Signals and Systems

#02: Continuous-time systems

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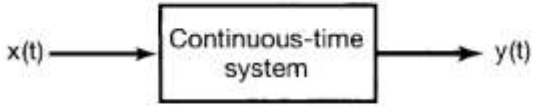
Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

Outline

- #01 Continuous-time signals
- **#02 Continuous-time systems**
- #03 Fourier series of periodic signals

Continuous-time systems

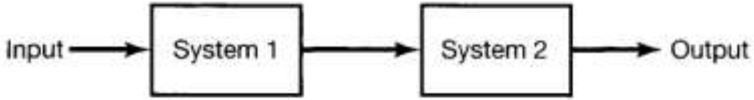
Applying a CT input signal $x(t)$ results in a CT output signal $y(t)$

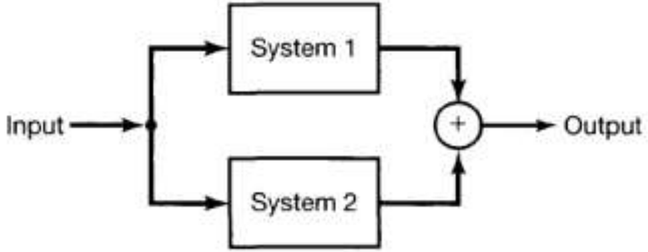


Systems can be interconnected:

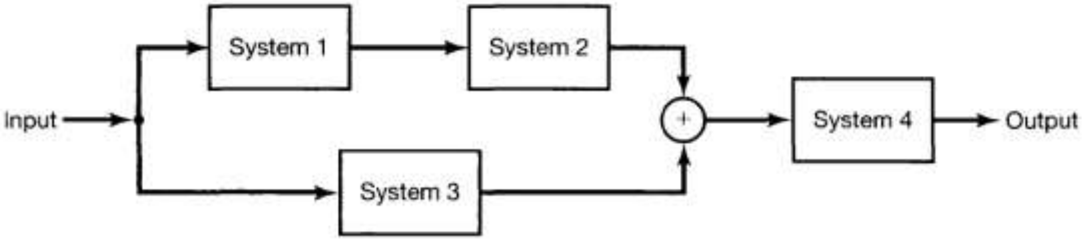
- series (cascade)
- parallel
- feedback

and combinations of the above

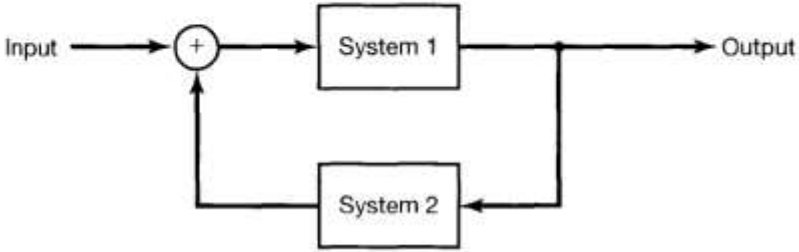




Series-parallel



Feedback



Basic system properties

Linearity, i.e., the property of superposition

Let $y_1(t)$ and $y_2(t)$ be the responses to $x_1(t)$ and $x_2(t)$, respectively

A system is linear if:

1. the response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$
2. the response to $ax_1(t)$ is $ay_1(t), \quad \forall a \in \mathbb{C}$

Combining those we have:

$$ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t), \;\; \forall a, b \in \mathbb{C}$$

Also note that a zero input produces a zero output.

