Optimal Power Control for LDPC Codes in Block-Fading Channels

Gottfried Lechner, Khoa D. Nguyen, Albert Guillén i Fàbregas
and Lars K. Rasmussen

***Abstract*—We study the error probability of LDPC codes in delay-limited block-fading channels with channel state information (CSI) at the transmitter and the receiver. We derive the optimal power allocation algorithms for LDPC codes with specific degree distributions using multi-edge-type density evolution error boundaries. The resulting performance approaches the outage probability for a number of power constraints. Furthermore, we adapt the algorithm for finite-length codes and show that the proposed algorithm enables gains larger than 10 dB over uniform power allocation. The method is valid for general, possibly correlated, fading distributions. This represents the first analysis of specific LDPC codes over block-fading channels with full CSI.**

I. Introduction

The block-fading channel [1], [2] has attracted attention over the past decade as a conveniently simple channel model that captures fundamental characteristics of practical wireless communications systems. The most popular example is an orthogonal frequency division multiplex (OFDM) system, where it is common to assume that the fading coefficient of a single frequency band is constant over a finite number of OFDM symbols. Other examples are frequency-hopping in GSM/EDGE systems, free-space optical systems [3], and hybrid optical-radio frequency systems [4] where the links can be modeled as (possibly correlated) slow-varying fading channels.

In practice, the number of independent fading blocks is predominantly quite limited. For example, in OFDM-based systems, there is a significant degree of frequency correlation, which implies that only groups of subcarriers can be considered (and treated, for code design purposes) as independent. Furthermore, for a large number of fading blocks it is usually not desirable to construct full diversity codes since the rate of the code is always upper bounded by $1/B$ where B is the number of blocks. Therefore, for practical reasons, we focus our examples on relevant scenarios where the number of fading blocks is relatively small.

G. Lechner and K. D. Nguyen are with the Institute for Telecommunications Research, University of South Australia, Mawson Lakes Boulevard, Mawson Lakes 5095, South Australia, Australia (e-mail: gottfried.lechner@unisa.edu.au, khoa.nguyen@unisa.edu.au).

A. Guillén i Fàbregas is with the Department of Engineering, University of Cambridge, Trumpington Street, CB2 1PZ, Cambridge, UK (e-mail: guillen@ieee.org).

L. K. Rasmussen is with the Communication Theory Laboratory, School of Electrical Engineering, and the ACCESS Linnaeus Center, Royal Institute of Technology, Osquldas väg 10, SE-100 44 Stockholm, Sweden (e-mail: Lars.Rasmussen@ieee.org).

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For delay-sensitive applications, the block-fading channel is delay-limited, implying that each codeword is transmitted over a finite number of fading blocks. An outage occurs when the instantaneous mutual information is less than the target transmission rate [1], [2]. It has been shown [5], that there exist codes whose error probability is arbitrarily close to the outage probability for large block lengths; conversely, the word error probability of any code is lower bounded by the outage probability for sufficiently long block lengths. Therefore, the outage probability is the natural fundamental limit of the channel.

An important characteristic of the outage probability is its SNR exponent or diversity gain. The outage SNR exponent is the asymptotic (for large SNR) slope of the outage probability as a function of the SNR, in a log-log scale. For discrete, fixed transmission alphabets, such as QAM signal constellations, the optimal SNR exponent is determined by the transmission and channel parameters through the Singleton bound [5]–[7]. Practical coding schemes based on powerful turbo-like codes [8], [9] and low-density parity-check (LDPC) codes [10], [11] have been proposed, and demonstrated to achieve full diversity.

When channel state information (CSI) is available only at the receiver, the available transmission power is uniformly distributed across fading blocks. In case CSI is available at the transmitter, the outage probability can be minimized through power allocation, i.e., the transmit power is allocated across blocks as a function of the channel realization subject to certain constraints. Optimal power allocation for delay-limited block-fading channels using continuous or discrete symbol alphabets has been studied in [2], [4], [12]–[17], where short-term, long-term, and short-to-long-term power ratio (SLPR) constraints have been of particular interest. In some cases, all outages can be removed, showing dramatic performance gains with respect to uniform power allocation.

The region of channel gain realizations causing an outage event has previously been characterized by an *outage boundary* [9], [10], [18]. Furthermore, a similar *error boundary* can be determined for practical coding schemes, providing a qualitative measure of the gap to the outage limit [9], [10], [18], [19].

