Signals and Systems

#05: Fourier transform of periodic signals

Giovanni Geraci
Universitat Pompeu Fabra, Barcelona
https://www.upf.edu/web/giovanni-geraci
giovanni.geraci@upf.edu

Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

Outline

- #01 Continuous-time signals
- #02 Continuous-time systems
- #03 Fourier series of periodic signals
- #04 Fourier transform of aperiodic signals
- **#05 Fourier transform of periodic signals**
- #06 Basic properties of the Fourier transform

Fourier transform of periodic signals

Recall:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier transform pair

representation of $x(t)$ as a linear combination of complex exponentials at different frequencies

Convergence if $x(t)$ has finite energy, i.e., $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$

This does not hold for periodic signals!

We can obtain the Fourier transform of periodic signals by using impulses

Note that for $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

Recall the property: $\delta(\omega - \omega_0)\,X(j\omega) = \delta(\omega - \omega_0)\,X(j\omega_0)$

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$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

More in general for $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \Rightarrow \quad \text{Fourier series representation!}$$

Given a periodic signal $x(t)$ with Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

its Fourier transform is the train of impulses:

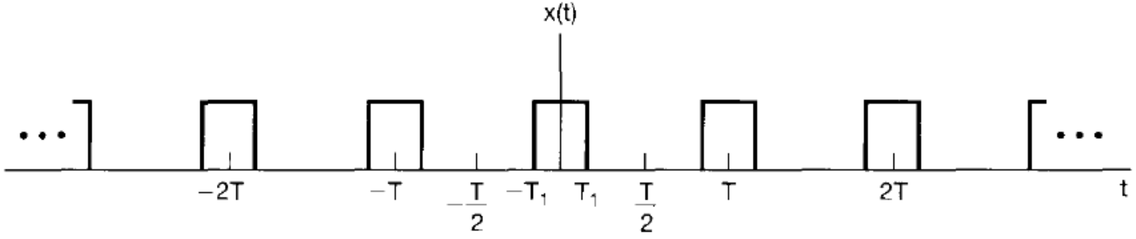
$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

centered at frequencies $k\omega_0$ and with amplitudes $2\pi a_k$

Examples

Example: periodic square wave

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases} \qquad a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = \frac{\sin(k\omega_0 T_1)}{k\pi}$$



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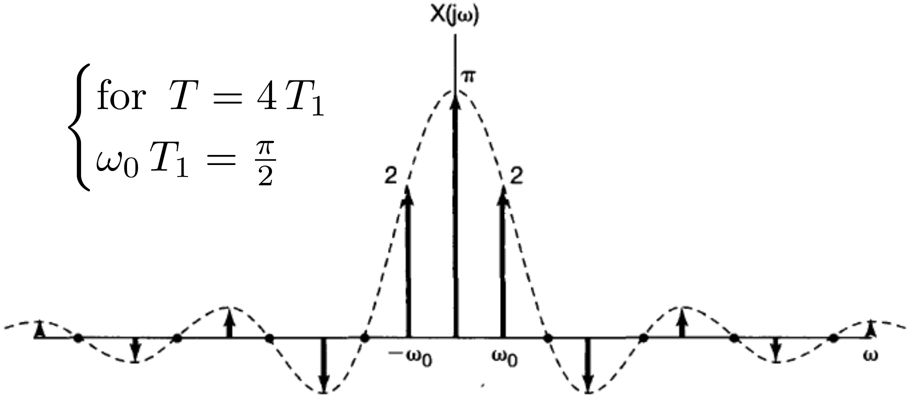
$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) = \sum_{k=-\infty}^{\infty} 2\frac{\sin(k\omega_0 T_1)}{k}\delta(\omega - k\omega_0)$$

centered at frequencies $k\omega_0$ and with amplitudes $2\pi a_k$

$$\begin{cases} \text{for } \; T = 4T_1 \\ \omega_0 T_1 = \frac{\pi}{2} \end{cases}$$



Example: sine wave

$$x(t) = \sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

Fourier series coefficients:

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}, \quad a_k = 0 \quad \text{otherwise}$$

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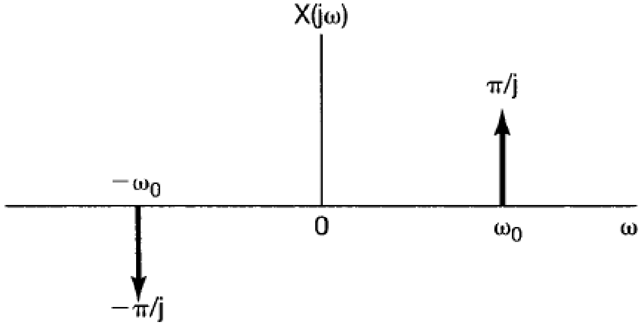


Fourier series coefficients:

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}, \quad a_k = 0 \quad \text{otherwise}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

centered at frequencies $k\omega_0$ and with amplitudes $2\pi a_k$

Example: cosine wave

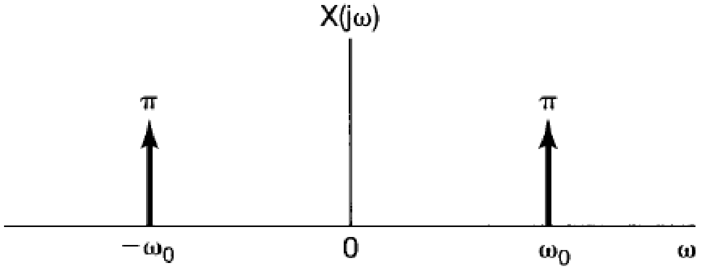
$$x(t) = \cos \omega_0 \, t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

Fourier series coefficients:

$$a_1 = a_{-1} = \frac{1}{2}, \quad a_k = 0 \quad \text{otherwise}$$

Example: cosine wave

$$x(t) = \cos \omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$



Fourier series coefficients:

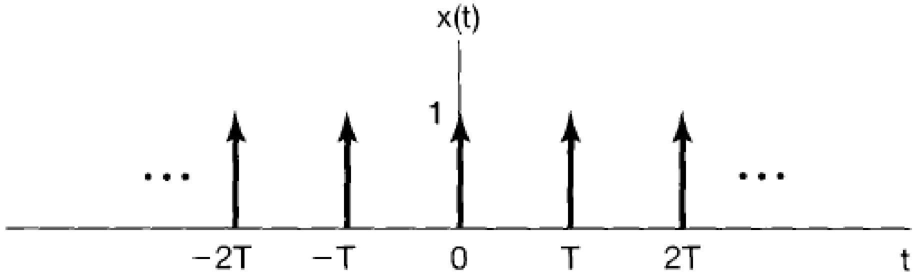
$$a_1 = a_{-1} = \frac{1}{2}, \quad a_k = 0 \quad \text{otherwise}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

centered at frequencies $k\omega_0$ and with amplitudes $2\pi a_k$

Example: impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \qquad a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-jk\omega_0 t}dt = \frac{1}{T}$$

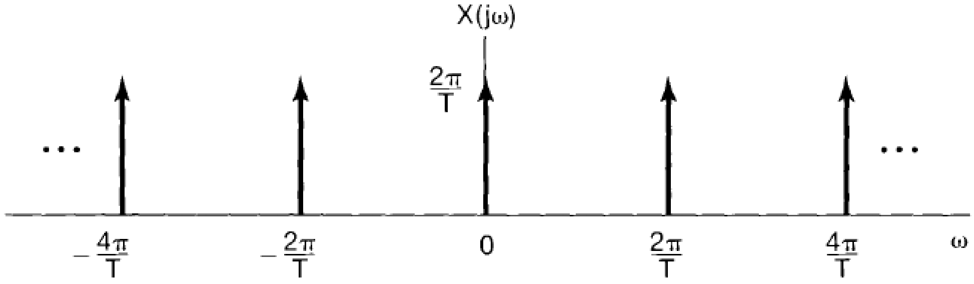


Example: impulse train

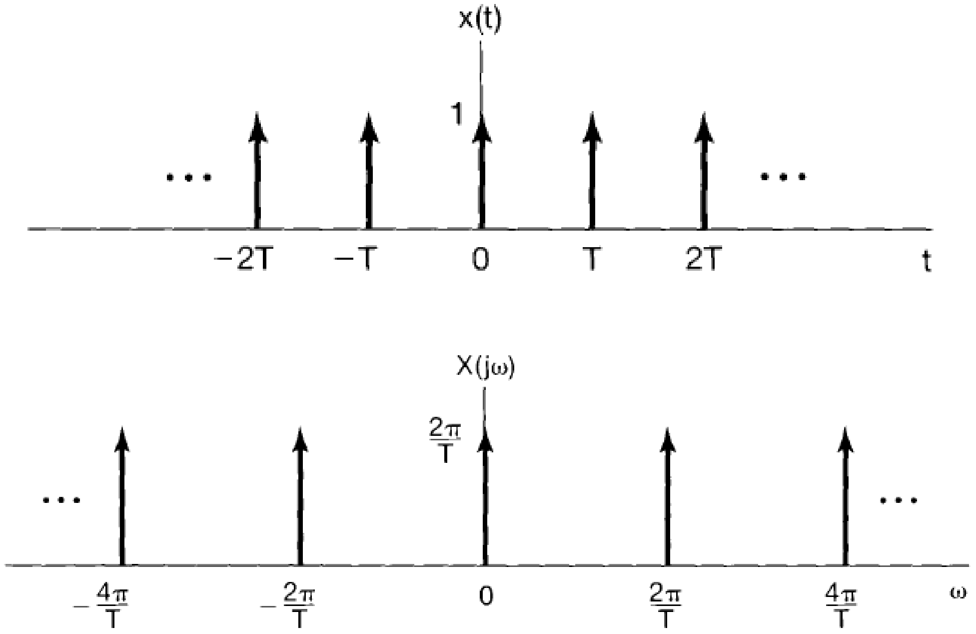
$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \qquad a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$$

centered at frequencies $k\omega_0$ and with amplitudes $2\pi a_k$



Relationship between time/frequency domain



larger spacing in time ⇒ smaller spacing in frequency

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