Signals and Systems

#01: Continuous-time signals

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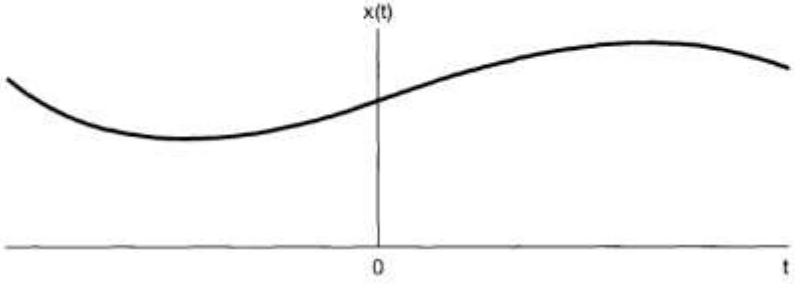
Some of the images in this presentation are from "Signals and Systems", A. V. Oppenheim, A. S. Wilsky, and S. H. Nawab, 2nd ed. Pearson.

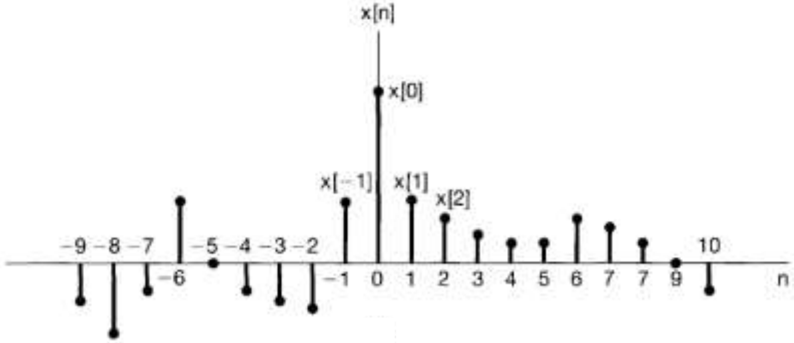
Outline

- **#01 Continuous-time signals**
- **#02 Continuous-time systems**

Signals describe physical phenomena:

- Continuous-time signals
- Discrete-time signals





Signal energy and power

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$ energy over finite interval

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$ total energy

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$ time-averaged power

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$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \qquad \text{time-averaged power}$$

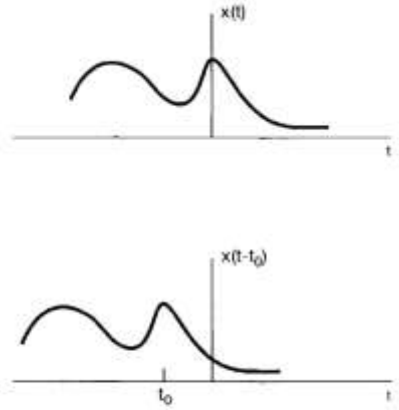
Three classes of signals:

- w/ finite total energy $\Rightarrow$ zero average power $P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$
- w/ finite average power, i.e., $P_\infty > 0 \Rightarrow E_\infty = \infty$
- w/ infinite power and energy, e.g., $x(t) = t$

Transformation of the independent variable

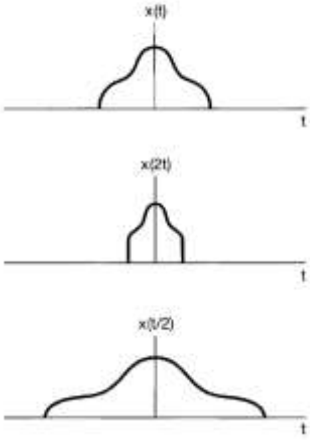
Time shift: $x(t - t_0)$

- delayed if $t_0 > 0$
- anticipated if $t_0 < 0$



Time scaling: $x(\alpha t)$

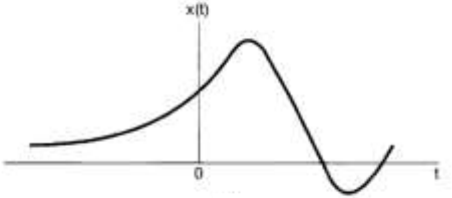
- stretched if $|\alpha| < 1$
- compressed if $|\alpha| > 1$

