Approaching the Outage Probability of the Amplify-and-Forward Relay Fading Channel

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Abstract—We study coding techniques for the single-relay non-orthogonal amplify-and-forward half-duplex relay fading channel. Unlike the multiple-antenna case, we show that 2×2 rotations induce large gains in outage probability with no increase in decoding complexity under iterative probabilistic decoding. We compare rotated and unrotated turbo-coded schemes and show that they both perform close to their corresponding outage limits.

I. INTRODUCTION

Due to the nature of wireless channels, effects such as fading, shadowing, and interference from other transmitters can cause the channel quality to fluctuate during transmission. One major way to combat static fading is to provide diversity in either time, frequency, or space [1]. In [2], [3], the authors set up a framework for *cooperative communications*, where multiple terminals use the resources of each other to form a virtual antenna array that provides spatial diversity. These works triggered a flurry of research on cooperative communications, and since, multiple works have proposed communication schemes and analyzed outage probability in slowly varying fading environments [4], [5], [6], [7], [8]. The main protocols that have been proposed are the *amplify-and-forward*, where the relay only amplifies the signal received from the source, before transmitting it to the destination, and the *decode-and-forward*, where the relay decodes the received signal before transmitting it to the destination. A number of recent works consider turbo coding for multiple scenarios, including full-duplex, decode-and-forward, orthogonal amplify-and-forward and distributed coding [9], [10], [11], [12], [13].

In this work, we study the performance of distributed coded modulation schemes in the single-relay non-orthogonal amplify-and-forward channel. We focus on the outage probability [14] of the channel with discrete input constellations. As opposed to the multiple-antenna case, we show that full-diversity rotations (see e.g. [15] and references therein) provide large performance gains without any increase in the overall decoding complexity of the system. We illustrate these effects by means of a turbo-coded [16] bit-interleaved coded modulation (BICM) [17] scheme that closely approaches the outage probability of the channel.

II. SYSTEM MODEL

We consider the amplify-and-forward fading relay channel. We impose the half-duplex constraint, for which terminals cannot transmit and receive signals in the same time slot. We consider the TDMA-based Protocol I from [7]: during the first time slot, the source terminal broadcasts a signal to the relay and destination terminals, the relay is silent. In the second time slot, both the relay and source terminals communicate with the destination (see [7] for details). This protocol is also known as non-orthogonal amplify-and-forward (NAF) protocol. According to the NAF protocol, the received signals at relay and destination are given by

$$y_r[2k] = \sqrt{\mathcal{E}_s^{(e)}} \, h_{sr} \, x[2k] + \eta_r[2k] \tag{1}$$

$$y_d[2k] = \sqrt{\mathcal{E}_s^{(e)}} \, h_{sd} \, x[2k] + \eta_d[2k] \tag{2}$$

$$y_d[2k+1] = \sqrt{\mathcal{E}_r} \, \beta h_{rd} \, y_r[2k]$$
$$+ \sqrt{\mathcal{E}_s^{(o)}} \, h_{sd} \, x[2k+1] + \eta_d[2k+1] \tag{3}$$

for $k = 0, \ldots, L-1$, where $y_r[2k], y_d[2k] \in \mathbb{C}$ are the received signals at the relay and destination respectively at even time slots, $y_d[2k+1] \in \mathbb{C}$ is the received signal at the destination at odd time slots, $x[2k], x[2k+1] \in \mathbb{C}$ are the transmitted symbols at even and odd time slots respectively, $\eta_r[2k], \eta_d[2k], \eta_d[2k+1] \sim \mathcal{N}_\mathbb{C}(0, N_0)$ are the corresponding circularly symmetric complex Gaussian noise samples at the relay and destination, $h_{sr}, h_{sd}, h_{rd} \sim \mathcal{N}_\mathbb{C}(0, 1)$ are the complex fading coefficients corresponding to *source-relay*, *source-destination*, and *relay-destination* links respectively, and

$$\beta = \left(\mathcal{E}_s^{(e)} |h_{sr}|^2 + N_0 \right)^{-\frac{1}{2}} \tag{4}$$

