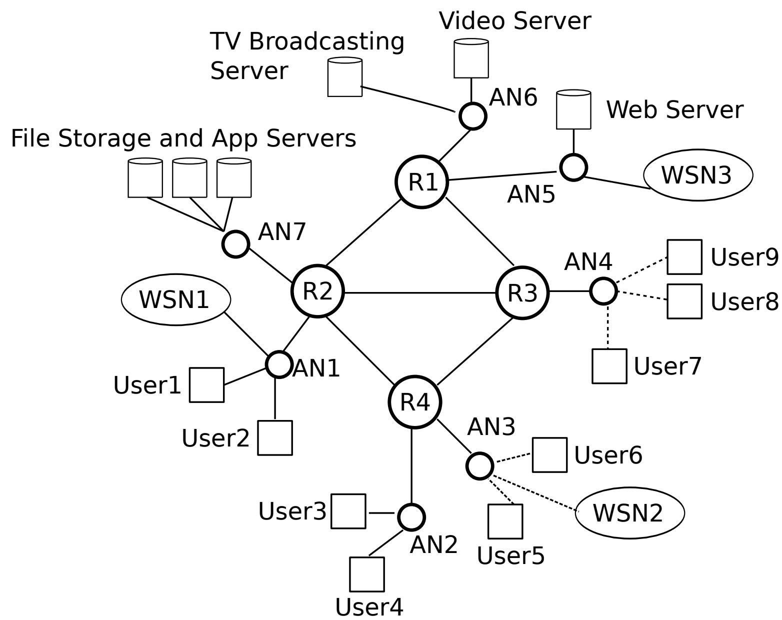

Figure 1: Basic Network

Seminar 2: Exercises

Exercise 1

Let us consider User 4 from Figure 1. The traffic load it generates can be modelled by a Markovian stochastic process $X(t)$ with state space $\mathcal{X} = \{1, 2, 4, 8\}$ Mbps. It takes a new value (from its state space) every $T = 1$ seconds, and has the following probability transition matrix:

$$P = \begin{bmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.5 & 0.0 & 0.5 & 0 \\ 0 & 0.6 & 0.0 & 0.4 \\ 0 & 0 & 1 & 0.0 \end{bmatrix}$$

1. Draw a possible realization of X(t) of 10 seconds.
2. Draw the Discrete Time Markov Chain (CTMC) modelling $X(t)$.

3. Explain why X(t) is a Markovian process. Is it a reversible Markovian process? Is it a birth and death Markovian process?
4. Write its global and local balance equations.
5. Obtain its Equilibrium distribution (the stationary probability distribution).
6. Compute the average traffic load generated by User 4, i.e., $E[X(t)]$.

Exercise 2

Consider the Web Server in Figure 1. It receives requests at a rate $\lambda_{\text{req}} = 10$ requests / second following a Poisson process. The expected time required to process each request is $\mathrm{E}[D_\mathrm{s}] = 0.15$ seconds and follows (the random variable representing the service time) an exponential distribution. The Web Server is only able to process one request simultaneously. Therefore, the received requests that can not be immediately processed wait in a buffer for their turn. The buffer size is $Q = 3$, and the maximum number of requests in the Web Server is $K = 4$. The considered system is depicted in Figure 2.

1. Given $X(t)$ is an stochastic process representing the number of requests in the Web Server at any arbitrary instant of time, draw the Continuous Time Markov Chain (CTMC) that describes the Web Server. Is it a birth and death process?
2. Write its local balance equations.
3. Obtain its Equilibrium distribution (the stationary probability distribution).
4. What is the probability that a new arriving request finds the Web Server with 4 requests and therefore, it is blocked?
5. What is the probability that a new arriving request finds the Web Server without any request?
6. Plot P_b for λ_{req} equal to 1, 5, 10, 15, 20 and 25 packets / second. Is P_b linear with λ_{req}?

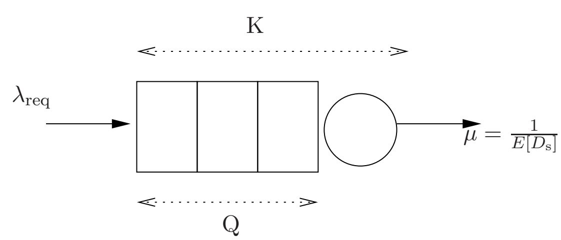

Figure 2: Queueing Model of the Web Server

Solutions

Exercise 1

A single realization of X(t) could be: ..., 4, 8, 2, 1, 2, 2, 4, 8, 4, 8, ... Note that transitions between non-contiguous values is not possible.