The aim of this paper is to study the performance of LDPC code ensembles over the block-fading channel with power allocation. We apply multi-edge-type density evolution to completely characterize the code ensemble by its error boundary. Power allocation schemes arising from various power constraints are easily incorporated into the framework. Optimal power allocation algorithms for infinite-length codes operate exactly at the error boundary of the code, which is the equivalent of the threshold for ergodic channels. In contrast to the asymptotic case of infinite block length, finite-length codes do not show a *threshold effect*. Therefore, we derive new power allocation algorithms that can be applied to finite-length codes. Our modified algorithm for finite-length codes leads to performance gains of more than 10 dB with respect to uniform power allocation for lengths as short as 200. Although we restrict ourselves to binary inputs, the results can be easily generalized to arbitrary input constellations.

The remainder of this paper is organized as follows. The system model and the power allocation schemes are introduced in Section II. In Section III, we define outage and error regions which allow for a unified treatment of code ensembles and the computation of outage/error probabilities. These regions are used in Section IV to derive results for systems with power allocation. Power allocation for finite-length codes is presented in Section V and concluding remarks can be found in Section VI.

II. System Model and Power Allocation

We consider transmission of codewords over B channels (blocks), where the channel coefficients α_b, $b = 1, \ldots, B$ are constant and chosen independently from a known distribution. Let $\boldsymbol{x}_b$ denote the input of channel b consisting of the elements $x_{b,\ell}$, $\ell = 1, \ldots, N/B$, where N (an integer multiple of B) denotes the overall codeword length. We assume that $x_{b,\ell}$ are chosen with equal probability from $\{+\sqrt{\gamma_b}, -\sqrt{\gamma_b}\}$ and therefore $\mathbb{E}\left[x_{b,\ell}^2\right] = \gamma_b$. The corresponding input-output relationship of the channel is given by

$$\boldsymbol{y}_b = \alpha_b \boldsymbol{x}_b + \boldsymbol{z}_b, \tag{1}$$

where $\boldsymbol{z}_b$ denotes zero-mean white Gaussian noise with variance $\sigma^2 = 1$. For simplicity, we assume that the fading coefficients α_b are Rayleigh distributed with $\mathbb{E}\left[\alpha_b^2\right] = 1$. However, our results hold for a wide variety of fading distributions. The average received SNR on block b is therefore γ_b.

When CSI is only available at the receiver, the transmit power is distributed uniformly across the fading blocks, i.e., $\gamma_b = P_{\text{x}}$ for $b = 1, \ldots, B$, where P_{x} denotes the average transmit power. In the case of CSI at the transmitter, power allocation subject to short-term or long-term constraints can be applied. For a short-term constraint, we have that

$$\langle \boldsymbol{\gamma} \rangle \leq P_{\text{ST}}, \tag{2}$$

where $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_B)$ and

$$\langle \boldsymbol{\gamma} \rangle \triangleq \frac{1}{B} \sum_{b=1}^{B} \gamma_b \tag{3}$$

denotes the arithmetic mean of the elements of the vector $\boldsymbol{\gamma}$. For a long-term constraint, the *expected* power per codeword is upper bounded by P_{LT}

$$\mathbb{E}\left[\langle \boldsymbol{\gamma} \rangle\right] \leq P_{\text{LT}}. \tag{4}$$

An example of a combination of a short- and long-term power constraint is the case of a short-term to long-term power ratio (SLPR) where

$$\frac{P_{\text{ST}}}{P_{\text{LT}}} = \frac{\langle \boldsymbol{\gamma} \rangle}{\mathbb{E}\left[\langle \boldsymbol{\gamma} \rangle\right]} \leq \text{SLPR}. \tag{5}$$

III. Outage and Error Regions

Let $I_B(\boldsymbol{\alpha}, \boldsymbol{\gamma})$ denote the *instantaneous* mutual information between the input and output vector of the block-fading channel normalized by the codeword length

$$I_B(\boldsymbol{\alpha}, \boldsymbol{\gamma}) = \frac{1}{B} \sum_{b=1}^{B} I_b(\gamma_b \alpha_b^2), \tag{6}$$

where the vector of channel coefficients is $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_B)$ and $I_b(\gamma_b \alpha_b^2)$ is the mutual information of an AWGN channel with binary inputs and SNR $\gamma_b \alpha_b^2$.

