# MIMO ARQ With Multibit Feedback: Outage Analysis

Khoa D. Nguyen, *Member, IEEE*, Lars K. Rasmussen, *Senior Member, IEEE*, Albert Guillén i Fàbregas, *Senior Member, IEEE*, and Nick Letzepis, *Member, IEEE*

***Abstract*—This paper studies the asymptotic outage performance of incremental redundancy automatic-repeat-request (INR-ARQ) transmission over multiple-input multiple-output (MIMO) block-fading channels with discrete input constellations. We first show that transmission with random codes using a discrete signal constellation across all transmit antennas achieves the optimal outage diversity given by the Singleton bound. The optimal SNR-exponent and outage diversity of INR-ARQ transmission over the MIMO block-fading channel are then analysed. We show that a significant gain in outage diversity is obtained by providing more than one bit feedback at each ARQ round. Thus, the outage performance of INR-ARQ transmission can be remarkably improved with minimal additional overhead. A practical feedback-and-power-adaptation rule is proposed for MIMO INR-ARQ, demonstrating the benefits provided by multibit feedback. Although the rule is sub-optimal in terms of outage performance, it achieves the optimal outage diversity.**

***Index Terms*—Automatic-repeat-request, multibit feedback, outage diversity, outage probability, power allocation, SNR-exponent.**

## I. Introduction

In this paper, we take an information-theoretic approach to analyzing and designing multiple-input multiple-output (MIMO) transmission strategies for incremental redundancy (INR) automatic-repeat-request (ARQ) schemes over a block-fading channel. In particular, we propose the use of multibit feedback for power adaptation and study the outage diversity of the resulting protocol over the MIMO block-fading channel, which characterizes the slope of the outage probability curve at high signal-to-noise ratio (SNR) in log-log scale.

### A. Prior Art

The block-fading channel [1], [2] is a useful mathematical model for many practical wireless communication scenarios. The channel consists of a finite number of consecutive or parallel transmission blocks, where each block is affected by an independent fading coefficient. The model approximates well the characteristics of delay-limited transmission over slowly varying channels, such as Orthogonal Frequency Division Multiplexing (OFDM) transmission over slowly-fading frequency-selective multipath channel, as well as narrowband transmission with frequency-hopping as encountered in the Global System for Mobile Communications (GSM) and the Enhanced Data rate for GSM evolution (EDGE) standards.

Due to the finite number of fading blocks, the information rate supported by the channel depends on the instantaneous channel realization and, therefore, is a random variable. When the instantaneous mutual information is less than the transmission rate, transmission is in outage [2]. In this case, it follows from the strong converse theorem (see, e.g., [3]–[5]) that messages are decoded in error with probability one [6], [7]. Furthermore, it is shown in [4], [8] that the use of sufficiently long random codes achieves an average error rate equal to the outage probability. Therefore, the outage probability is a fundamental limit on the performance of block-fading channels.

MIMO transmission has revolutionized modern wireless communications, and is now a key technology used in most current standards, e.g., WiFi (IEEE 802.11) and WiMax (IEEE 802.16) [9], [10]. Moreover, due to the randomness of the communication rate supported by the channel, it is essential to use adaptive techniques to enable high-rate reliable communication, where the transmission rate and/or power is adjusted to the channel realization. The use of adaptive techniques depends strongly on the availability of channel state information (CSI) at the transmitter and the receiver. In most communication systems, CSI can be estimated at the receiver, while CSI is usually not directly available at the transmitter. The use of ARQ transmission techniques is, therefore, a powerful approach for providing transmitter CSI, which in turn can be used to significantly improve the performance over block-fading channels [11].

The optimal diversity-multiplexing tradeoff for a MIMO channel with optimal (Gaussian) input distribution has been characterized in [12]. For systems with discrete input constellations, the rank criterion for the optimal outage diversity was

Manuscript received June 01, 2009; revised March 30, 2011; accepted June 22, 2011. Date of publication October 06, 2011; date of current version February 08, 2012. This work was presented in part at the 2009 Australian Communication Theory Workshop, Sydney, Australia, and the 2009 IEEE International Symposium on Information Theory. This work was supported in part by the Australian Research Council under ARC Grants RN0459498, DP0558861, and DP0881160, in part by the Swedish Research Council under VR Grant 621-2009-4666, and in part by the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013)/ERC Grant agreement no. 228044.