The DTMC is shown in Figure 3.

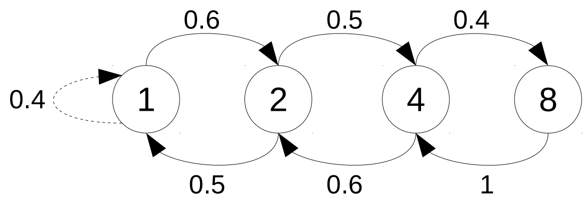

Figure 3: DTMC representing the behaviour of User 4

It is a Markovian process since next future value at $t+1$ only depends on current value at t, i.e., the Markov property holds:

$$P(X(t+1) = x_{t+1}|X(t) = x_t) = P(X(t+1) = x_{t+1}|X(t) = x_t, X(t-1) = x_t - 1, X(t-2) = x_{t-2}, \ldots)$$

as it can be seen from P and the DTMC.

Yes, it is a reversible Markovian process since any pair of states have bidirectional transitions. No, it is not a birth and death model even if we move only in steps of one state forward or backward, as the increases / decreases are not like +1/-1.

The global balance equations are:

$$\begin{aligned} 0.6\pi_1 &= 0.5\pi_2 \\ \pi_2 &= 0.6\pi_1 + 0.6\pi_4 \\ \pi_4 &= 0.5\pi_2 + \pi_8 \\ \pi_8 &= 0.4\pi_4 \end{aligned}$$

Since the Markovian process is reversible, local balance exists. Therefore, the local balance equations are:

$$\begin{aligned} 0.6\pi_1 &= 0.5\pi_2 \\ 0.5\pi_2 &= 0.6\pi_4 \\ 0.4\pi_4 &= \pi_8 \end{aligned}$$

Then, given the condition that $\pi_1 + \pi_2 + \pi_4 + \pi_8 = 1$, we can solve the system of equations as follows:

$$\begin{aligned}
\pi_2 &= \frac{0.6}{0.5}\pi_1 \\
\pi_4 &= \frac{0.5}{0.6}\pi_2 = \frac{0.5}{0.6}\frac{0.6}{0.5}\pi_1 = \pi_0 \\
\pi_8 &= \frac{0.4}{1}\pi_4 = 0.4\frac{0.5}{0.6}\frac{0.6}{0.5}\pi_1 = 0.4\pi_0
\end{aligned}$$

We find π_1 as follows

$$\pi_1 + \frac{0.6}{0.5}\pi_1 + \pi_1 + 0.4\pi_1 = 1 \rightarrow \pi_1 = \frac{1}{1 + \frac{0.6}{0.5} + 1 + 0.4} = 0.2778$$

as well as the rest of stationary probabilities

$$\begin{aligned}
\pi_2 &= \frac{0.6}{0.5}\pi_1 = 0.3333 \\
\pi_4 &= \frac{0.5}{0.6}\pi_2 = \frac{0.5}{0.6}\frac{0.6}{0.5}\pi_1 = \pi_1 = 0.2778 \\
\pi_8 &= \frac{0.4}{1}\pi_4 = 0.4\frac{0.5}{0.6}\frac{0.6}{0.5}\pi_1 = 0.4\pi_1 = 0.1111
\end{aligned}$$

The average traffic load generated by User 4 is given by

$$E[X] = \pi_1 1 + \pi_2 2 + \pi_4 4 + \pi_8 8 = 2.94 \text{ Mbps} \tag{1}$$

Exercise 2

The CTMC chain modelling the Web Server operation is shown in Figure 4.