Following [12], we define the *outage region* as the set of all realizations of the channel coefficients where the channel does not support the transmission rate r of the code

$$\mathcal{R}_{\text{out}}(\boldsymbol{\gamma}; r) = \left\{ \boldsymbol{\alpha} \in \mathbb{R}_+^B : I_B(\boldsymbol{\alpha}, \boldsymbol{\gamma}) < r \right\}, \tag{7}$$

and the boundary $\mathcal{B}_{\text{out}}(\boldsymbol{\gamma}, R)$ of this outage region is given by

$$\mathcal{B}_{\text{out}}(\boldsymbol{\gamma}; r) = \left\{ \boldsymbol{\alpha} \in \mathbb{R}_+^B : I_B(\boldsymbol{\alpha}, \boldsymbol{\gamma}) = r \right\}. \tag{8}$$

The outage probability is obtained by integrating the density function of the fading parameters over the outage region

$$P_{\text{out}}(\boldsymbol{\gamma}; r) = \int_{\boldsymbol{\alpha} \in \mathcal{R}_{\text{out}}(\boldsymbol{\gamma}; r)} p(\boldsymbol{\alpha}) d\boldsymbol{\alpha}. \tag{9}$$

We define the outage diversity as

$$d_{\text{out}} = -\lim_{P \to \infty} \frac{\log P_{\text{out}}(\boldsymbol{\gamma}; r)}{\log P}, \tag{10}$$

where P denotes the average power. We will denote by $d_{\text{out,ST}}$ and $d_{\text{out,LT}}$ the outage diversity with short- and long-term power constraints, respectively.

In the same way, we compute the word error probability of an LDPC code ensemble by replacing the *outage region* by the *error region* of the code. This error region, i.e., the region of channel realizations for which the decoder is unable to decode successfully, can be computed using density evolution for multi-edge-type codes [20].

It is important to note that density evolution allows the computation of the bit error rate but we are interested in the computation of the word error rate. In the case where all variable node degrees are larger than two, it has been shown in [21] that the iterative decoding thresholds of bit and block error probability coincide. Jin and Richardson [22] extended this result to the case where degree two variable nodes exist but possess a certain structure. To be precise, the degree two variable nodes have to be arranged in a chain which ensures that the number of nodes in the neighborhood of a variable node in the degree two subgraph grows at most linearly in the distance from the node.

Let LDPC(L, R) define an LDPC ensemble in the context of the multi-edge-type framework (for a detailed description of multi-edge-type density evolution we refer to [20, Chapter 7]).

The multinomials L and R are associated with variable nodes and check nodes, respectively. Furthermore, assume that the ensemble satisfies the constraints for the degree two variable nodes as stated above, i.e., the iterative decoding threshold for bit and word error probabilities are identical. We define the error region of an LDPC(L, R) ensemble as

$$\mathcal{R}_{\text{err}}(\boldsymbol{\gamma}; L, R) = \left\{ \boldsymbol{\alpha} \in \mathbb{R}_+^B : \lim_{i \to \infty} P_{\text{b}}(i) > 0 \right\}, \qquad (11)$$

where i denotes the number of iterations and $P_{\text{b}}(i)$ denotes the bit error probability after i iterations. In other words, the region $\mathcal{R}_{\text{err}}$ consists of all realizations of the fading coefficients where the iterative decoder is not able to converge to zero errors. In contrast to the outage region, which depends only on the code rate r, the error region of an LDPC ensemble depends on the multi-edge degree distributions L and R (which in turn define the code rate).

Similarly to the outage probability, the word error probability of an LDPC ensemble is given by the integral of the distribution of the fading coefficients over the error region

$$P_{\text{err}}(\boldsymbol{\gamma}; L, R) = \int_{\boldsymbol{\alpha} \in \mathcal{R}_{\text{err}}(\boldsymbol{\gamma}; L, R)} p(\boldsymbol{\alpha}) d\boldsymbol{\alpha}. \qquad (12)$$

We similarly define the code diversity as

$$d_{\text{c}} = -\lim_{P \to \infty} \frac{\log P_{\text{err}}(\boldsymbol{\gamma}; L, R)}{\log P}. \qquad (13)$$

We will denote by $d_{\text{c,ST}}$ and $d_{\text{c,LT}}$ the code diversity with short- and long-term power constraints, respectively.

As an example for the rest of this paper, we consider two blocks ($B = 2$) and a full-diversity ($d_{\text{c,ST}} = 2$) root-LDPC code [10] which is defined by the parity-check matrix

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{H}_1 & \boldsymbol{H}_2 \\ \boldsymbol{H}_3 & \boldsymbol{H}_4 & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}, \qquad (14)$$

where all sub-matrices $\boldsymbol{H}_j$ ($j = 1, \ldots, 4$) have variable and check node degree of two and $\boldsymbol{I}$ and $\boldsymbol{0}$ denote the identity and zero matrix, respectively. Therefore, the overall code of rate $r = 1/2$ consists of variable nodes of degree two and three and check nodes of degree five. The edges in the sub-matrices $\boldsymbol{H}_2$ and $\boldsymbol{H}_4$ are placed such that these variable nodes form a chain, therefore satisfying the condition in [22] for equal bit and block error probability thresholds. This particular structure allows the code to achieve full diversity [10].