K. D. Nguyen and N. Letzepis are with the Institute for Telecommunications Research, University of South Australia, Australia (e-mail: khoa.nguyen@unisa.edu.au; nick.letzepis@ieee.org).

L. K. Rasmussen was with the Institute for Telecommunications Research, University of South Australia, Australia. He is now with the Communication Theory Laboratory, KTH Royal Institute of Technology, and the ACCESS Linnaeus Center, Stockholm, Sweden (e-mail: lars.rasmussen@ieee.org).

A. Guillén i Fàbregas is with the with the Institució Catalana de Recerca i Estudis Avançats (ICREA), the Department of Information and Communication Technologies, Universitat Pompeu Fabra, Barcelona, Spain, and the Department of Engineering, University of Cambridge, Cambridge, United Kingdom (e-mail: guillen@ieee.org).

Communicated by G. Taricco, Associate Editor for Communications.

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Digital Object Identifier 10.1109/TIT.2011.2170919

TABLE I
SUMMARY OF NOTATIONS

| | | | |
|---|---|---|---|
| $B$ | No. of blocks per round | $L$ | Maximum No. rounds |
| $N_t$ | No. transmit antennas | $N_r$ | No. receive antennas |
| $R$ | Rate (bits per channel uses) | $J$ | No. channel uses per block |
| $P$ | Power constraint | $P_\ell$ | Transmit power |
| $\mathcal{X}$ | input constellation | $M$ | Constellation size |
| $K$ | No. feedback levels | $k, \boldsymbol{k}$ | Feedback index, vector |
| $I$ | Mutual information | $\hat{I}, \overline{I}$ | Quantization thresholds of $I$ |
| $\boldsymbol{X}, \boldsymbol{x}, x$ | Transmited signal | $\boldsymbol{Y}, \boldsymbol{y}, y$ | Received signal |
| $\boldsymbol{H}$ | Channel gain matrix | $\boldsymbol{W}$ | AWGN noise |
| $P_e, p$ | Error, outage probability | $d$ | Outage diversity, SNR exponent |

derived in [13] from a worst-case analysis of the pair-wise error probability (PEP). References [14], [15] establish the Singleton bound on the optimal SNR-exponent of quasi-static MIMO channels with discrete input constellations. The Singleton bound is achievable by a wide range of input constellations via a unified code construction method proposed in [15].

In an INR-ARQ scheme, transmission starts with a high-rate codeword, and additional redundancy is requested via a feedback link when the codeword is not successfully decoded. Transmission is in outage if the codeword is not decodable within the maximum delay constraint allowed by the system. Traditional INR-ARQ systems implement one-bit feedback from the receiver, indicating whether additional redundancy is required. However, due to the accumulative nature of INR-ARQ schemes, performance improvements are possible when additional information regarding the status of the current transmission is provided through the feedback link. Several multibit feedback INR-ARQ schemes have been proposed in the literature. In particular, [16] shows that the throughput performance of ARQ systems can be improved by multibit feedback prior to each transmission round. The proposed system is equivalent to a conventional ARQ system with quantized CSI at the transmitter (CSIT). For systems with no CSIT, [17], [18] propose transmission using convolutional codes, while [19] proposes a multilayer broadcasting strategy for multibit feedback ARQ. Both approaches show that multibit feedback can significantly improve the throughput performance of ARQ transmission. There is, however, no unified approach for designing multibit feedback INR-ARQ transmission schemes.