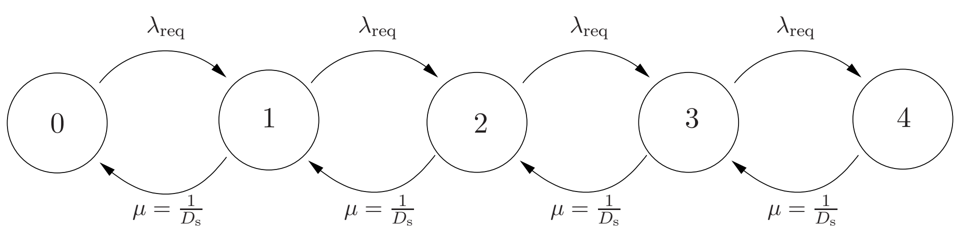

Figure 4: CTMC that models the operation of the Web Server

The local balance equations are as follows:

$$\pi_0\lambda_{\text{req}} = \pi_1\mu \tag{2}$$
$$\pi_1\lambda_{\text{req}} = \pi_2\mu \tag{3}$$
$$\pi_2\lambda_{\text{req}} = \pi_3\mu \tag{4}$$
$$\pi_3\lambda_{\text{req}} = \pi_4\mu \tag{5}$$
$$\tag{6}$$

We can write the probability of all the states in terms of π_0.

$$\pi_1 = \pi_0 \frac{\lambda_{\text{req}}}{\mu} \tag{7}$$

$$\pi_2 = \pi_1 \frac{\lambda_{\text{req}}}{\mu} = \pi_0 \left(\frac{\lambda_{\text{req}}}{\mu}\right)^2 \tag{8}$$

$$\pi_3 = \pi_2 \frac{\lambda_{\text{req}}}{\mu} = \pi_0 \left(\frac{\lambda_{\text{req}}}{\mu}\right)^3 \tag{9}$$

$$\pi_4 = \pi_3 \frac{\lambda_{\text{req}}}{\mu} = \pi_0 \left(\frac{\lambda_{\text{req}}}{\mu}\right)^4 \tag{10}$$

$$\tag{11}$$

Using the normalization condition, $\sum_{q=0}^{K} \pi_q = 1$, we can compute π_0

$$1 = \pi_0 + \pi_0 \left(\frac{\lambda_{\text{req}}}{\mu}\right) + \pi_0 \left(\frac{\lambda_{\text{req}}}{\mu}\right)^2 + \pi_0 \left(\frac{\lambda_{\text{req}}}{\mu}\right)^3 + \pi_0 \left(\frac{\lambda_{\text{req}}}{\mu}\right)^4 \tag{12}$$

and

$$\pi_0 = \frac{1}{1 + \left(\frac{\lambda_{\text{req}}}{\mu}\right) + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^2 + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^3 + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^4} = 0.075829 \tag{13}$$

Therefore, including also the numerical result in each case:

$$\pi_1 = \frac{\left(\frac{\lambda_{\text{req}}}{\mu}\right)}{1 + \left(\frac{\lambda_{\text{req}}}{\mu}\right) + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^2 + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^3 + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^4} = 0.113744 \tag{14}$$

$$\pi_2 = \frac{\left(\frac{\lambda_{\text{req}}}{\mu}\right)^2}{1 + \left(\frac{\lambda_{\text{req}}}{\mu}\right) + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^2 + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^3 + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^4} = 0.170616 \tag{15}$$

$$\pi_3 = \frac{\left(\frac{\lambda_{\text{req}}}{\mu}\right)^3}{1 + \left(\frac{\lambda_{\text{req}}}{\mu}\right) + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^2 + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^3 + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^4} = 0.255924 \tag{16}$$

$$\pi_4 = \frac{\left(\frac{\lambda_{\text{req}}}{\mu}\right)^4}{1 + \left(\frac{\lambda_{\text{req}}}{\mu}\right) + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^2 + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^3 + \left(\frac{\lambda_{\text{req}}}{\mu}\right)^4} = 0.383886 \tag{17}$$

$$\tag{18}$$

By using the PASTA property of Poisson arrivals, we can simply say that $P_{\text{b}} = \pi_4 = 0.383886$ and $P_{\text{b}} = \pi_0 = 0.075829$.

Respectively, for each value of λ_{req}, the P_{b} is: $4.3035 \cdot 10^{-4},\ 1.0371 \cdot 10^{-1},\ 3.8389 \cdot 10^{-1},\ 5.6536 \cdot 10^{-1},\ 6.6942 \cdot 10^{-1},\ 7.3432 \cdot 10^{-1}$. For the results, it can be clearly observed that the relation between P_{b} and λ_{req}. Increases of λ_{req} have different effects on P_{b} depending on the value of λ_{req}.