is the energy normalization coefficient at the relay [7]. Let $\mathcal{E}_s^{(e)}$ and $\mathcal{E}_s^{(o)}$ denote the energies transmitted by the source at even and odd time instances respectively, while $\mathcal{E}_r$ is the energy transmitted by the relay. We assume that the signal constellation is normalized in energy, namely $\mathbb{E}[|x|^2] = 1$. We also assume that the channel coefficients are perfectly known at the receiver and that they remain constant for the transmission of one frame (i.e., for $k = 0, \ldots, L-1$) but change independently from frame to frame (quasi-static assumption). From the NAF model described above, we obtain that the time duration corresponding to the transmission of a frame is $2L$. By inserting (1) into (3) we obtain that

$$y_d[2k+1] = \sqrt{\mathcal{E}_s^{(e)} \mathcal{E}_r} \, \beta h_{rd} h_{sr} \, x[2k] + \sqrt{\mathcal{E}_s^{(o)}} h_{sd} \, x[2k+1]$$
$$+ \sqrt{\mathcal{E}_r} \, \beta h_{rd} \, \eta_r[2k] + \eta_d[2k+1]. \tag{5}$$

Conditioned on a perfect knowledge of the fading coefficients at the receiver, the noise term in (5) has a Gaussian distribution

with zero mean and variance $\sigma^2 = N_0(1 + \mathcal{E}_r\beta^2|h_{rd}|^2)$. Therefore, by multiplying $y_d[2k+1]$ by

$$\alpha = \left(1 + \mathcal{E}_r\beta^2|h_{rd}|^2\right)^{-\frac{1}{2}} \qquad (6)$$

we obtain the equivalent row vector model

$$\widetilde{\mathbf{y}}_d[k] = \mathbf{x}[k]\,\mathbf{H} + \widetilde{\boldsymbol{\eta}}[k] \qquad (7)$$

where

$$\mathbf{H} = \begin{pmatrix} \sqrt{\mathcal{E}_s^{(e)}}\,h_{sd} & \sqrt{\mathcal{E}_s^{(e)}\mathcal{E}_r}\;\alpha\beta h_{rd}h_{sr} \\[2mm] 0 & \sqrt{\mathcal{E}_s^{(o)}}\;\alpha h_{sd} \end{pmatrix} \qquad (8)$$

and $\widetilde{\mathbf{y}}_d[k] = \left(y_d[2k]\;,\;\alpha y_d[2k+1]\right) \in \mathbb{C}^2$, $\mathbf{x}[k] = \left(x[2k]\;,\;x[2k+1]\right) \in \mathcal{X} \subset \mathbb{C}^2$, with $\mathcal{X}$ being the bidimensional signal constellation, and $\widetilde{\boldsymbol{\eta}}[k] \in \mathbb{C}^2$ is a circularly symmetric complex Gaussian noise vector with entries $\sim \mathcal{N}_{\mathbb{C}}(0, N_0)$, for $k = 0, \ldots, L-1$. Based on (7), the channel transition probabilities conditioned on the channel realization are

$$p\left(\widetilde{\mathbf{y}}_d[k]|\mathbf{x}[k], \mathbf{H}\right) = \frac{1}{(\pi N_0)^2} \exp\left(-\frac{\left\|\widetilde{\mathbf{y}}_d[k] - \mathbf{x}[k]\mathbf{H}\right\|^2}{N_0}\right) \qquad (9)$$

III. OUTAGE PROBABILITY AND ROTATIONS

The Shannon capacity of the channel described by (7) is zero, since for sufficiently large frame length, the word error probability of any coding scheme is lowerbounded by the *information outage probability* [14]

$$P_{\text{out}}(R) = \Pr\left(\frac{1}{2}I(\mathbf{x}; \widetilde{\mathbf{y}}_d|\mathbf{H}) < R\right) \qquad (10)$$

where R is the target information rate, and $I(\mathbf{x}; \widetilde{\mathbf{y}}_d|\mathbf{H})$ is the input-output instantaneous mutual information of the channel (7) for a fixed channel realization $\mathbf{H}$. The factor $\frac{1}{2}$ comes from the fact that one channel use of the equivalent channel (7) corresponds to 2 temporal channel uses, one per slot. Minimum outage is achieved with Gaussian inputs, for which

$$I(\mathbf{x}; \widetilde{\mathbf{y}}_d|\mathbf{H}) = \log_2 \det\left(\mathbf{I}_2 + \frac{1}{N_0}\mathbf{H}\mathbf{H}^\dagger\right). \qquad (11)$$