Time invariance, i.e., the behavior of the system is fixed over time

A time shift in the input results in an identical time shift in the output

$$x(t) \to y(t) \Rightarrow x(t - t_0) \to y(t - t_0)$$

We denote **LTI** a system that is both:

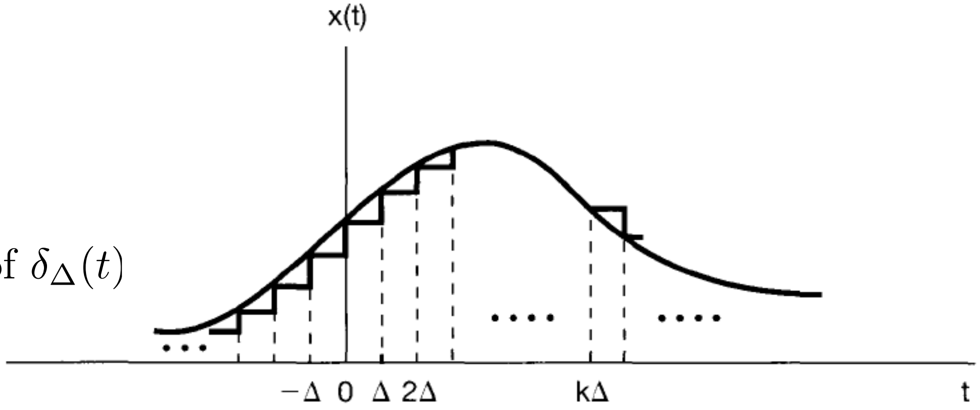
1. Linear
2. Time invariant

Our focus will mostly be on the analysis of LTI systems, as many real systems possess these properties.

LTI systems

Let us represent a signal in terms of impulses through an approximation

staircase approximation:
sum of scaled and shifted versions of $\delta_\Delta(t)$

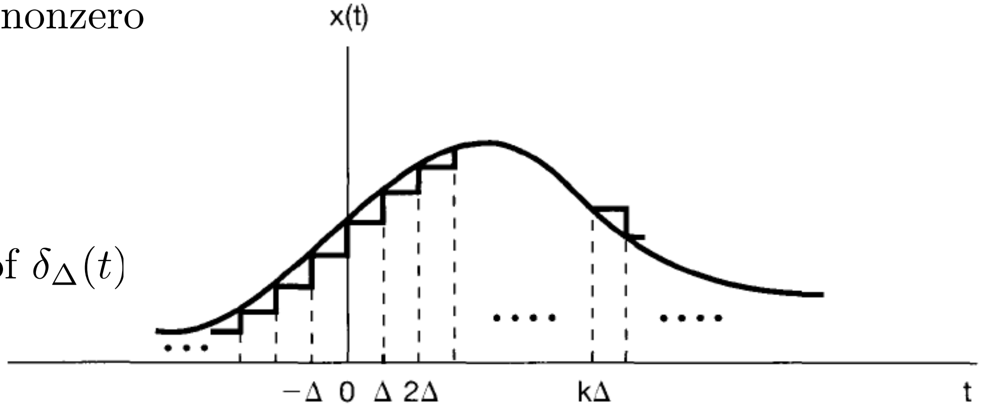


Let us represent a signal in terms of impulses through an approximation

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\, \delta_\Delta(t - k\Delta)\, \Delta \ , \quad \text{where } \delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 \le t < \Delta \\ 0 & \text{otherwise} \end{cases}$$

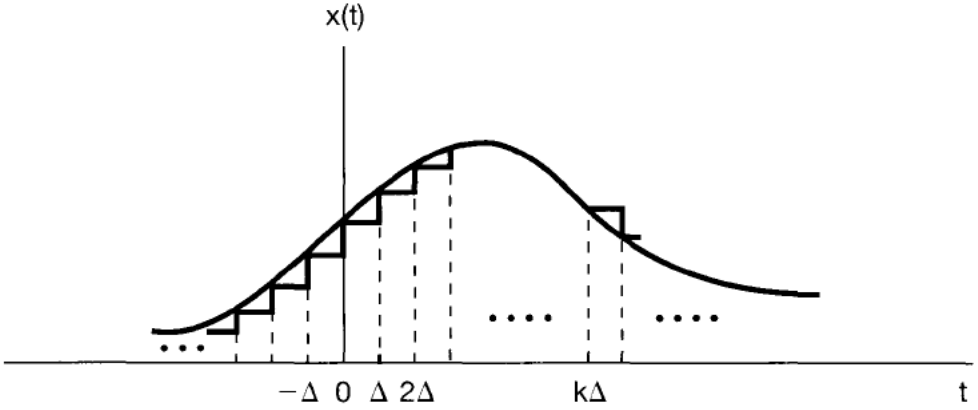
and $\forall t$ only one term in the sum is nonzero

staircase approximation:
sum of scaled and shifted versions of $\delta_\Delta(t)$



