Signals and Systems

#04: Fourier transform of aperiodic signals

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Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

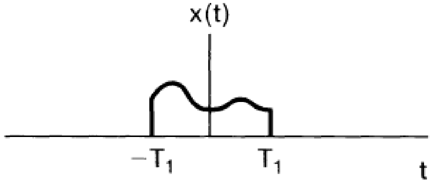
Outline

- **#01 Continuous-time signals**
- **#02 Continuous-time systems**
- **#03 Fourier series of periodic signals**
- **#04 Fourier transform of aperiodic signals**
- **#05 Fourier transform of periodic signals**

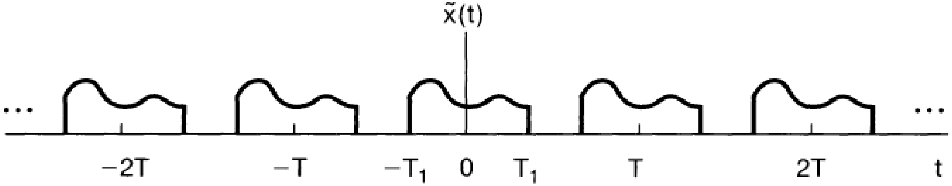
Fourier transform of aperiodic signals

Let us extend the concept of Fourier series to an aperiodic signal $x(t)$

we think of $x(t)$ as a periodic signal $\tilde{x}(t)$ for $T \to \infty$



$x(t) = \tilde{x}(t)$ over one period $[-\frac{T}{2}, \frac{T}{2}]$

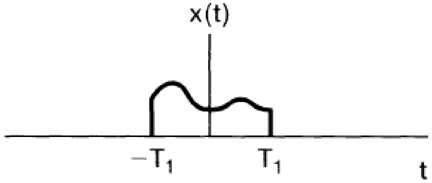


Let us extend the concept of Fourier series to an aperiodic signal $x(t)$
we think of $x(t)$ as a periodic signal $\tilde{x}(t)$ for $T \to \infty$

Recall the Fourier series:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad (\omega_0 = \frac{2\pi}{T})$$

$$a_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t) e^{-jk\omega_0 t} dt$$



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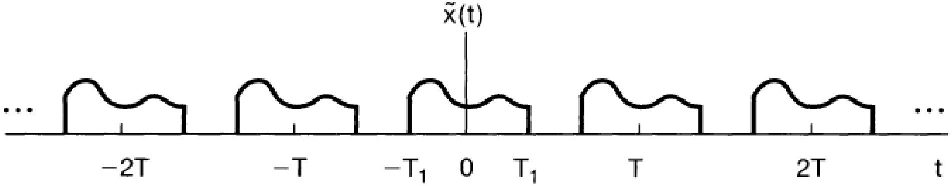

Recall the Fourier series:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt$$

$x(t) = \tilde{x}(t)$ over one period $\left[-\frac{T}{2}, \frac{T}{2}\right]$

Define: $X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

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As $T \to \infty$: $\tilde{x}(t) \to x(t)$, $\omega_0 \to 0$, and the sum becomes an integral

Recall the Fourier series:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0 \xrightarrow{T\to\infty}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Therefore:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier transform pair

representation of $x(t)$ as a linear combination of complex exponentials at different frequencies

Therefore:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Fourier transform pair

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

representation of $x(t)$ as a linear combination of complex exponentials at different frequencies

Convergence for a very broad class of signals:

if $x(t)$ has finite energy, i.e., $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$

Examples

Example: right-sided exponential $x(t) = e^{-at}u(t) \quad a > 0$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = -\frac{e^{-(a+j\omega)t}}{a+j\omega}\Bigg|_{0}^{\infty} = \frac{1}{a+j\omega}$$

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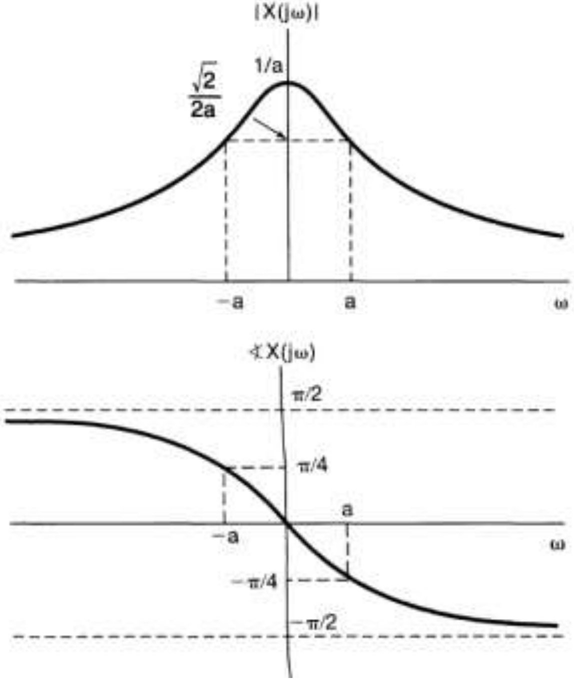
$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = -\frac{e^{-(a+j\omega)t}}{a+j\omega}\bigg|_0^{\infty} = \frac{1}{a+j\omega}$$