Figure 1 shows the outage boundary for $r = 1/2$ and the error boundary for this LDPC ensemble. The gap between the boundaries corresponds to the gap between outage probability and error rate of the LDPC code.

IV. Power Allocation for Infinite-Length Codes

Depending on the system parameters and the constraints on the transmit power, optimal power allocation can be determined to minimize the outage/error probability, given that CSI is available at the transmitter [2], [4], [12]–[16]. In this section we further develop the outage/error-region framework to deal with optimal power allocation based on various power constraints. In particular, we derive expressions to numerically evaluate the effective average transmit power and corresponding word error probability, and compare the performance of root-LDPC codes with optimal power allocation based on short-term, long-term, and short-to-long-term power ratio constraints.

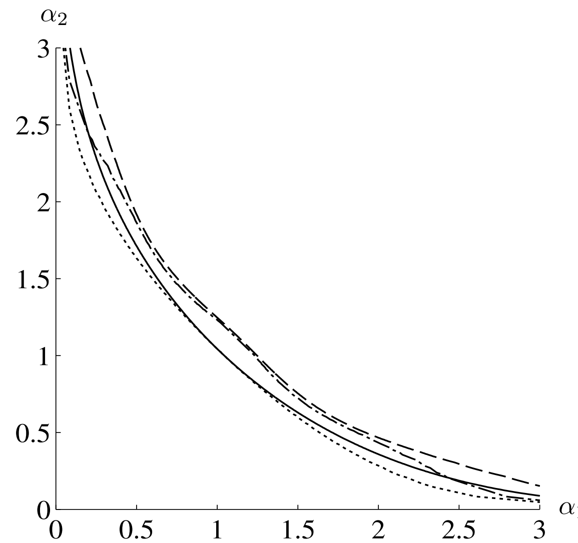

Fig. 1. Outage boundary (solid line) and error boundary for root-LDPC code with uniform power allocation (dashed line) at SNR = 0 dB and rate $r = 1/2$. The modified boundaries for the outage and error region due to power allocation are shown as dotted and dash-dotted lines, respectively.

Instead of allocating the same transmit power P_{x} on each block, the transmitter allocates power γ_b on block b, where $\gamma_b \geq 0$, $b = 1, \ldots, B$ are chosen such that the required mean power $\langle \boldsymbol{\gamma} \rangle$ for successful transmission is minimized. For the computation of the outage probability, successful transmission requires that the average mutual information is larger than the code rate r, whereas for the computation of the error rate, convergence of the LDPC decoder to zero error probability is required. By requiring that the realization of the channel coefficients $\boldsymbol{\alpha}$ does not belong to the regions defined in (7) and (11), we can study both cases and formulate the optimization problem as

$$\boldsymbol{\gamma}^*(\boldsymbol{\alpha}) = \arg \min_{\boldsymbol{\gamma} \in \mathbb{R}_+^B} \langle \boldsymbol{\gamma} \rangle \qquad \text{s.t. } \boldsymbol{\alpha} \notin \mathcal{R}(\boldsymbol{\gamma}). \qquad (15)$$

where $\mathcal{R}(\boldsymbol{\gamma})$ denotes $\mathcal{R}_{\text{out}}(\boldsymbol{\gamma}; r)$ or $\mathcal{R}_{\text{err}}(\boldsymbol{\gamma}; L, R)$, respectively. It has been shown in [23] that the solution of (15) is optimal in the sense that it minimizes the outage/error probability. This is because the above problem and the maximum mutual information subject to a power constraint problem are equivalent in terms of outage probability. The optimization of the power allocation algorithm in (15) depends only on the region of interest allowing outage and error regions to be treated in exactly the same way. Power allocation effectively modifies the outage/error region since it allows successful transmission for channel realizations which would cause an error for uniform power allocation. For the example of the previous section these modified outage/error regions are shown in Figure 1.

