Signals and Systems

#06: Basic properties of the Fourier transform

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Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

Outline

- #01 Continuous-time signals
- #02 Continuous-time systems
- #03 Fourier series of periodic signals
- #04 Fourier transform of aperiodic signals
- #05 Fourier transform of periodic signals
- **#06 Basic properties of the Fourier transform**
- #07 Convolution and multiplication

Linearity and time shifting

Recall the Fourier transform pair:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Linearity:

$$\text{if} \quad \begin{cases} x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega) \\ \\ y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(j\omega) \end{cases}$$

then

$$a\,x(t) + b\,y(t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad a\,X(j\omega) \; + \; b\,Y(j\omega)$$

Time shifting:

$$\begin{array}{lrcl} \text{if} & x(t) & \overset{\mathcal{F}}{\longleftrightarrow} & X(j\omega) \\ \text{then} & x(t-t_0) & \overset{\mathcal{F}}{\longleftrightarrow} & X(j\omega)\, e^{-j\omega t_0} \end{array}$$

magnitude not altered
phase shift linear in ω

Time shifting:

$$\text{if} \qquad x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\text{then} \qquad x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)\, e^{-j\omega t_0}$$

[Proof:]

$$\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt \qquad \text{replace } t-t_0 \text{ by } \tau$$

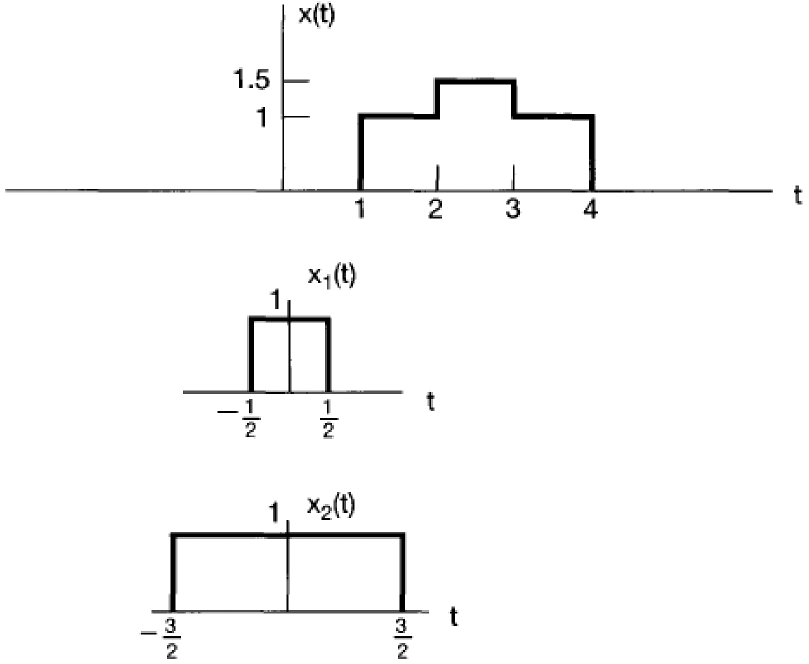
$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau+t_0)} d\tau = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$$

$$= e^{-j\omega t_0} \mathcal{F}\{x(t)\}$$

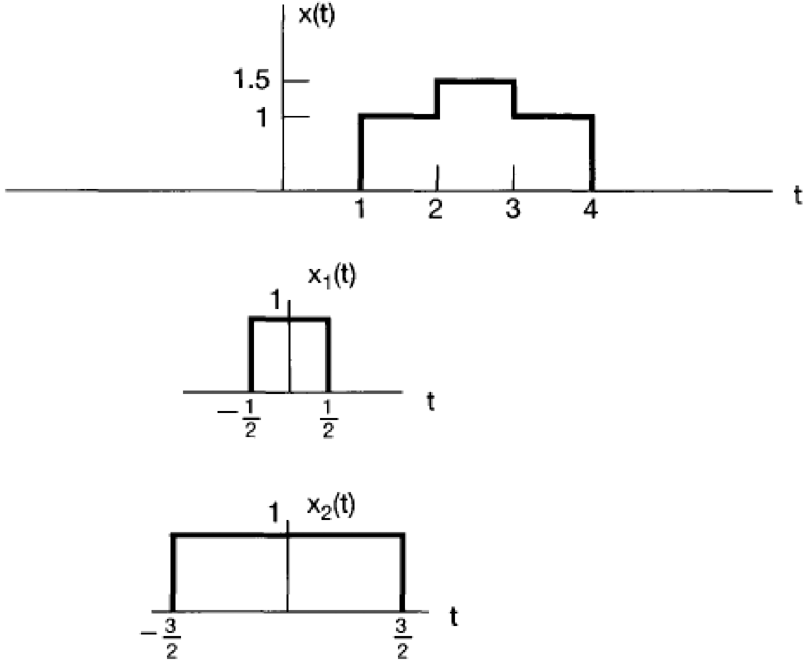
Example

Example: linearity and time shift

$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$



Example: linearity and time shift



$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$

Recall:

$$X_1(j\omega) = \mathcal{F}\{x_1(t)\} = \frac{2\sin(\frac{\omega}{2})}{\omega}$$

$$X_2(j\omega) = \mathcal{F}\{x_2(t)\} = \frac{2\sin(\frac{3\omega}{2})}{\omega}$$

then:

$$X(j\omega) = \mathcal{F}\{x(t)\} = e^{-j\frac{5}{2}\omega}\left[\frac{\sin(\frac{\omega}{2}) + 2\sin(\frac{3\omega}{2})}{\omega}\right]$$