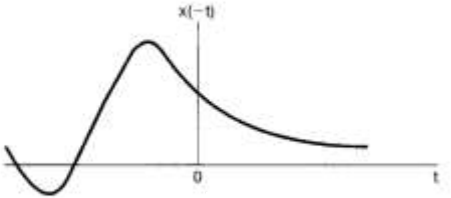


They can be combined: $x(\alpha t + \beta)$

Time reversal: $x(-t)$

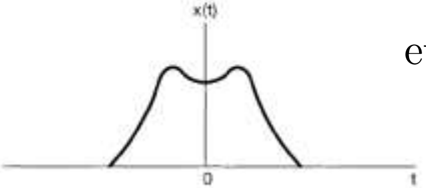
- e.g., audio played backwards



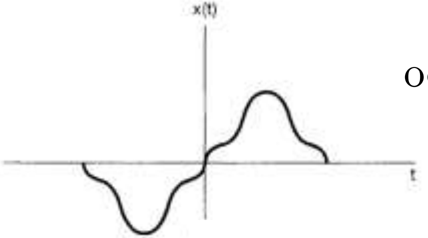


Even (odd) signals

- identical (opposite) to their time-reversed counterpart



even: $x(t) = x(-t)$

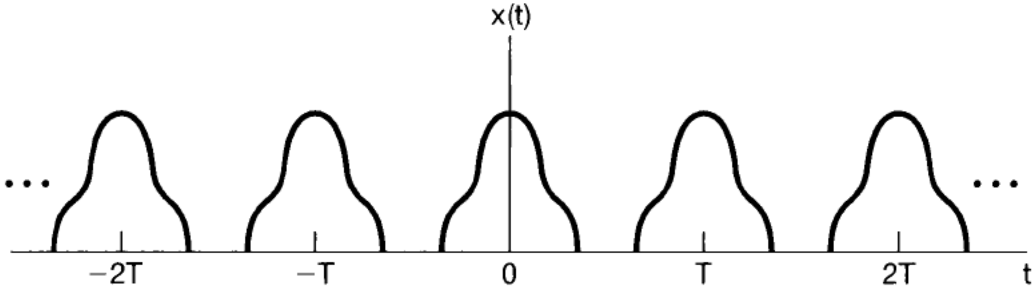


odd: $x(t) = -x(-t)$

Periodic, exponential, and sinusoidal signals

Periodic signals:

$x(t) = x(t+T)$ is periodic with period T



As a consequence, $x(t) = x(t+mT)$ for any integer m
$\Rightarrow x(t)$ is also periodic with period mT

The smallest T for which it holds is the fundamental period T_0

Exponential and sinusoidal signals:

$x(t) = Ce^{at}$ with C and a complex

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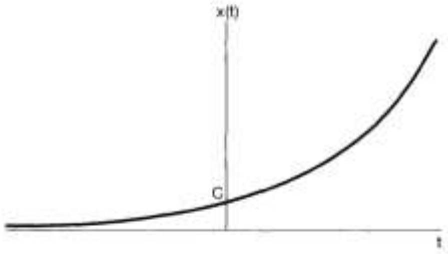
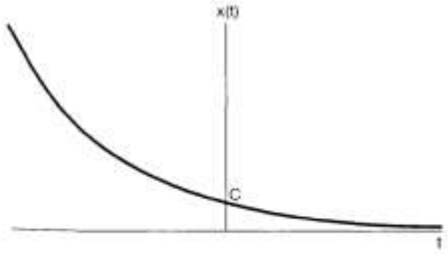
- **Real exponential**
  C and a are real
  $a > 0$ growing
  $a < 0$ decaying





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- **Periodic complex exponential**
  a is purely imaginary: $x(t) = e^{j\omega_0 t}$
  it's periodic, what's the period?
  $e^{j\omega_0 t} = e^{j\omega_0 (t+T_0)} \Rightarrow e^{j\omega_0 T_0} = 1 \Rightarrow T_0 = \frac{2\pi}{|\omega_0|}$

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- **Sinusoidal signal**
  $x(t) = A\cos(\omega_0 t + \phi) \quad \omega_0 = 2\pi f_0$
  t [s], ϕ [rad], ω_0 [rad/s], f_0 [Hz]

Relationship between complex exponential and sinusoidal signals:

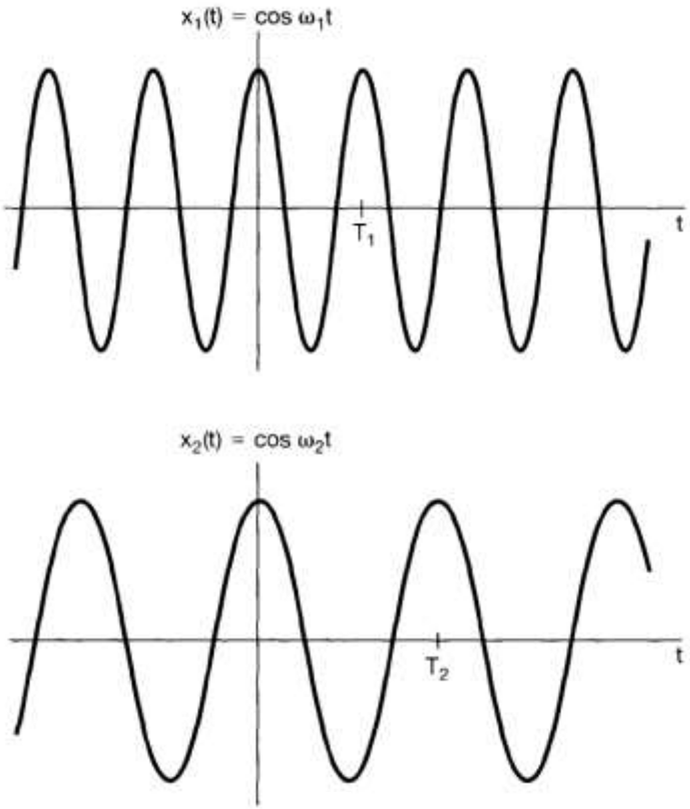
$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$

ω_0 is the fundamental frequency

$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$

$A\cos(\omega_0 t + \phi) = A\Re\{e^{j(\omega_0 t+\phi)}\}$

$A\sin(\omega_0 t + \phi) = A\Im\{e^{j(\omega_0 t+\phi)}\}$



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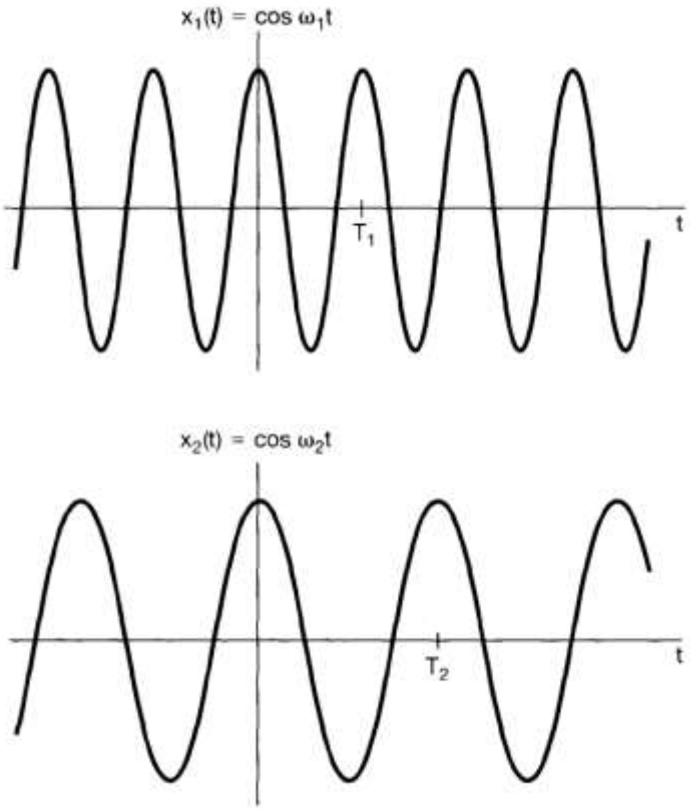
finite energy in one period
infinite total energy
finite average power



General complex exponentials:

both a and C are complex

$C = |C|e^{j\theta} \quad a = r + j\omega_0$

$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$
$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$