In practice, Gaussian codebooks are not feasible, and we usually resort to discrete signal constellations. In this case, assuming uniformly distributed inputs over $\mathcal{X}$ we have that

$$I(\mathbf{x}; \widetilde{\mathbf{y}}_d|\mathbf{H}) = \log_2|\mathcal{X}| - \mathbb{E}\left[\log_2 \frac{\sum_{\mathbf{x}' \in \mathcal{X}} p(\widetilde{\mathbf{y}}_d|\mathbf{x}', \mathbf{H})}{p(\widetilde{\mathbf{y}}_d|\mathbf{x}, \mathbf{H})}\right] \qquad (12)$$

where the expectation is over the joint distribution $p(\mathbf{x}, \widetilde{\mathbf{y}}_d|\mathbf{H})$.

When quadrature-amplitude modulation (QAM) is used, the signal constellation is $\mathcal{X} = \mathcal{X}_{\text{QAM}} \times \mathcal{X}_{\text{QAM}}$ where $\mathcal{X}_{\text{QAM}} \in \mathbb{C}$ is a standard QAM constellation of size $|\mathcal{X}_{\text{QAM}}| = M$. In some cases, it can be beneficial to precode the M-QAM symbols with a unit rate rotation $\boldsymbol{\Phi} \in \mathbb{C}^{2\times2}$ [15], [18]. In this case, the constellation $\mathcal{X}$ is a higher-order bidimensional complex constellation, and detection has to be performed in $\mathbb{C}^2$. In order to compute the outage probability (10) with rotated QAM constellations, we compute (12) bearing in mind that now input vectors $\mathbf{x} \in \mathcal{X}$ belong now to the rotated constellation. This is equivalent to letting $\mathbf{x} \in \mathcal{X}_{\text{QAM}} \times \mathcal{X}_{\text{QAM}}$ and replacing $\mathbf{H}$ by $\boldsymbol{\Phi}\mathbf{H}$. Remark that using a 2×2 rotation entails no increase
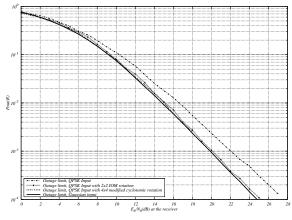

Fig. 1. Outage probability for QPSK and $R = 1$ bit per channel use.

in decoding complexity using probabilistic iterative decoding, since in order to compute the channel transition probabilities in (9), the decoder uses a list of candidates of size $|\mathcal{X}| = M^2$, with and without rotation. This stands in stark contrast to the case of $n_T \times n_R$ multiple-antenna channels with i.i.d. fading, where rotations are of dimension $n_T^2 \times n_T^2$ (see e.g., [18], [19]) and hence increase the decoding complexity with respect to the unrotated system. In our case, the zero entry in the equivalent channel matrix (8) enables decoding with the same complexity of the unrotated system. In this work, we design the rotation matrix $\boldsymbol{\Phi}$ that minimizes the outage probability, namely

$$\boldsymbol{\Phi}_{\text{IOM}} = \arg\min_{\boldsymbol{\Phi} \in \mathcal{G}} P_{\text{out}}(R) \qquad (13)$$

following the method outlined in [20], where $\mathcal{G} \subseteq \mathbb{C}^{2\times2}$ is the set of complex unitary matrices and IOM stands for information outage minimization.

Figure 1 shows $P_{\text{out}}(R)$ with Gaussian input, QPSK input with no rotation, IOM rotation of dimensions 2×2 and modified cyclotomic rotation [18] of dimensions 4×4. In numerical examples in this letter, we have chosen $\mathcal{E}_s^{(o)} = \mathcal{E}_r = \frac{\mathcal{E}_s^{(e)}}{2}$. Remark that the model (7), the transition probabilities (9), outage probabilities (10) and coding schemes remain valid for general power allocation rules [7]. As we observe, the inclusion of the rotation gives a significant gain. We also see that 4×4 rotations are not needed, as 2×2 rotations are sufficient. Therefore, coded modulation schemes constructed using 2×2 rotations can achieve a remarkable gain over unrotated modulations, *with no increase in decoding complexity*.

