

# Network Engineering

## Lecture 3. Markovian processes + Discrete Time Markov Chains

Boris Bellalta  
boris.bellalta@upf.edu

# Markov process

A Markov process is a dependent and stationary stochastic process  $X(t)$  characterized by:

- A state space of  $X(t)$ , called  $\mathcal{X}$ .
- The time at which  $X(t)$  changes, which can be at specific time instants (time-discrete) or at any arbitrary time (time-continuous).
- A transition matrix, which is called  $\mathbf{P}$  if the Markov process is time-discrete, called **probability transition matrix**, or  $\mathbf{Q}$  if the Markov process is time-continuous, called **rate transition matrix**. In both cases, the transition matrix represents the possible transitions from any state  $i$  to any state  $j$ .
- The equilibrium distribution (if exists),  $\boldsymbol{\pi}$ .  $\longrightarrow$  where  $\pi_i = P(X = x_i)$

# Markov property

- Next value of the stochastic process depends only on the current value.

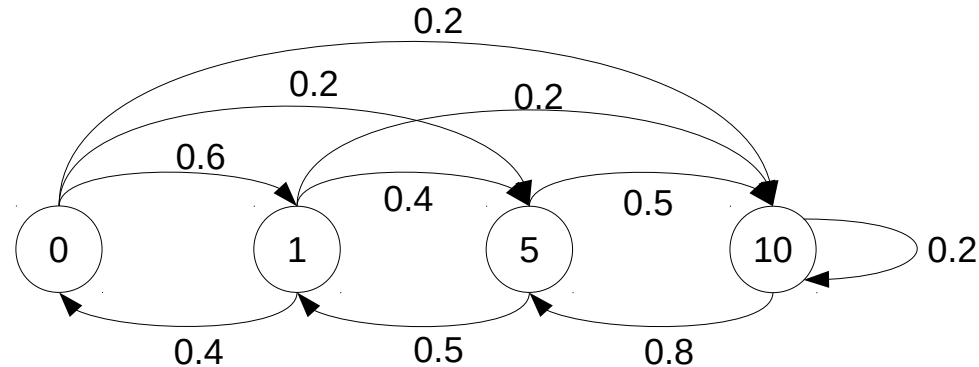
$$Pr\{X_n = x_n | X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \dots, X_0 = x_0\} = Pr\{X_n = x_n | X_{n-1} = x_{n-1}\} \quad (3.1)$$

# Markov chains

A Markov chain is a tool to represent a Markov process. It is composed by 'circles' and 'arrows', with circles representing  $\mathcal{X}$ , i.e., the possible outcomes of the random variable  $X(t)$ , and the arrows represent the transitions, i.e., the dependences between states.

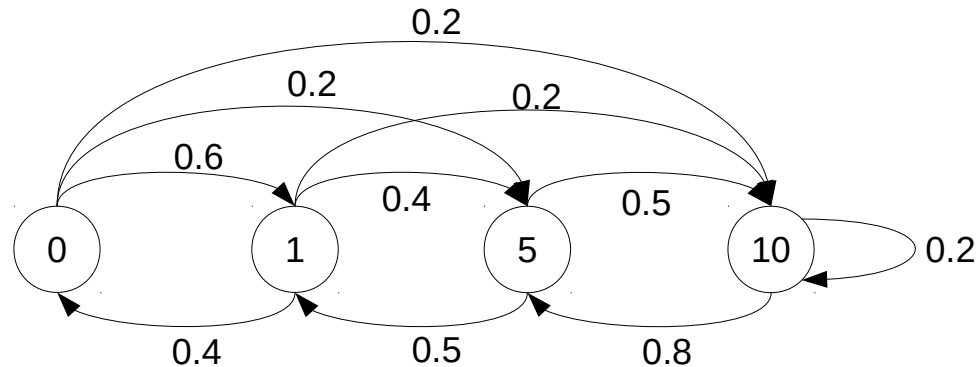
A Markov Chain, as a representation of a Markov process satisfies the Markov property, which states that next state only depends of the current state.