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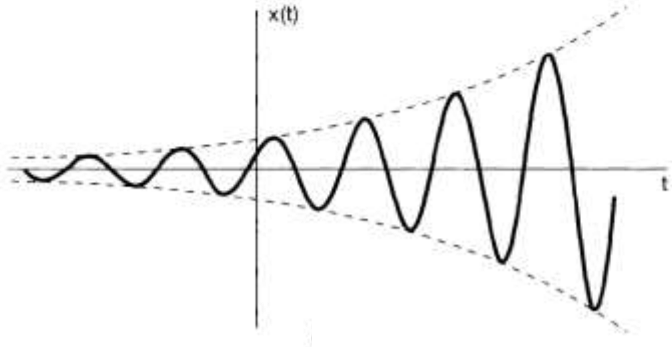
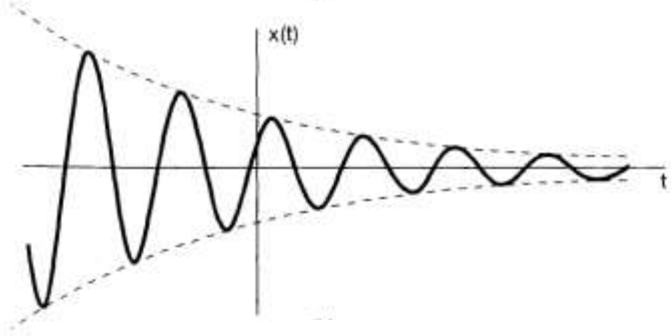
if $r = 0 \Rightarrow$ real and imaginary parts are sinusoidal
if $r > 0 \Rightarrow$ multiplied by a growing exponential
if $r < 0 \Rightarrow$ multiplied by a decaying exponential



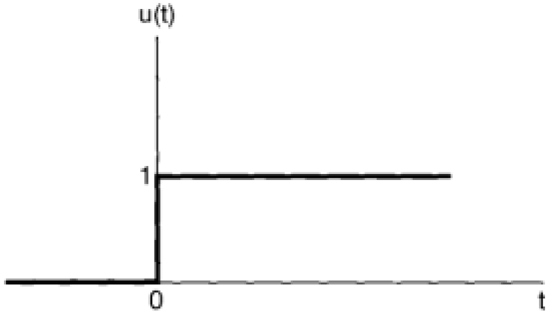


Unit impulse and unit step function

Unit step function or "Heaviside"

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

discontinuous at $t = 0$
$u(t)$ formally not differentiable



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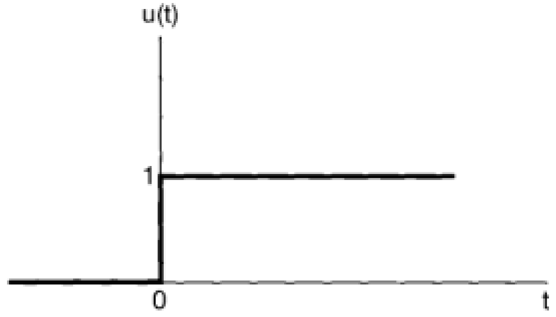

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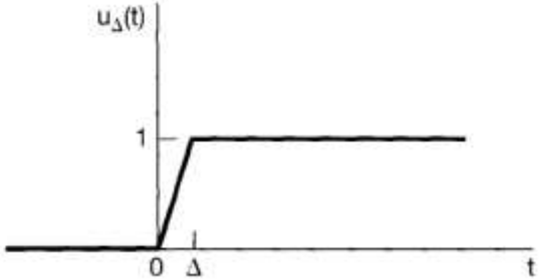
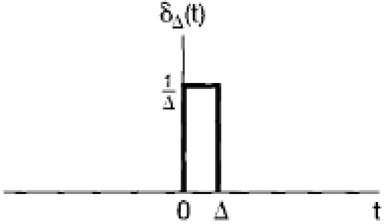
$u(t)$ formally not differentiable

Consider an approximation $u_\Delta(t)$
then $\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$

$\delta_\Delta(t)$ is a pulse with duration Δ
as $\Delta \to 0$, $\delta_\Delta(t)$ is narrower and higher

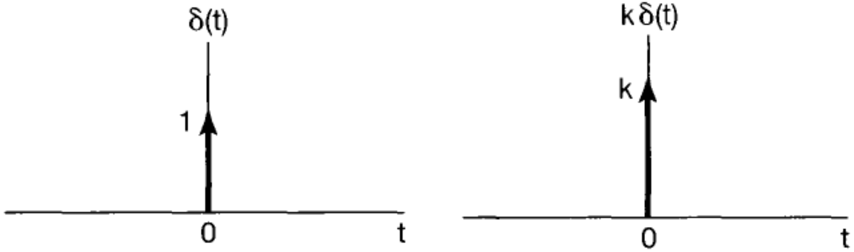




In the limit:

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$ unit impulse or "Dirac"

and the area is concentrated at $t = 0$



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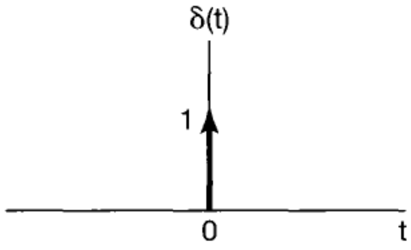
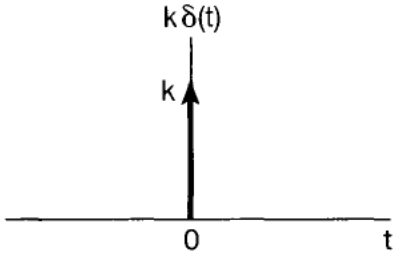
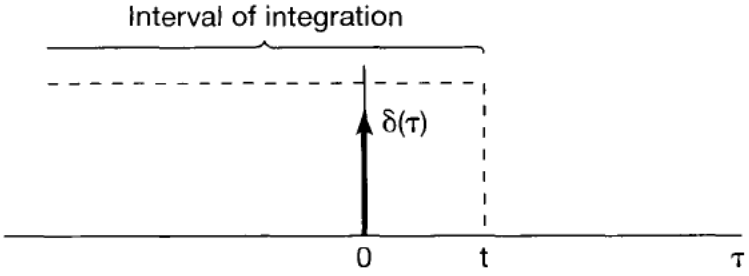




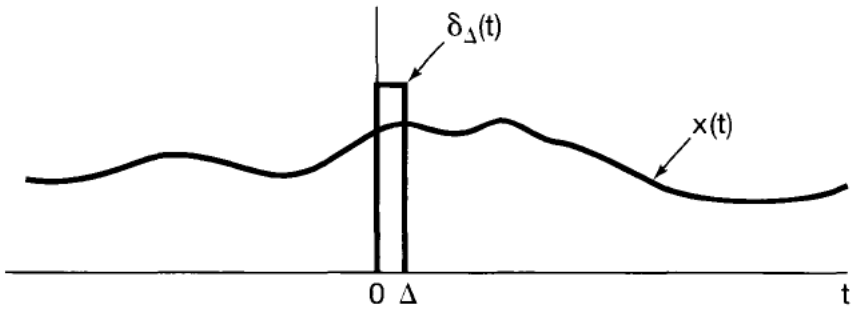
We have that $\frac{du(t)}{dt} = \delta(t)$

and conversely $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$



Sampling property of the impulse:

Consider the product $x(t) \cdot \delta_\Delta(t)$



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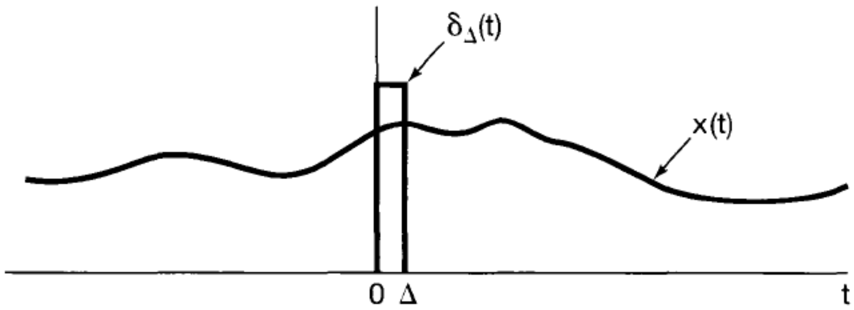
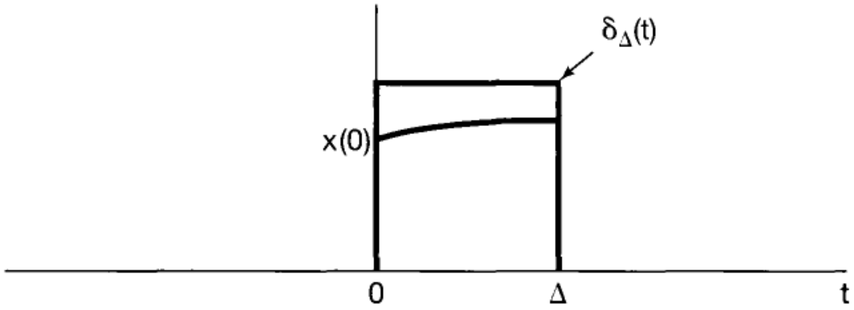


For small Δ, $x(t)$ is approximately constant in $[0, \Delta]$

hence: $x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$

in the limit: $x(t)\delta(t) = x(0)\delta(t)$



More in general:
$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$

Outline

- **#01 Continuous-time signals**
- **#02 Continuous-time systems**