The optimization problem in (15) can be solved in an efficient way. For a given $\boldsymbol{\alpha}$, let $\mathcal{R}_\gamma$ denote the region of all power-allocation vectors $\boldsymbol{\gamma}$ which lead to an outage

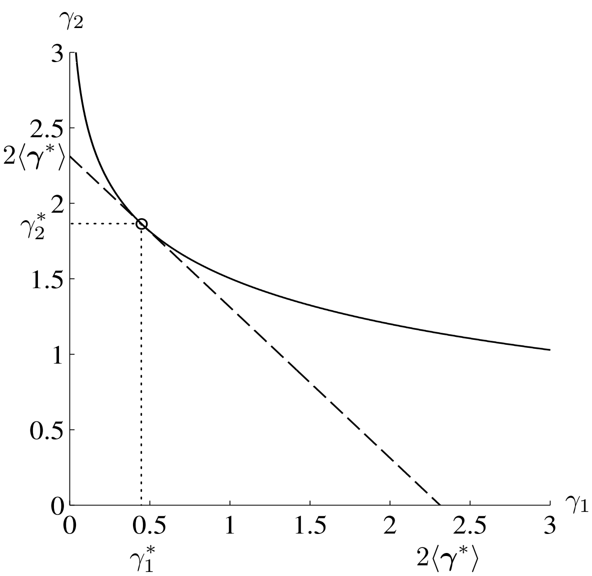

Fig. 2. Power allocation for two blocks with $\alpha_1 = 0.2$ and $\alpha_2 = 2.0$. The solid line corresponds to the boundary of $\mathcal{R}_\gamma$ and the dashed line to the mean power $\langle\boldsymbol{\gamma}\rangle = 0.63$ dB.

(the corresponding error region for LDPC codes is defined according to (11))

$$\mathcal{R}_\gamma(\boldsymbol{\alpha}; r) = \left\{\boldsymbol{\gamma} \in \mathbb{R}_+^B : I_B(\boldsymbol{\alpha}, \boldsymbol{\gamma}) < r\right\}. \tag{16}$$

In the block-fading channel, the instantaneous mutual information of a block is just a function of the product $\gamma_b\alpha_b^2$ and therefore, the region $\mathcal{R}_\gamma$ can be obtained directly from $\mathcal{R}_{\text{out}}$. For two blocks, the optimization problem is illustrated in Figure 2 with $\alpha_1 = 0.2$ and $\alpha_2 = 2.0$. The solid line represents the boundary of $\mathcal{R}_\gamma$ and the dashed line corresponds to vectors $\boldsymbol{\gamma}$ with the same mean power. The point of tangency of both functions is the solution of (15) and the values of γ_1^* and γ_2^* can directly be obtained. For higher dimensions, the optimization can be performed numerically. We note that the complexity of such an optimization grows exponentially in the number of dimensions (i.e., fading blocks). Since we focus on a relatively small number of blocks this is not an issue in practice.

We start by considering a short-term power constraint where the mean power over one codeword is upper bounded by P_{ST}. The optimal solution for this problem has been obtained in [13] and it is based on the relationship between mutual information of Gaussian channels and the minimum mean-square error [24]. Let $\mathcal{R}_{\text{ST}}$ denote the modified region under a short-term power constraint defined as

$$\mathcal{R}_{\text{ST}}(P_{\text{ST}}) = \left\{\boldsymbol{\alpha} \in \mathbb{R}_+^B : \langle\boldsymbol{\gamma}^*(\boldsymbol{\alpha})\rangle > P_{\text{ST}}\right\}. \tag{17}$$

The outage/error probability under a short-term power constraint is computed in the same way as before by integration over the probability density function of the channel parameters

$$P_{\text{err,ST}}(P_{\text{ST}}) = \int_{\boldsymbol{\alpha}\in\mathcal{R}_{\text{ST}}(P_{\text{ST}})} p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) d\boldsymbol{\alpha}. \tag{18}$$

It has been shown in [14], [16] that the diversity achieved by this power allocation algorithm is given by the Singleton bound, i.e., the same as if no power allocation was employed.

We next consider a long-term power constraint. In contrast to a short-term power constraint, where the mean power over one codeword is upper bounded, a long-term power constraint upper bounds the *expected* power per codeword by P_{LT}. The optimal power allocation for this case was determined in [12] for Gaussian distributed inputs and in [14], [16], [17] for arbitrary constellations. In [14] it was shown that a long-term power constraint P_{LT} can be enforced by imposing a corresponding short-term power constraint ${P_{\text{ST}}}^* > P_{\text{LT}}$ (see [14] for more details). Therefore, the outage/error region can be defined via the short-term power constraint as

$$\mathcal{R}_{\text{LT}}(P_{\text{LT}}) = \left\{\boldsymbol{\alpha} \in \mathbb{R}_+^B : \langle\boldsymbol{\gamma}^*(\boldsymbol{\alpha})\rangle > {P_{\text{ST}}}^*\right\}. \tag{19}$$

The optimal power allocation under a long term constraint [14] is

$$\boldsymbol{\gamma}_{\text{LT}}^*(\boldsymbol{\alpha}) = \begin{cases} \boldsymbol{\gamma}^*(\boldsymbol{\alpha}), & \langle\boldsymbol{\gamma}^*(\boldsymbol{\alpha})\rangle \le {P_{\text{ST}}}^*, \\ \mathbf{0}, & \text{otherwise} \end{cases} \tag{20}$$

i.e., if the required power for successful transmission of a codeword is larger than ${P_{\text{ST}}}^*$, the transmitter allocates zero power on that codeword, thereby saving transmit power.