Example:



# Stationary Markov Process

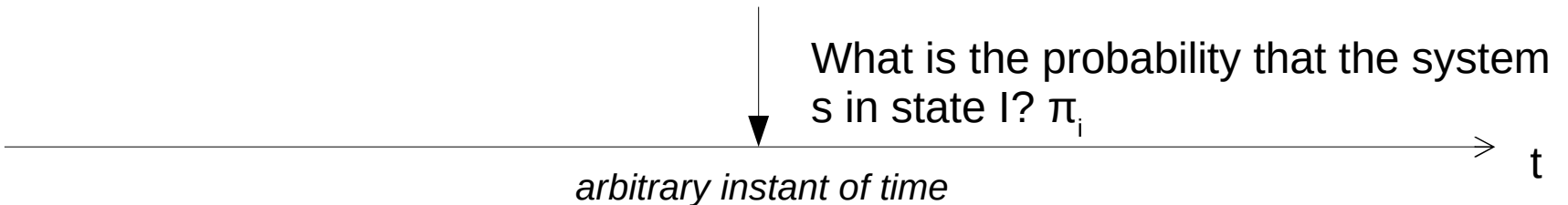
- The equilibrium distribution exists if:
  - Transition probabilities / rates are time independent
  - All states can be reached (the Markov chain is irreducible)
  - Once departing one state, we will return in the future to it (the Markov chain is positive recurrent)
  - There are different possible paths between any two states (the Markov chain is aperiodic)



# Solving a Markov chain

- Markov chains can be very useful to model real systems
- It is easy to fix the state space and the transition/rate matrices based on design and functional requirements of the real system
- So, the goal is to be able to obtain the equilibrium distribution

$\pi_i \rightarrow$  Probability that the system is  
in state 'i' at an arbitrary instant of time

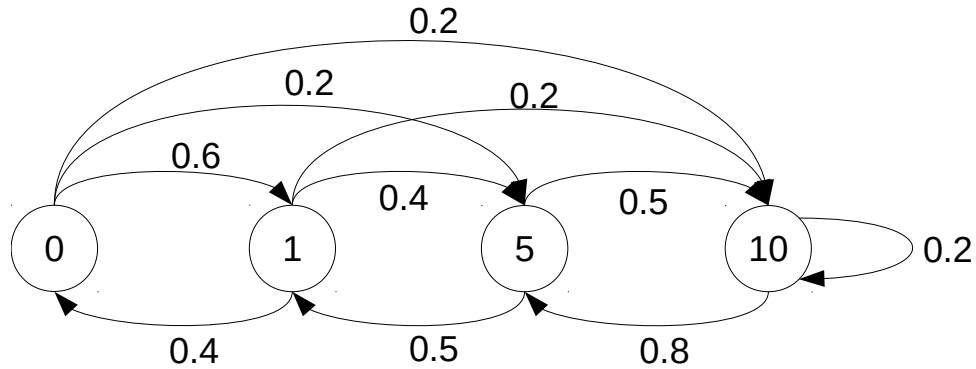


# How to find the state space and P

- It can be done from a data set
- Data set = [a,b,a,b,a,a,b,a,b,b,b,a,a,b,a,a,a,b]; N=18 samples, N-1 transitions
  - Values that appear: State space: {a,b}
  - Prob of the value t+1 after a value t, given we are at state 'i'
    - From a→b: 6 times →  $p_{a,b} = 6/10$
    - From a→a: 4 times →  $p_{a,a} = 4/10$
    - From b→a: 5 times →  $p_{b,a} = 5/7$
    - From b→b: 2 times →  $p_{b,b} = 2/7$

# Solving a Markov chain

- The equilibrium distribution is:
  - $\pi_0, \pi_1, \pi_5, \pi_{10}$





# Examples

- Weather model (sunny, cloudy, raining, etc.)
- Number of packets waiting for transmission in a buffer
- Number of bikes in a bike station
- People waiting to pay in a supermarket
- Number of active users in a WIFI network

# Discrete and Continuous Markov chains

Markov chains can be classified in two types depending on when  $X(t)$  changes:

- A transition between two states happens only at specific instants of time. We will refer to these Markov Chains as Discrete Time Markov Chains (DTMC). For DTMC the transition matrix is called  $\mathbf{P}$  and contains the probabilities to move from any state  $i$  to any state  $j$ .
- A transition between two states can be done at any arbitrary time. We will refer to these Markov Chains as Continuous Time Markov Chains (CTMC). For CTMC the transition matrix is called  $\mathbf{Q}$  and contains the rates to move from any state  $i$  to any state  $j$ .