Magnitude and phase of $X(j\omega)$ are:

$$|X(j\omega)| = \left|\frac{a-j\omega}{a^2+\omega^2}\right| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$\sphericalangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$



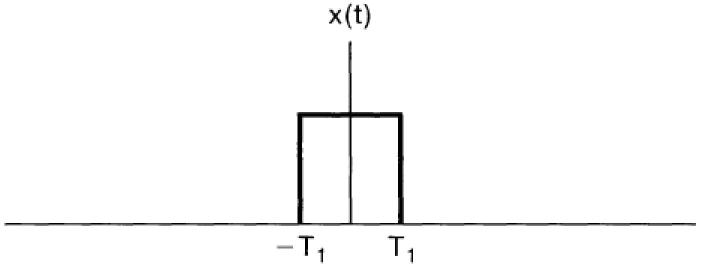
Example: unit impulse $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$

$X(j\omega)$ has equal contributions at all frequencies

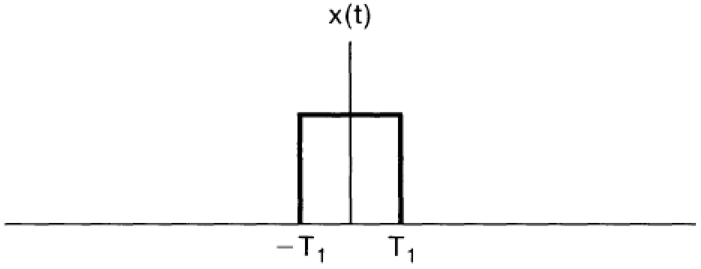
Recall the property: $\delta(t - t_0)\, x(t) = \delta(t - t_0)\, x(t_0)$

Example: rectangular pulse $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$



Example: rectangular pulse $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = -\frac{e^{-j\omega t}}{j\omega}\Bigg|_{-T_1}^{T_1} = -\frac{e^{-j\omega T_1} - e^{j\omega T_1}}{j\omega} = \frac{2\sin(\omega T_1)}{\omega}$$



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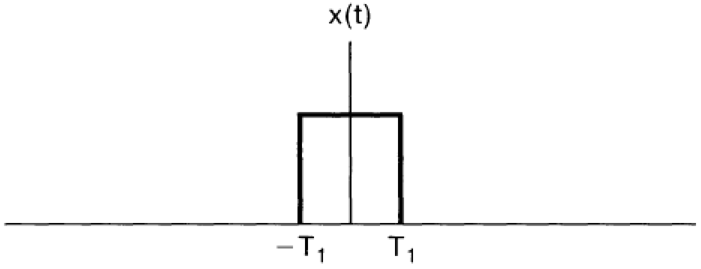
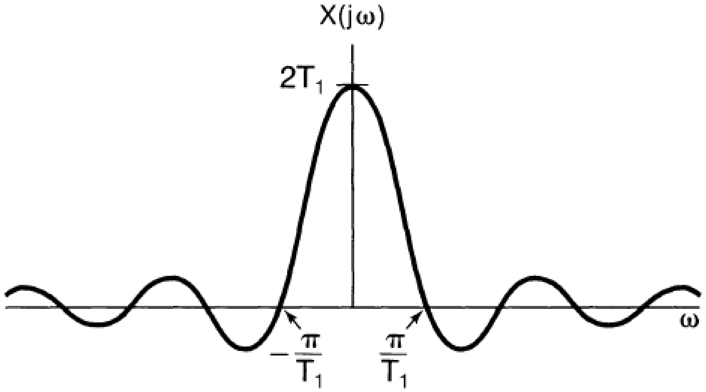
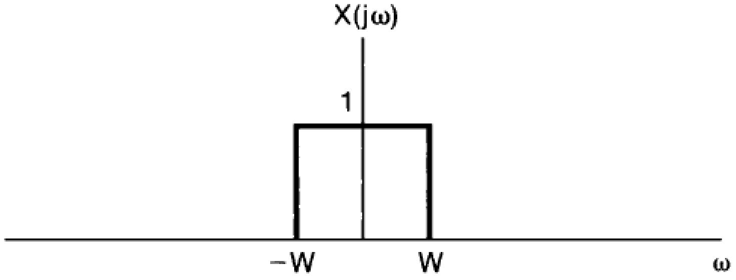
$$\text{sinc}(\theta) \triangleq \frac{\sin(\pi\theta)}{\pi\theta} \quad \Rightarrow \quad X(j\omega) = 2\,T_1\,\text{sinc}\left(\frac{\omega T_1}{\pi}\right)$$





Example: ...