The average transmit power P_{avg} is given by the integral over all fading gains outside the error region imposed by the short-term constraint ${P_{\text{ST}}}^*$, and the short-term power constraint ${P_{\text{ST}}}^*$ is determined such that the average transmit power does not exceed the long-term power constraint, i.e.,

$$P_{\text{avg}} = \int_{\boldsymbol{\alpha}\notin\mathcal{R}_{\text{ST}}({P_{\text{ST}}}^*)} \langle\boldsymbol{\gamma}^*(\boldsymbol{\alpha})\rangle p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) d\boldsymbol{\alpha} \le P_{\text{LT}}. \tag{21}$$

It is now straightforward to determine the error probability for the case of a long-term power constraint by setting $P_{\text{ST}} = {P_{\text{ST}}}^*$ in (18). In a similar manner, we can determine the error probability for the case of short-term to long-term power ratio constraints.

In [23], it is shown that the outage diversity under a long-term power constraint can be obtained from the diversity of the corresponding system with a short-term constraint. The proof in [23] is based on the concept of outage regions which translate to error regions in a straightforward way. Therefore, the relation between short-term and long-term diversity also holds for the word error rates of LDPC code ensembles: If the short-term diversity $d_{\text{c,ST}}$ is larger than one, there exists a P_0 such that the delay-limited capacity is positive for $P_{\text{LT}} \ge P_0$. Therefore, for $d_{\text{c,ST}} > 1$, the long-term diversity $d_{\text{c,LT}}$ is infinite and the average transmit power converges to a finite value. On the other hand, if $d_{\text{ST}} < 1$, the diversity of the system under a long-term constraint is given by

$$d_{\text{c,LT}} = \frac{d_{\text{c,ST}}}{1 - d_{\text{c,ST}}}. \tag{22}$$

Note, however, that the diversity will always be $d_{\text{c,ST}}$ for any finite SLPR.

As an example, we show the outage probability and the word error rates of the root-LDPC code in (14) under a long-term power and SLPR constraint in Figure 3. We assume $B = 2$ fading blocks and the fading coefficients α_b are distributed according to a Rayleigh distribution. In the case of no short-term constraint, the error probability can be made

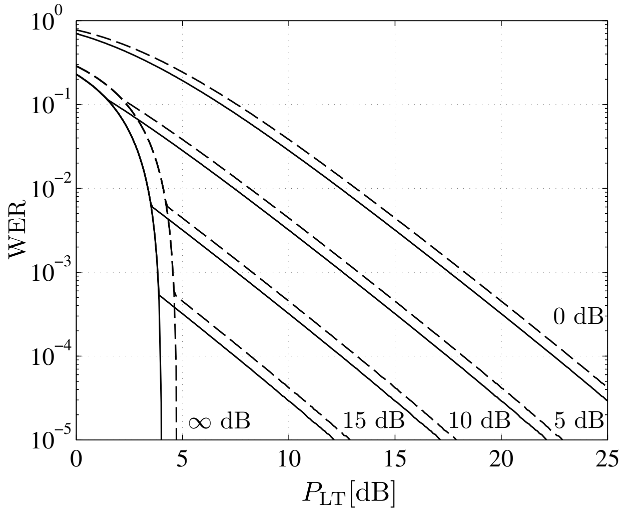

Fig. 3. Outage probability (solid lines) and word error rates of the root-LDPC code (dashed lines) for long-term power and SLPR constraints. Numbers on the curves denote the SLPR.