# Equilibrium distribution

- The stationary probability distribution is also called equilibrium distribution.
- It represents the probability to find the Markov process in state 'i' when we observe it at an arbitrary instant of time.
- We use  $\pi_i = P(X=i)$  to represent the probability that the Markov chain is in state 'i'

$$\sum_{\forall i \in S} \pi_i = 1$$

# Discrete Time Markov chains (DTMC)

We will use DTMC to model systems in which  $X(t)$  only takes new values at specific time instants. Examples of cases that can be modelled by DTMC are

- Measuring the traffic on a link and checking its characteristics (i.e., number of packets) only at specific intervals.

Therefore, the main considerations for DTMCs are:

- In DTMC, transitions between states occur at specific time instants,  $t_1, t_2, t_3, t_4, t_5, t_6$ , etc. At each time instant, a transition from state  $i$  to state  $j$  happens with probability  $p_{i,j}$ .
- The transition probabilities from any state  $i$  to any state  $j$  define the matrix  $\mathbf{P}$ , called probability transition matrix.

# Global Balance Equations

In equilibrium, for any state  $i$ , the next condition is satisfied:

$$\pi_i \sum_{\forall j \neq i} p_{i,j} = \sum_{\forall j \neq i} \pi_j p_{j,i} \quad (3.4)$$

which are the global balance equations. If the transition probabilities  $p_{i,j}$  are known, the global balance equations provide as many equations as variables, and therefore, solving the system of equations, the equilibrium distribution of the Markov chain can be found.

