$

$$Y(j\omega) = H(j\omega) \cdot X(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)} = \ldots$$

For $b \neq a$

$$Y(j\omega) = \frac{1}{(a+j\omega)(b+j\omega)} = \frac{A}{a+j\omega} + \frac{B}{b+j\omega} = \dots$$

$$A\,b + A\,j\omega + B\,a + B\,j\omega = 1 \quad \Rightarrow \quad \begin{cases} A = \frac{1}{b-a} \\ B = \frac{1}{a-b} \end{cases}$$

$$\dots = \frac{1}{b-a}\left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad y(t) = \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]u(t)$$

For $b = a$

$$Y(j\omega) = \frac{1}{(a + j\omega)^2} = \ldots$$

recall

$$e^{-at}\, u(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1}{a + j\omega}$$

$$t\, e^{-at}\, u(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad j\, \frac{d}{d\omega} \left[ \frac{1}{a + j\omega} \right] = \frac{1}{(a + j\omega)^2}$$

$$\ldots = j\, \frac{d}{d\omega} \left[ \frac{1}{a + j\omega} \right] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad y(t) = t\, e^{-at}\, u(t)$$

# Outline

- #01 Continuous-time signals
- #02 Continuous-time systems
- #03 Fourier series of periodic signals
- #04 Fourier transform of aperiodic signals
- #05 Fourier transform of periodic signals
- #06 Basic properties of the Fourier transform
- **#07 Convolution property**
- #08 Multiplication property