Signals and Systems

#03: Fourier series of periodic signals

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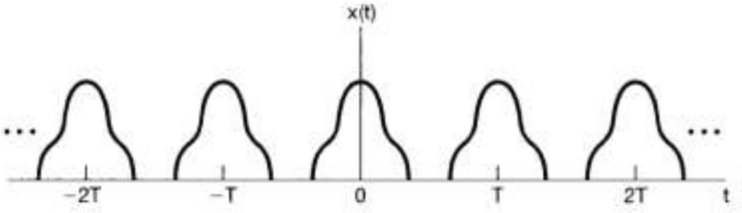
Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

Outline

Recall a periodic signal $x(t) = x(t + T)\ \forall t$

$$T_0 = \min T \qquad \text{fundamental period}$$

$$\omega_0 = 2\pi / T_0 \qquad \text{fundamental frequency}$$



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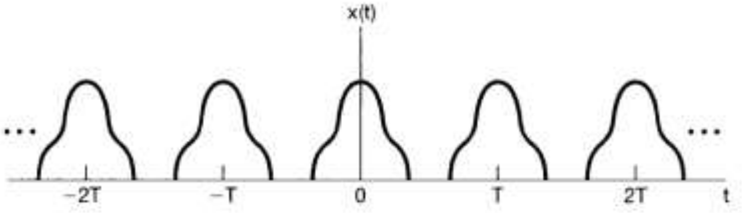

Recall the complex exponential $x(t) = e^{j\omega_0 t}$

The set of harmonically related exponentials is:

$$\Phi_k(t) = e^{jk\omega_0 t} = e^{jk\frac{2\pi}{T}t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

each with fundamental frequency multiple of ω_0

each periodic with period T

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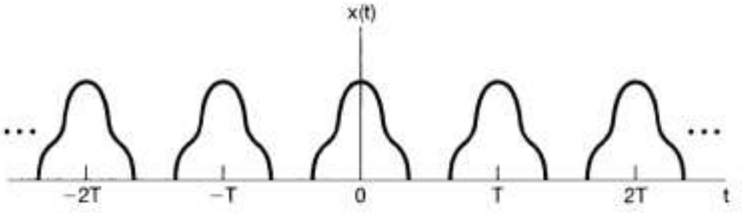

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Construct a linear combination like:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$ **Fourier series** representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$ we want to determine the **Fourier series coefficients** a_k

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Multiply both sides by $e^{-jn\omega_0 T}$ and integrate in $[0, T]$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt \overset{(a)}{=} T \cdot a_n \Rightarrow a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$

where (a) follows from $\int_0^T e^{j(k-n)\omega_0 t}dt = \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$ (integral over one period)

Therefore we have

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

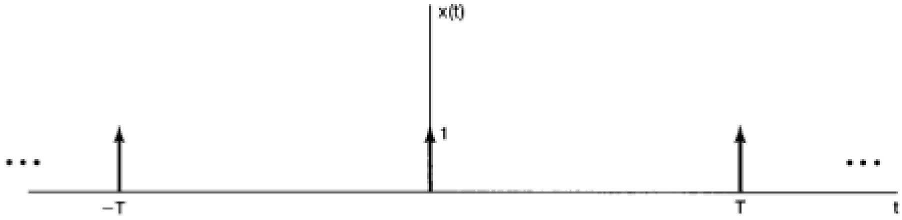
$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}dt \quad \text{(the integral is on any period } T\text{)}$$

a_k are the Fourier series or *spectral* coefficients
the coefficients a_k exist and the series converges for a large class of signals

In particular

$$a_0 = \frac{1}{T}\int_T x(t)dt \quad \text{is the average of } x(t)$$

Example: impulse train



$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \delta(t - kT) e^{-jk\frac{2\pi}{T}t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) dt = \frac{1}{T} \;\; \forall k$$

All the Fourier series coefficients are identical

Example: sine wave

$x(t) = \sin \omega_0 t$ we can use the formula or simply note that

$$x(t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

Thus $a_1 = \frac{1}{2j}$, $a_{-1} = -\frac{1}{2j}$, and $a_k = 0$ otherwise

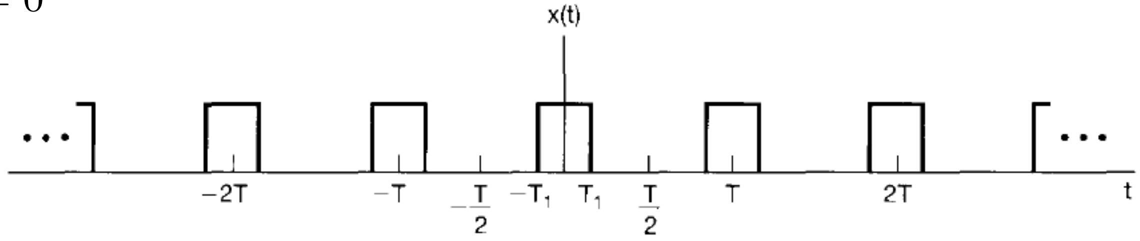
Example: periodic square wave

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt = \left.\frac{e^{-jk\omega_0 t}}{-jk\omega_0 T}\right|_{-T_1}^{T_1} = \frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{jk\omega_0 T} = \frac{2}{k\omega_0 T}\sin(k\omega_0 T_1)$$

$$= \frac{\sin(k\omega_0 T_1)}{k\pi} \quad \text{for } k \neq 0$$



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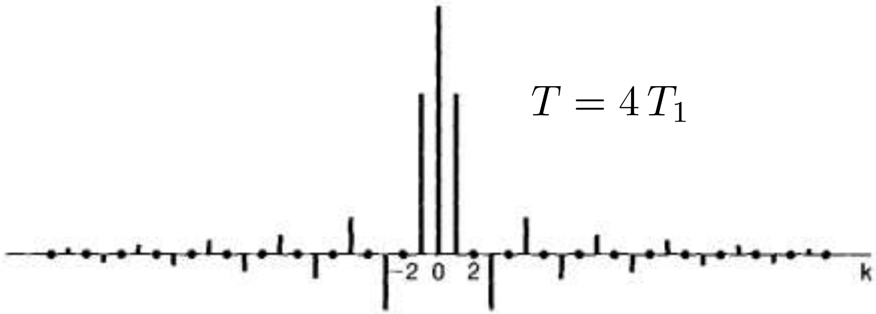

$T = 4\,T_1$

Outline

- #01 Continuous-time signals
- #02 Continuous-time systems
- **#03 Fourier series of periodic signals**
- #04 Fourier transform of aperiodic signals