Conjugation, differentiation, and integration

Conjugation:

$$\text{if} \qquad x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\text{then} \qquad x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega)$$

Conjugation:

$$\text{if} \qquad x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\text{then} \qquad x^*(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega)$$

If $x(t)$ is real $\Rightarrow \quad X(-j\omega) = X^*(j\omega)$ (conjugate symmetry)

Conjugation:

$$\text{if} \qquad x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\text{then} \qquad x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega)$$

[Proof:]

$$X^*(j\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]^* = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt \qquad \text{replace } \omega \text{ by } -\omega$$

$$X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt = \mathcal{F}\left\{x^*(t)\right\}$$

Differentiation:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega\, X(j\omega)$$

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[Proof:]

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{differentiate both sides}$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, j\omega\, e^{j\omega t}d\omega = \mathcal{F}^{-1}\left\{j\omega X(j\omega)\right\}$$

useful to study systems described by differential equations

Differentiation:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega\, X(j\omega)$$

Integration:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}\, X(j\omega) + \pi\, X(0)\, \delta(\omega)$$

Examples

Example: $x(t) = u(t)$

$$\text{we know that} \quad g(t) = \delta(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad G(j\omega) = 1$$

$$\text{and that} \quad x(t) = \int_{-\infty}^{t} g(\tau)\, d\tau$$

therefore

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi\, G(0)\, \delta(\omega) = \frac{1}{j\omega} + \pi\, \delta(\omega)$$

Example: $x(t) = u(t)$

$$\text{we know that} \quad g(t) = \delta(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad G(j\omega) = 1$$

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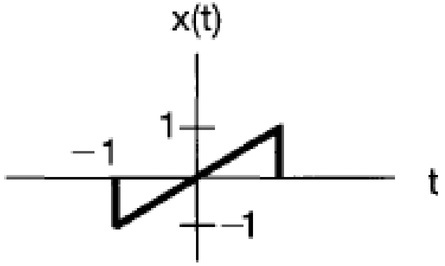
therefore

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi\, G(0)\, \delta(\omega) = \frac{1}{j\omega} + \pi\, \delta(\omega)$$

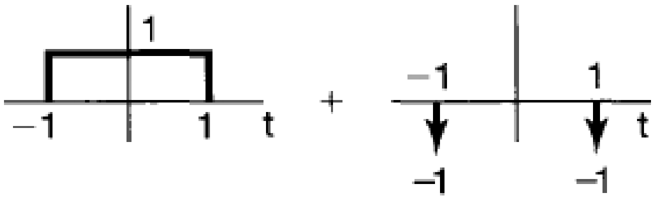
if we differentiate, we recover

$$\delta(t) = \frac{d\, u(t)}{dt} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad j\omega \left[\frac{1}{j\omega} + \pi\, \delta(\omega) \right] = 1$$

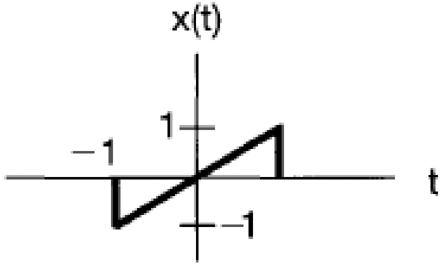
Example:



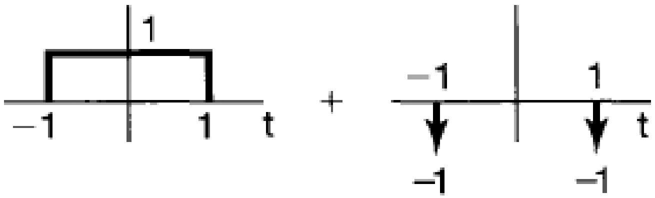
$$g(t) = \frac{d\,x(t)}{dt}$$ can be expressed as



Example:



$g(t) = \dfrac{d\,x(t)}{dt}$ can be expressed as



we have

$$G(j\omega) = \frac{2\sin\omega}{\omega} - e^{j\omega} - e^{-j\omega} \quad \text{and} \quad G(0) = 0$$

therefore

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi\, G(0)\, \delta(\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

Time/frequency duality

Time and frequency scaling:

$$\begin{aligned} x(t) &\xleftrightarrow{\mathcal{F}} X(j\omega) \\ x(at) &\xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right) \quad \text{with } a \text{ nonzero real} \end{aligned}$$

compression in time $\Rightarrow$ expansion in frequency

and vice versa

Time and frequency scaling:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right) \quad \text{with } a \text{ nonzero real}$$

[Proof:]

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt \qquad \text{replace } \tau = at$$

$$= \begin{cases} \dfrac{1}{a} \displaystyle\int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega}{a}\tau} d\tau & a > 0 \\ -\dfrac{1}{a} \displaystyle\int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega}{a}\tau} d\tau & a < 0 \end{cases}$$

Time shifting, differentiation, and integration in the frequency domain:

$$-j\,t\,x(t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \frac{dX(j\omega)}{d\omega}$$

$$e^{j\omega_0 t}\,x(t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad X\left(j(\omega-\omega_0)\right)$$

$$-\frac{1}{jt}\,x(t) + \pi\,x(0)\,\delta(t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \int_{-\infty}^{\omega} X(j\eta)\,d\eta$$

Parseval's relation:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

the total energy can be computed over time or frequency

Parseval's relation:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

[Proof:]

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t) \, x^*(t) dt = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

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