$\hat{x}(t)$ is a sum of scaled and shifted versions of $\delta_\Delta(t)$

$\hat{y}(t)$—the response of a linear system to $\hat{x}(t)$—will be the superposition of the responses to scaled and shifted versions of $\delta_\Delta(t)$

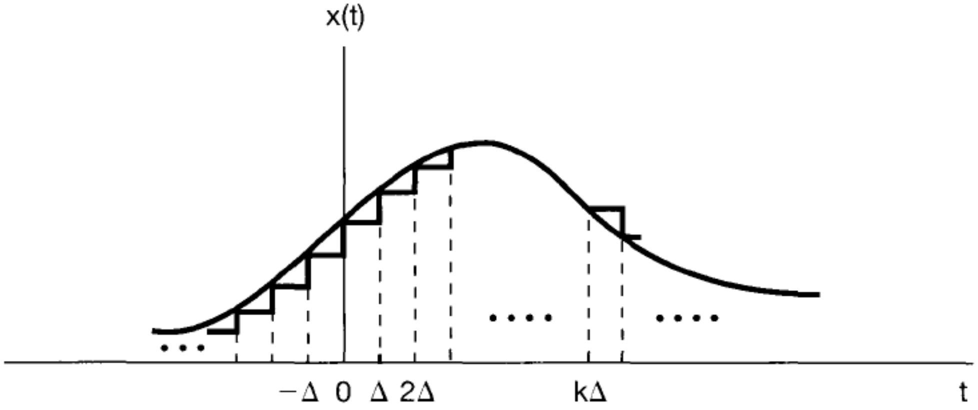


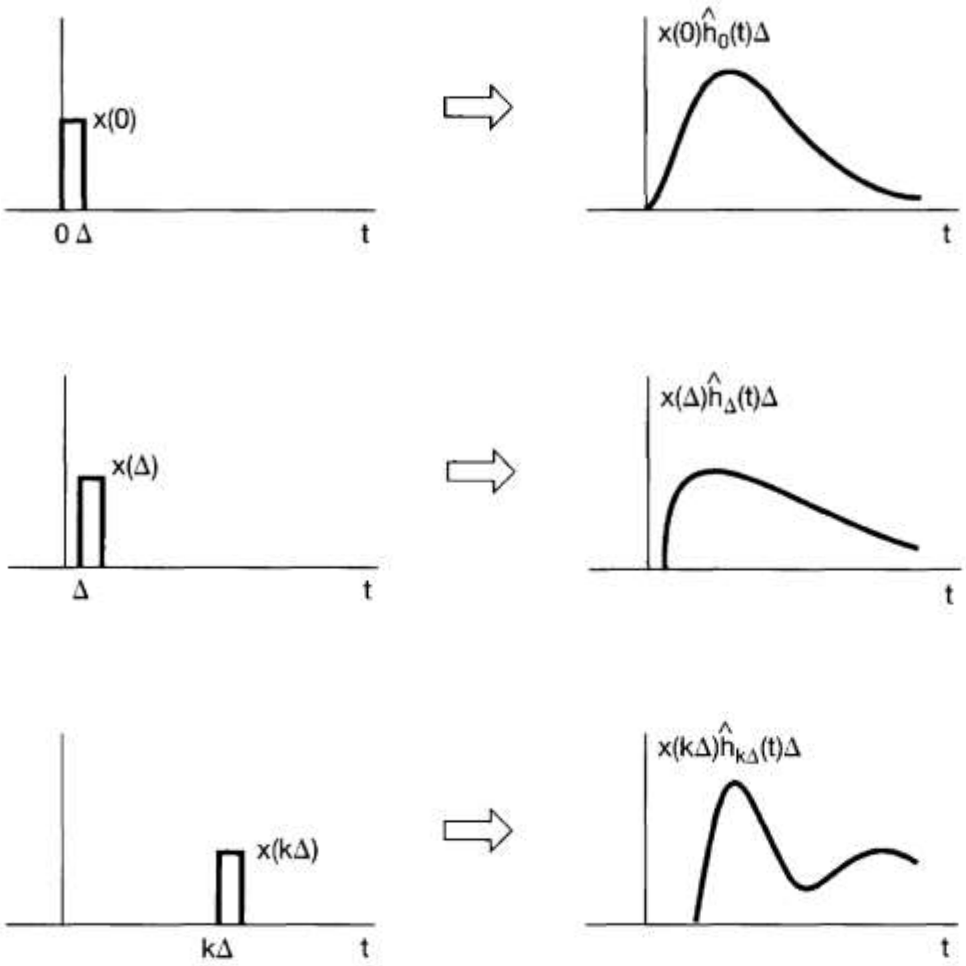
$\hat{x}(t)$ is a sum of scaled and shifted versions of $\delta_\Delta(t)$