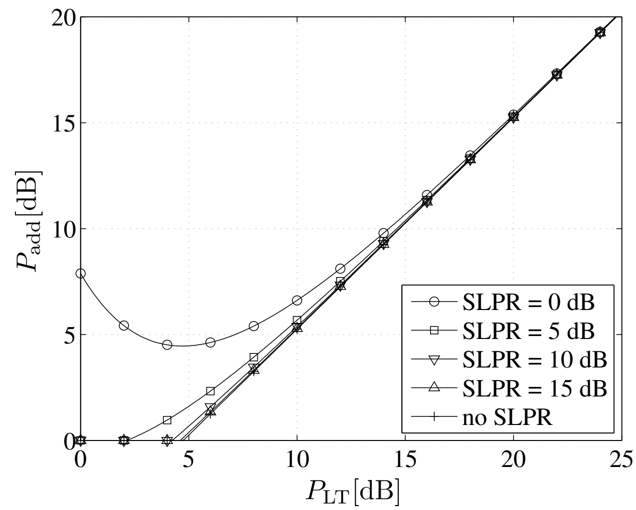

Fig. 4. Difference between the long-term constraint and the average transmit power.

arbitrarily small with finite P_{LT}. This is due to the fact, that $d_{\mathrm{c,ST}} = 2$ for this LDPC code ensemble [10] and therefore, the delay-limited capacity is larger than zero. For high P_{LT}, the outage/error probability is dominated by the short-term constraint and therefore, the slope of the curve corresponds to the diversity order under a short-term power constraint. This has been shown in [16] for the outage probability and holds also for the error probability.

V. Power Allocation for Finite-Length Codes

Finite-length codes have a non-zero error rate even outside the error region which requires a modification of the algorithms of Section IV. We note that an exact analysis would require the error probability of the finite-length code for every vector of fading gains. This is not feasible in general and we therefore follow a suboptimal approach. However, simulation results at the end of this section show that our method performs close to the asymptotic limits and that it achieves gains of more than 10 dB.

The simplest approach is to increase the transmit power, i.e., computing the necessary transmit power for the asymptotic case of infinite block length and then adding a power margin that is sufficiently large to allow the finite-length decoder to converge. This section shows how this additional margin should be allocated to the individual fading blocks and how large it should be.

First, we discuss the allocation of the additional transmit power on the fading blocks. One approach is to distribute it uniformly over the blocks. However, we argue that this is not a good approach as shown in the following example: Assume a block-fading scenario with two blocks where the first block is received error-free (i.e., at high SNR) and the other block at low SNR. Adding additional transmit power to the first block (as done by a uniform allocation) will not help the decoder. We therefore propose an allocation scheme that maximizes the mutual information.

We use the power allocation algorithm for the infinite block length case (20) with an appropriate constraint (see below) leading to a power allocation vector $\boldsymbol{\gamma}^*_{\mathrm{LT}}(\boldsymbol{\alpha})$. If the transmitter allocates non-zero power, we add additional transmit power P_{add} such that the mutual information is maximized

$$\boldsymbol{\gamma}_{\mathrm{add}} = \arg \max_{\substack{\boldsymbol{\gamma}' \\ \langle \boldsymbol{\gamma}' \rangle = P_{\mathrm{add}}}} I_B(\boldsymbol{\alpha}, \boldsymbol{\gamma}^*_{\mathrm{LT}}(\boldsymbol{\alpha}) + \boldsymbol{\gamma}'), \tag{23}$$

and transmit using the power allocation $\boldsymbol{\gamma}^*_{\mathrm{LT}}(\boldsymbol{\alpha}) + \boldsymbol{\gamma}_{\mathrm{add}}$.

The remaining question is how large P_{add} should be. To answer this question, we consider a system with a long-term power constraint P_{LT} and an additional SLPR constraint. Systems with a short-term power constraint can be obtained by setting the SLPR to 0 dB. In such a setting, the maximum short-term transmit power P_{ST} is either limited by the long-term or by the SLPR constraint. According to (21), the long-term constraint can be translated into a corresponding short-term constraint P^*_{ST}. Therefore, the effective short-term constraint is given by

$$P_{\mathrm{ST}} = \min \left\{ P^*_{\mathrm{ST}}, P_{\mathrm{LT}} \cdot \mathrm{SLPR} \right\}. \tag{24}$$

Consider the case where the short-term transmit power is limited by P^*_{ST}. If the power allocation (20) allocates non-zero power, we add an additional power margin P_{add}. To maintain the long-term power constraint in (4), we choose P^*_{ST} such that the average transmit power due to P^*_{ST} (21) and the additional power P_{add} satisfies

$$P_{\mathrm{avg}} + P_{\mathrm{add}} \left(1 - P_{\mathrm{err,ST}}(P^*_{\mathrm{ST}})\right) = P_{\mathrm{LT}}. \tag{25}$$

This allows to apply power allocation for the infinite block-length case followed by an additional margin while still not violating the long-term power constraint.