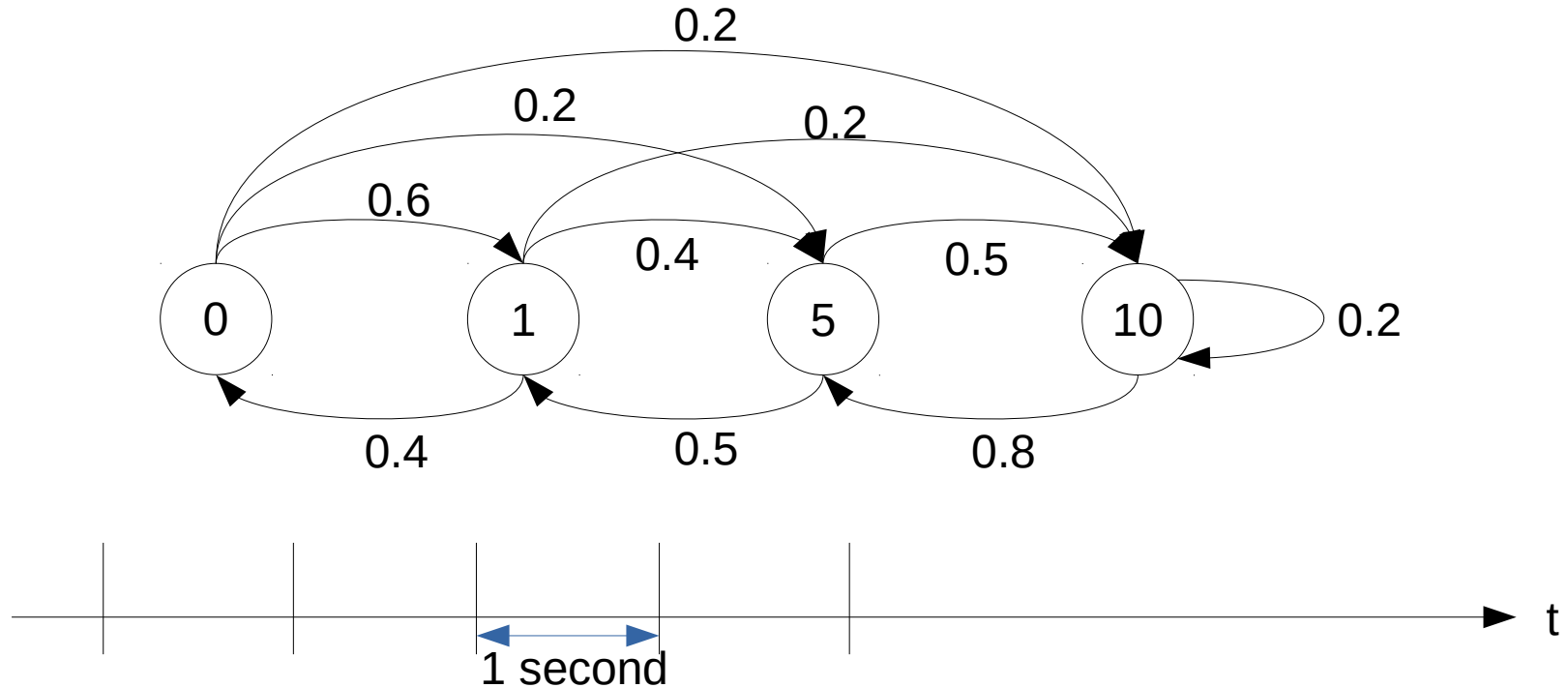
# Example

- The traffic load (Mbps) generated by a user changes every second. We are interested to know the probability the user is generating a given load at an arbitrary instant of time, and then calculate the expected traffic load, and its variance.
- To answer this question, we must calculate the equilibrium distribution of a DTMC (the system changes at specific time instants, every second) with the following state space and transition probabilities:
  - $X=\{0, 1, 5, 10\}$  Mbps;
  - $P=[0 \ 0.6 \ 0.2 \ 0.2; 0.4 \ 0 \ 0.4 \ 0.2; 0 \ 0.5 \ 0 \ 0.5; 0 \ 0 \ 0.8 \ 0.2;]$

$$P = \begin{bmatrix} 0 & 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.4 & 0.2 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.8 & 0.2 \end{bmatrix}$$

# Example

- We can describe the stochastic process using a Markov chain:



Example: We write the balance equations

- Balance Equations

$$\pi_0 = 0.4\pi_1$$

$$\pi_1 = 0.6\pi_0 + 0.5\pi_5$$

$$\pi_5 = 0.2\pi_0 + 0.4\pi_1 + 0.8\pi_{10}$$

$$0.8\pi_{10} = 0.2\pi_0 + 0.2\pi_1 + 0.5\pi_5$$

- We have a system of equations to solve

$$\pi_0 - 0.4\pi_1 = 0$$

$$-0.6\pi_0 + \pi_1 - 0.5\pi_5 = 0$$

$$-0.2\pi_0 - 0.4\pi_1 + \pi_5 - 0.8\pi_{10} = 0$$

$$-0.2\pi_0 - 0.2\pi_1 - 0.5\pi_5 + 0.8\pi_{10} = 0$$



Example: We solve the DTMC (i.e., find the stat. dist)

- Alternative way (Matlab code):
  - `[V D]=eigs(P.');` `st=V(:,1).';` `stat_dist = st./sum(st);`
- Stationary distribution
  - $\pi_0=0.0948$   $\pi_1=0.2370$   $\pi_5=0.3602$   $\pi_{10}=0.3081$

# Example

Using the stationary probability distribution, i.e.,  $\pi_x \forall x \in \mathcal{X}$ , we can compute other metrics of relevance, such as the expected traffic load:

$$E[X] = \sum_{\forall x \in \mathcal{X}} x\pi_x = 0\pi_0 + 1\pi_1 + 5\pi_5 + 10\pi_{10} = 5.1185 \text{ [Mbps]}$$

The second moment:

$$E[X^2] = \sum_{\forall x \in \mathcal{X}} x^2\pi_x = 0^2\pi_0 + 1^2\pi_1 + 5^2\pi_5 + 10^2\pi_{10} = 40.0474 \text{ [Mbps}^2\text{]}$$

And the variance:

$$V[X] = E[X^2] - E^2[X] = 13.8485 \text{ [Mbps}^2\text{]}$$

# Example

```
function ExampleUserLoadModel()
```

```
X=[0 1 5 10]; % Mbps
```

```
P=[0 0.6 0.2 0.2; 0.4 0 0.4 0.2; 0 0.5  
0 0.5; 0 0 0.8 0.2];
```

```
pi_dist = SolvingDTMC(P);
```

```
EX=X*pi_dist';
```

```
EX2 = (X.^2)*pi_dist';
```

```
VX = EX2 - EX^2;
```

```
disp([EX EX2 VX]);
```

```
function stat_dist = SolvingDTMC(P)
```

```
[V D]=eigs(P.')
```

```
st=V(:,1).';
```

```
stat_dist = st./sum(st);
```