$\hat{y}(t)$—the response of a linear system to $\hat{x}(t)$—will be the superposition of the responses to scaled and shifted versions of $\delta_\Delta(t)$

Let $\hat{h}_{k\Delta}(t)$ be the response to $\delta_\Delta(t - k\Delta)$, then

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\hat{h}_{k\Delta}(t)\,\Delta$$





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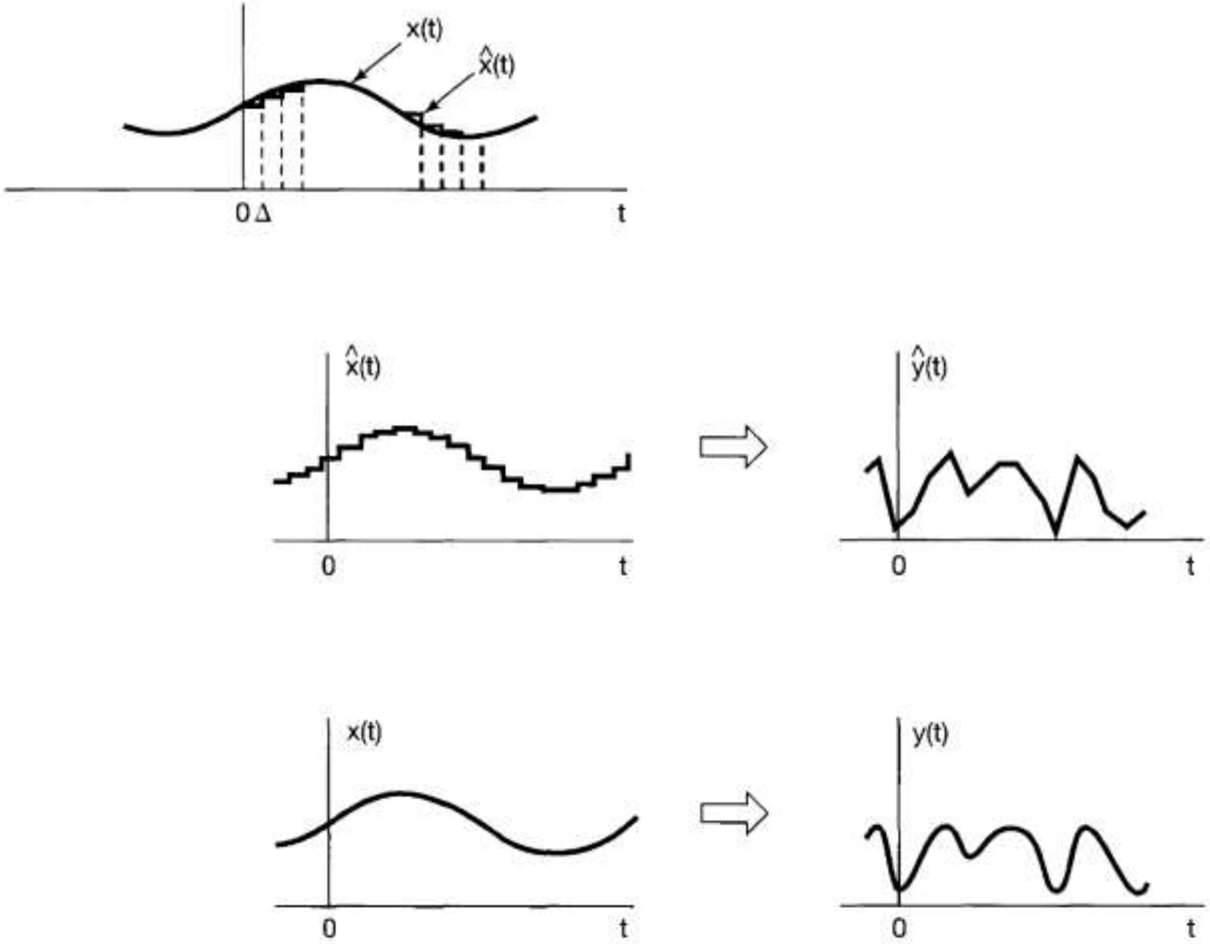
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As $\Delta \to 0$, $\hat{x}(t) \to x(t)$ and its response $\hat{y}(t) \to y(t)$