Consider now the case where the short-term transmit power is limited by $P_{\mathrm{LT}} \cdot \mathrm{SLPR}$. Using (21) allows the computation of the average transmit power P_{avg} associated with this short-term power constraint. This average transmit power is less than P_{LT} because otherwise, the short-term constraint would be

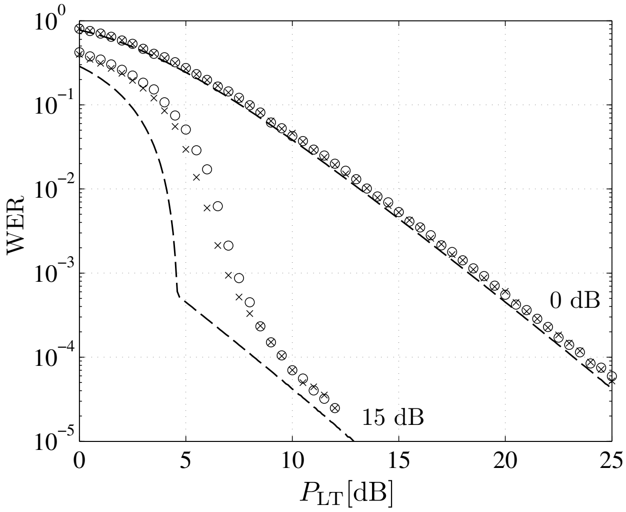

Fig. 5. The dashed lines show the word error rate of infinite length root-LDPC codes for long-term power and SLPR constraints. Simulation results with $\circ$ are for $N = 200$ and results with $\times$ for $N = 2,000$.

limited by P^*_{ST}. Therefore, if we make sure that the peak-power constraint $P_{\mathrm{LT}}\cdot$SLPR is not exceeded, we can add an additional margin to every transmitted codeword which is given by

$$P_{\text{add}} = \frac{P_{\mathrm{LT}} - P_{\text{avg}}}{1 - P_{\text{err,ST}}(P_{\mathrm{LT}} \cdot \text{SLPR})}. \quad (26)$$

This additional margin P_{add} is shown in Figure 4 for the case of the root-LDPC code in (14). It can be seen that it is zero as long as P^*_{ST} is the active constraint and increases with the long-term constraint. For the case of a short-term constraint only (i.e., SLPR $= 0$ dB) we see that the additional margin is always larger than approximately 5 dB. Therefore, it is possible to achieve error rates close to the asymptotic case even with short block lengths. Furthermore, for larger values of SLPR, we can expect to achieve error rates close to the asymptotic case (in block length) if the long-term constraint (and therefore also the additional power margin) is sufficiently large.

To demonstrate the performance gains of our method, we use the same example as in Section IV and set the SLPR to 15 dB. For this scenario, we constructed parity-check matrices of length $N = 200$ and $N = 2,000$ according to (14). For the case where P^*_{ST} is the limiting quantity in (24), we set $P_{\text{add}} =$ 1.6 dB and $P_{\text{add}} = 1.3$ dB for the code of length $N = 200$ and $N = 2,000$, respectively. The resulting error rates are shown in Figure 5. It can be observed that the longer code shows a better performance in the waterfall region. However, when the SLPR constraint is the limiting quantity, both codes perform close to the results predicted in Section IV. As a comparison we also show the error probability for a short-term power constraint (SLPR = 0 dB). As we observe, at error rate 10^{-3} our method achieves more than 10 dB gain with respect to the short-term power constraint algorithm even with a short code of $N = 200$.

VI. Conclusions

We presented an efficient method to study the word error rates of LDPC code ensembles over the block-fading channel with power allocation. The approach is based on a complete characterization of the ensemble by an error boundary which is the first analysis of specific LDPC codes over block-fading channels with optimal power control. Our framework allows for the incorporation of short-term, long-term and short-term to long-term power ratio constraints.

For two fading blocks, the gain achieved by optimal power allocation based on a short-term power constraint is limited but significant gains can be obtained by optimal power allocation strategies based on a long-term power constraint. Furthermore, we conclude that codes which show a good performance for uniform power allocation are also good for systems with optimal power allocation based on short-term and/or long-term power constraints. This is in line with information-theoretic conclusions stating that an optimal transmission strategy can be based on an outage-achieving coding scheme (for uniform power allocation), followed by an optimal power allocation rule [12]. To further support this claim, the example of a root-LDPC code without optimized degree distribution shows that this code already performs within 0.5 dB of the outage probability.

Finite length codes do not exhibit a threshold behavior, i.e., even for channel realizations above the decoding threshold, the decoder is not guaranteed to converge to zero errors. This has to be considered for the derivation of optimal power allocation algorithms for finite length codes. We proposed a suboptimal algorithm that performs close to the infinite-length results and that allows for gains of more than 10 dB with respect to uniform or short-term constrained power allocation.

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