Signals and Systems 2

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Tutorial #4: Sampling/reconstruction and modulation/demodulation

1. Consider two signals $x_1(t)$ and $x_2(t)$, bandlimited so that $X_1(j\omega) = 0, |\omega| \geq \omega_1$ and $X_2(j\omega) = 0, |\omega| \geq \omega_2$. Both signals are multiplied, so that $w(t) = x_1(t) \cdot x_2(t)$, and then sampled by a periodic impulse train $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n \cdot T_s)$. Determine the maximum sampling period T_s so that $w(t)$ can be perfectly reconstructed by an ideal LPF.

2. Determine the Nyquist rate of the signals below:
   (a) $x(t) = 1 + \cos(2000\pi t) + \sin(4000\pi t)$
   (b) $x(t) = \frac{\sin(4000\pi t)}{\pi t}$
   (c) $x(t) = (\frac{\sin(4000\pi t)}{\pi t})^2$

3. Consider a signal $x(t)$ such that its Nyquist rate is ω_0. Determine the Nyquist rate of the signals below:
   (a) $x(t) + x(t-1)$
   (b) $x^2(t)$
   (c) $x(t) \cdot \cos(\omega_0 t)$

4. Consider a signal $x(t)$ such that $X(j\omega) = 0, |\omega| \geq \omega_b$. The signal is modulated by multiplication with a carrier $p(t) = \cos(\omega_p t)$, resulting in $x_m(t)$. If we sample $x_m(t)$ with an impulse train of period T_s, determine the maximal value of ω_p so that we can perfectly reconstruct $x_m(t)$.

5. Determine whether the following statements are true or false:

(a) The signal $x(t) = u(t + T_0) - u(t - T_0)$ can be sampled correctly (without aliasing) with an impulse train if the sampling period is $T_s < 2T_0$.

(b) The signal with Fourier transform $X(j\omega) = u(\omega + \omega_0) - u(\omega - \omega_0)$ can be sampled correctly (without aliasing) with an impulse train if the sampling period is $T_s < \frac{\pi}{\omega_0}$.

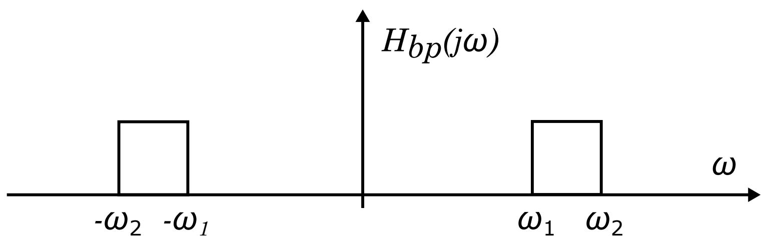
(c) The signal with Fourier transform $X(j\omega) = u(\omega) - u(\omega - \omega_0)$ can be sampled correctly (without aliasing) with an impulse train if the sampling period is $T_s < \frac{2\pi}{\omega_0}$.

6. Consider a signal $x(t)$ satisfying $X(j\omega) = \begin{cases} \neq 0, & \omega_1 \leq |\omega| \leq \omega_2 \\ 0, & \text{otherwise} \end{cases}$, where $\omega_1 > \omega_2 - \omega_1$.

Such signals, named *passband signals*, can be perfectly reconstructed after sampling with an impulse train with sampling frequency ω_s **below the Nyquist rate**, by using a reconstruction filter $H(j\omega) = \begin{cases} A, & \omega_a \leq |\omega| \leq \omega_b \\ 0, & \text{otherwise} \end{cases}$.

Determine ω_a, ω_b, and the range of ω_s so that perfect reconstruction is possible.

7. Consider the ideal bandpass filter (with unitary amplitude) shown below:



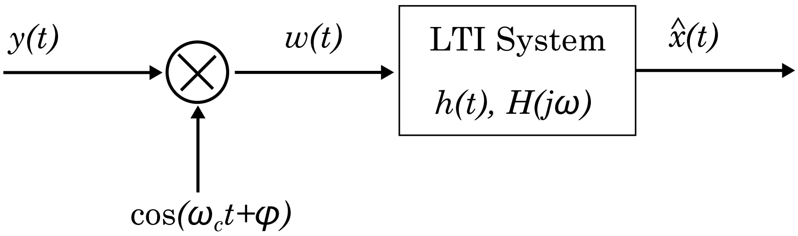
Show that its impulse response, $h_{bp}(t)$, can be represented as any of the following three ways:

(a) A cosine-modulated lowpass filter: $h_{bp}(t) = A\frac{sin(\omega_{co}t)}{\pi t}cos(\omega_0 t)$ (Note: determine A, ω_{co}, and ω_0).