$$y(t) = \lim_{\Delta \to 0} \hat{y}(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\, h_{k\Delta}(t)\, \Delta$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t) \quad \textbf{convolution} \text{ integral}$$

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$$
\begin{aligned}
y(t) &= \lim_{\Delta \to 0} \hat{y}(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\, h_{k\Delta}(t)\, \Delta \\
&= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t) \quad \textbf{convolution} \text{ integral}
\end{aligned}
$$

where $h(t-\tau)$ is the response to $\delta(t-\tau)$

$h(t)$ is the **impulse response**, i.e., the response to $\delta(t)$

The LTI system is completely characterized by $h(t)$

Example: $h(t) = \delta(t - t_0)$ (time shift)

$$x(t) * h(t) = x(t) * \delta(t - t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau - t_0)d\tau = x(t - t_0)$$

Example: $h(t) = u(t - t_0)$ (integrator)

$$x(t) * h(t) = x(t) * u(t - t_0) = \int_{-\infty}^{\infty} x(\tau)u(t - \tau - t_0)d\tau = \int_{-\infty}^{t-t_0} x(\tau)d\tau$$

Properties of LTI systems

Commutative

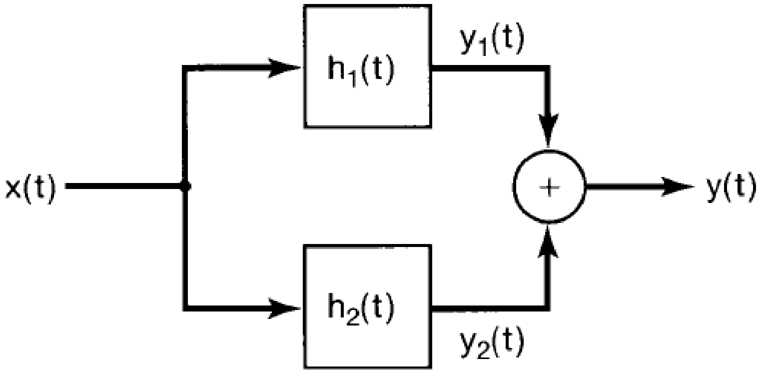
$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau) = d\tau$$

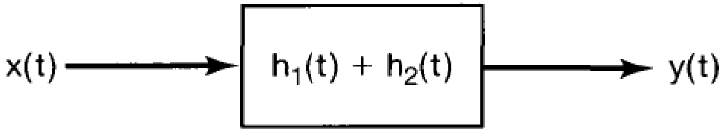
- A system w/ input $x(t)$ and impulse response $h(t)$

  produces the same output as
- A system w/ input $h(t)$ and impulse response $x(t)$