IV. CODE CONSTRUCTION AND EXAMPLES

We consider coded modulation schemes $\mathcal{M} \subset \mathcal{X}^L \subset \mathbb{C}^{2L}$ of rate R. We denote the codewords of $\mathcal{M}$ by $\mathbf{X} \in \mathcal{M}$, $\mathbf{X} = \left(\mathbf{x}[0], \ldots, \mathbf{x}[L-1]\right)$. We consider that $\mathcal{M}$ is a BICM scheme [17] over $\mathcal{X}$, that uses a binary parallel turbo code of rate R_c and an interleaver of size N [16]. The component encoders are assumed to be identical recursive systematic convolutional (RSC) codes. Under this setting, we have that $L = \frac{N}{2R_c \log_2 M}$. BICM symbols are then rotated by a unit symbol rate rotation matrix $\boldsymbol{\Phi}$. Since the rotation has unit rate, the overall transmission rate in bits per channel use is $R = \frac{N}{2L} = R_c \log_2 M$. The presence of a rotation guarantees full diversity for any $0 < R_c < 1$.

We will compare the above system with a dual scheme that uses no rotations. In this case, the BICM codewords
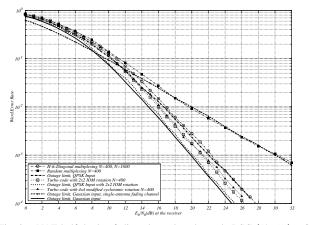
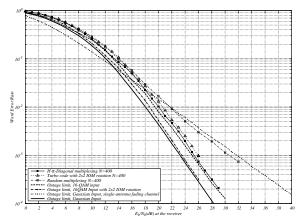Fig. 2.   NAF half-duplex fading channel, turbo code RSC $(17, 15)_8$, QPSK.

Fig. 3.   NAF half-duplex fading channel, turbo code RSC $(17, 15)_8$, 16QAM.

of length $2L$ are then reordered, according to the channel multiplexer pattern [21]. A channel multiplexer distributes the turbo-coded bits to the time slots, in such a way that full diversity and maximum coding gain are always guaranteed. In our simulations we use the h-π-diagonal multiplexer from [21], which was proved to be the most powerful multiplexer for parallel turbo codes of rate $R_c = \frac{1}{2}$. The achievable diversity with this scheme relies on the properties of the code and the channel multiplexer, and is bounded by the Singleton bound [22], [23]. This implies that the largest R_c that can be transmitted with diversity 2 is $R_c = \frac{1}{2}$.

Figures 2 and 3 show the word (or frame) error rate simulations for the aforementioned schemes with QPSK and 16-QAM using a parallel turbo code of rate $R_c = \frac{1}{2}$ with two identical RSC component encoders with generators $(17, 15)_8$ and pseudo-random interleaver of size N. As we see, the benefits of the rotated construction are clear. Furthermore, we observe that the unrotated system with the h-π-diagonal multiplexer performs within tenths of a dB of its corresponding outage limit, and that when it is used with random multiplexing a diversity loss is incurred. We note that due to the suboptimality of BICM with 16-QAM, the rotated and unrotated performances with the h-π-diagonal multiplexer are very similar. Note also that the multiplexer is designed to lead the turbo code to optimal performance, while the rotation is designed to minimize the outage probability, independently from the code structure. As shown in [21], [23] the error probability is independent of the interleaver length N.

V. CONCLUSIONS

We have studied coding schemes for the single-relay non-orthogonal amplify-and-forward half-duplex fading channel. We have shown that rotations provide a large performance gain over unrotated constellations with no extra cost in decoding complexity. We have compared the performance of rotated and unrotated turbo-coded BICM schemes and have shown that they both perform very close to their respective outage limits.

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