(b) The difference of two ideal lowpass filters: $h_{bp}(t) = \frac{sin(\omega_2 t)}{\pi t} - \frac{sin(\omega_1 t)}{\pi t}$.

(c) Convolution of the impulse responses of an ideal highpass filter and an ideal lowpass filter: $h_{bp}(t) = \left[\delta(t) - \frac{sin(\omega_1 t)}{\pi t}\right] * \frac{sin(\omega_2 t)}{\pi t}$.

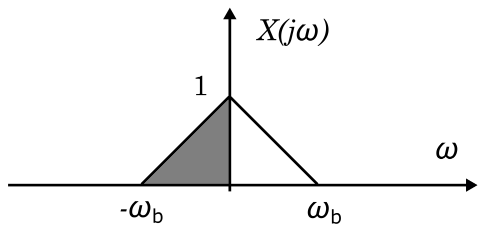
8. Consider the demodulation system below:



where

$$H(j\omega) = \begin{cases} 2, & |\omega| \leq \omega_{co} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

We have shown that when $\phi = 0$, if we apply this system to the modulated signal $y(t) = x(t)cos(\omega_c t)$, we can recover the original signal $x(t)$ (i.e., $\hat{x}(t) = x(t)$), provided that $x(t)$ is bandlimited and the frequencies ω_{co} and ω_c are properly chosen.
Assume that $x(t)$ has the following bandlimited Fourier transform representation:



and that $\omega_c > \omega_b$.

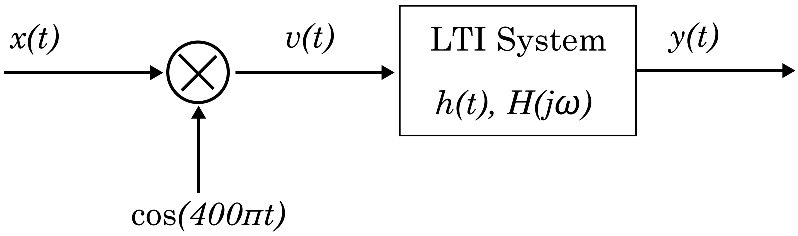
(a) For $\phi \neq 0$, use Euler's formula to show that: $w(t) = \frac{1}{2}x(t)cos(\phi) + \frac{1}{2}x(t)cos(2\omega_c t + \phi)$.

(b) Use the result in part (a) to obtain $W(j\omega)$ and plot it.

(c) Assuming $\omega_{co} = \omega_b$, plot $\hat{X}(j\omega)$.

(d) Obtain an equation for $\hat{x}(t)$ in terms of $x(t)$ and ϕ.

9. Consider the amplitude modulation system below:



and the bandlmited input signal $x(t)$ of exercise 8 with $\omega_b = 100\pi$. Assume that the LTI system has the frequency response of an ideal bandpass filter:

$$H(j\omega) = \begin{cases} 1, & 300\pi < |\omega| \leq 400\pi \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

(a) Plot $H(j\omega)$ and $V(j\omega)$.

(b) Use the plots in part (a) to plot $Y(j\omega)$. Note that the negative frequency portion of $X(j\omega)$ is shaded. Mark the corresponding regions in your plot of $V(j\omega)$ and $Y(j\omega)$.

(c) The output signal is called a “single-sideband” signal. Justify this terminology.

(d) How could you recover $x(t)$ from the single-sideband signal $y(t)$? Draw a block diagram of the system that recovers the original signal.

10. Consider the signal $x(t)$ as in exercise 8, with $\omega_b = 80\pi$. We sample the signal $x(t)$ at frequency ω_s, and then we reconstruct it with a filter that has the following frequency response:

$$H_r(j\omega) = \begin{cases} T, & |\omega| \leq \pi/T_s \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

(a) Choose the value of ω_s so that it is equal to the Nyquist rate and plot $X_s(j\omega)$.

(b) If $\omega_s = 2\pi/T_s = 100\pi$, plot $X_s(j\omega)$ and use that to plot $X_r(j\omega)$. What do you observe?