Distributive

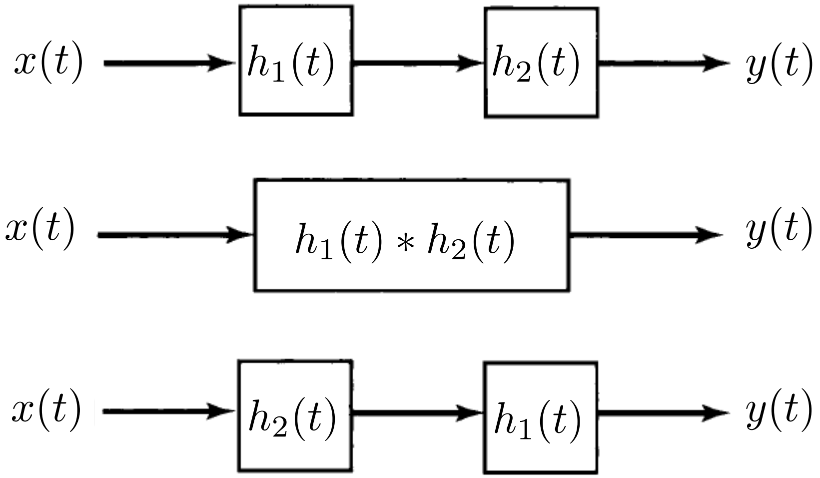
$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$





Associative

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$



Memory

A system is memoriless if its output at any time depends only on the value of the input at the same time

For LTI systems:

$$h(t) = 0 \quad \forall t \neq 0$$

$$y(t) = k\, x(t) \quad \text{and} \quad h(t) = k\, \delta(t)$$

Invertibility

An inverse system exists that, when connected in series with the original system, produces an output equal to the input of the first system:

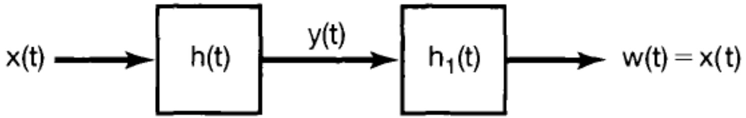
$$\text{system } h(t), \quad \text{inverse } h_1(t), \quad h(t) * h_1(t) = \delta(t)$$

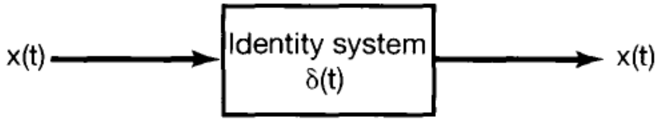
Example: time shift $h(t) = \delta(t - t_0)$

$$y(t) = x(t) * \delta(t - t_0) = x(t - t_0)$$

to invert, use $h_1(t) = \delta(t + t_0)$

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$





Causality

The output depends only on the present and past values of the input

For LTI systems:

$$h(t) = 0 \quad \forall t < 0$$

Example:

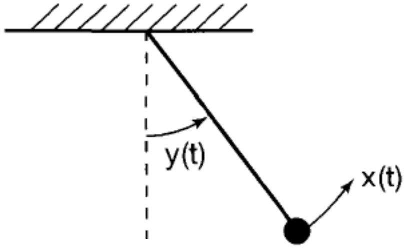
time shift $\quad h(t) = \delta(t - t_0) \quad$ is causal if $\quad t_0 > 0$

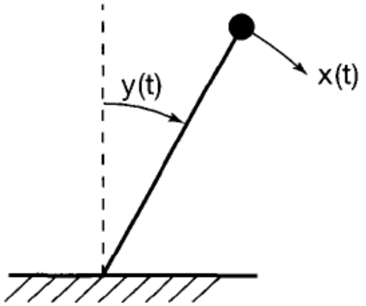
Stability

Every bounded input produces a bounded output, i.e.,

If $|x(t)| < B \quad \forall t,$ then

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right|$$
$$\leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau \leq B \int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$$





For LTI systems, stability is equivalent to the condition:

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty \quad \text{(impulse response absolutely integrable)}$$

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