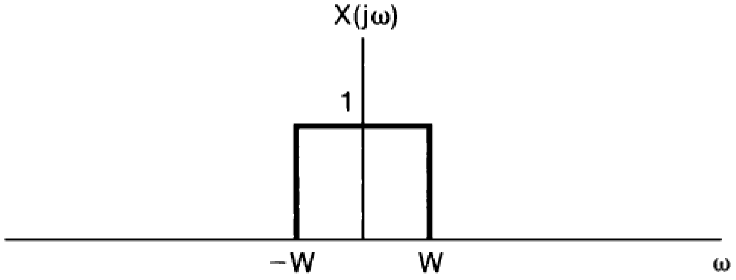
$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

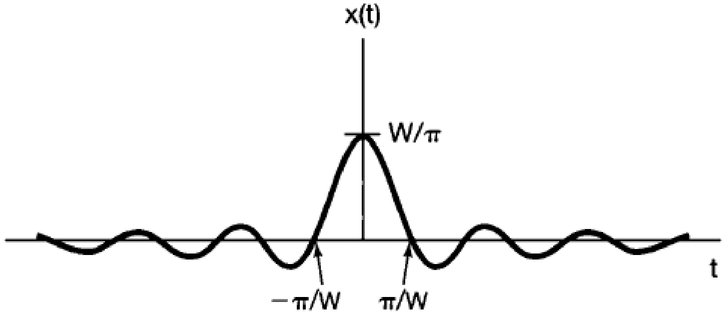


Example: sinc function

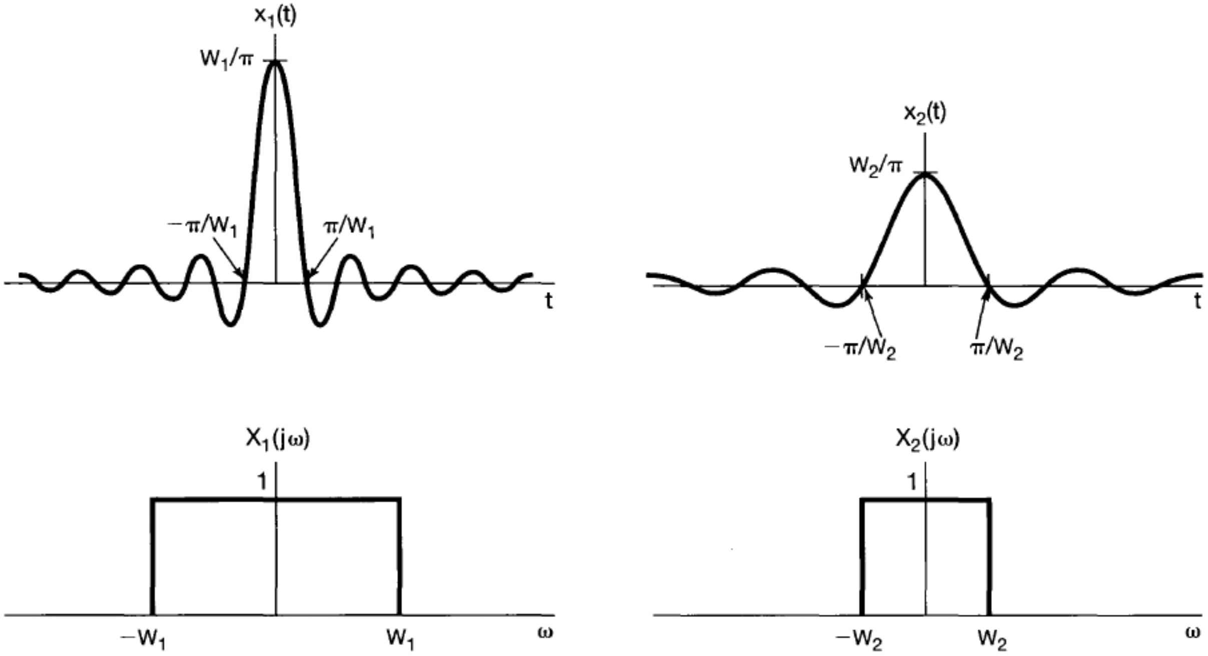
$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t} d\omega = \ldots = \frac{\sin(Wt)}{\pi t} = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right)$$





Relationship between time/frequency domain



As $W \to \infty$, $X(j\omega) \to 1$ and $x(t) \to \delta(t)$

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