## Example [Note]

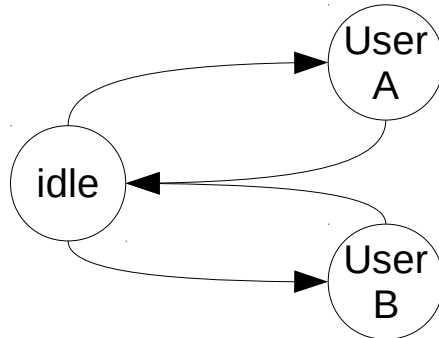
- In the 1<sup>st</sup> lecture we had a question: how to select the capacity of a link.
- For instance, let's assume that the criterion is the following:
  - $C = 4 \cdot$  The expected traffic load
- Given information of the process representing the load of a link, such as the state space, and  $P$ , which can be obtained by just observing the dynamics of the process, we can easily answer that question.

# Reversible Markov chains

- We say a Markov chain is reversible if the following condition is satisfied for all its states:

$$\pi_i P\{X_n = j | X_{n-1} = i\} = \pi_j P\{X_n = i | X_{n-1} = j\}$$

- In practice, we have **bidirectional transitions between two given states**.
- Example: Two users. Each user can be active or idle. When one user is active, the other is idle.



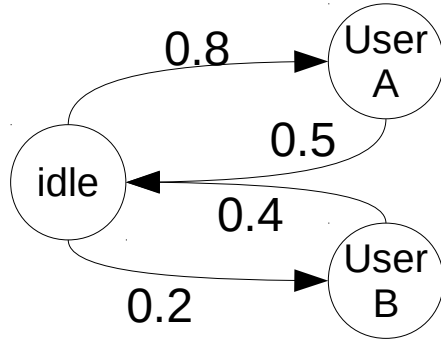
# Local Balance Equations

If the DTMC is reversible, the local balance equations can be formulated and used.

$$\pi_i p_{i,j} = \pi_j p_{j,i} \quad (3.5)$$

which together with the normalization condition,  $\sum_{\forall i} \pi_i = 1$ , allows to solve the system of equations and find the equilibrium distribution.

# Example



Self-transitions are not shown

The local balance equations are:

$$0.8\pi_0 = 0.5\pi_A$$

$$0.4\pi_0 = 0.2\pi_B$$

Since there are 3 unknown variables and two equations, we need to add also the following condition:

$$\pi_0 + \pi_A + \pi_B = 1$$

Arranging terms:

$$\pi_A = \frac{0.8}{0.5}\pi_0$$

$$\pi_B = \frac{0.4}{0.2}\pi_0$$

# Example

Substituting here:

$$\pi_0 + \frac{0.8}{0.5}\pi_0 + \frac{0.4}{0.2}\pi_0 = 1$$

Results in:

$$\pi_0 = \frac{1}{1 + \frac{0.8}{0.5} + \frac{0.4}{0.2}} = 0.2174$$

$$\pi_A = \frac{0.8}{0.5}\pi_0 = 0.3478$$

$$\pi_B = \frac{0.4}{0.2}\pi_0 = 0.4348$$



# Birth-and-death Markov processes

- Transitions between states are only between consecutive states (forward (births) or backward (deaths)).
  - Births increase by 1 the current value of the state space.
  - Deaths decrease by 1 the current value of the state space.
- They are reversible Markov process  $\rightarrow$  local balance equations
- We will use them to capture the buffer dynamics in network interfaces:
  - The state space of the random variable is the number of packets in the buffer.
  - Forward transitions: a packet arrival
  - Backward transitions: a packet departure

# Example of a birth and death Markovian process

- Consider a network interface where 'events' happen only at predefined time instants.
- The network interface has a single transmitter, and the maximum buffer size is  $Q=2$  packets.
- At each predefined time instant, only one event may occur:
  - It is an arrival with probability  $p$
  - It is a departure with probability  $q$
  - Nothing happens with probability  $1-p-q$ ,  $1-p$  or  $1-q$