An important performance measure for INR-ARQ transmission in the MIMO block-fading channel is the rate-diversity-delay tradeoff. This tradeoff has only been studied for INR-ARQ systems with one-bit ACK/NACK feedback in [20]–[22]. In particular, [20] characterizes the rate-diversity-delay tradeoffs of Gaussian input MIMO INR-ARQ systems with both short-term and long-term average power constraints. For systems with discrete input constellations, the optimal rate-diversity tradeoff for systems with short-term power constraints was characterized in [21], [23]. For ARQ systems with discrete input constellation and long-term power constraints, an optimal power allocation rule has been derived in [23], providing significant improvement on outage performance. However, the rate-diversity tradeoff of the corresponding system was not studied.

### B. Contributions

As a first contribution we consider fixed-rate transmission over the MIMO block-fading channel. We show that the outage diversity is given by the Singleton bound, and that it is achievable with random codes constructed over arbitrary discrete input constellations. This rigorously proves that the Singleton bound is the optimal SNR-exponent of MIMO transmission with discrete input constellations. The result will also prove instrumental in designing and analyzing INR-ARQ transmission over the MIMO block-fading channel.

As our main contribution we study the rate-diversity tradeoff of the MIMO ARQ system with multibit feedback under long-term power constraints. The analysis shows that multibit feedback and optimal power adaptation provide significant outage diversity gains for ARQ transmission over the block-fading channel. It is shown that a finite number of feedback bits is sufficient to achieve the maximal outage diversity. The optimal rate-diversity tradeoff for the one-bit feedback case is also presented, which characterizes the asymptotic gains provided by the optimal power allocation rule proposed in [23]. As a further contribution a practically feasible feedback-and-power-adaptive rule is proposed. Although the rule is sub-optimal in terms of outage performance, it can achieve the optimal outage diversity, thus clearly illustrating the benefits offered by multibit feedback.

### C. Notation and Organization

The following notations are used in the paper. Boldface uppercase ($\boldsymbol{A}$) and lowercase ($\boldsymbol{a}$) variables correspondingly denote matrices and vectors; while scalar variables are denoted by lightface ($a$ or $A$). Sets are denoted by calligraphic letters; while the sets of natural, real and complex numbers are correspondingly denoted with $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$. The mathematical expectation of a random variable is denoted by $\mathbb{E}[\cdot]$. Nonconjugate transpose of matrices are denoted by $(\cdot)'$. The operation $\lfloor\cdot\rfloor$ ($\lceil\cdot\rceil$) returns the maximum (minimum) integer smaller (larger) than a real number. For convenience, the physical meanings of commonly used parameters are summarized in Table I.

The remainder of the paper is organized as follows. Section II describes the MIMO block-fading channel model. Section III proposes the multibit feedback INR-ARQ system based on mutual information and information outage. Sections IV and V discuss system design and performance analysis. Finally, concluding remarks are given in Section VI and proofs are collected in the Appendices.

## II. System Model

Consider INR-ARQ transmission over a MIMO block-fading channel with $N_t$ transmit and $N_r$ receive antennas. Each ARQ round is transmitted over $B$ additive white Gaussian noise (AWGN) blocks of $J$ channel uses each, where block $b$ at ARQ round $\ell$ is affected by a flat fading channel gain matrix $\boldsymbol{H}_{\ell,b} \in \mathbb{C}^{N_r \times N_t}$. The baseband equivalent of the channel in the $\ell$th ARQ round is given by

$$\boldsymbol{Y}_\ell = \sqrt{\frac{P_\ell}{N_t}}\boldsymbol{H}_\ell \boldsymbol{X}_\ell + \boldsymbol{W}_\ell \tag{1}$$

where $P_\ell$ is the transmit power in round $\ell$, $\boldsymbol{X}_\ell \in \mathbb{C}^{BN_t \times J}, \boldsymbol{Y}_\ell, \boldsymbol{W}_\ell \in \mathbb{C}^{BN_r \times J}$ are correspondingly the transmitted signal, the received signal, and the additive noise; while $\boldsymbol{H}_\ell \in \mathbb{C}^{BN_r \times BN_t}$ is a block diagonal channel gain matrix at round $\ell$ with

$$\boldsymbol{H}_\ell = \mathrm{diag}(\boldsymbol{H}_{\ell,1}, \ldots, \boldsymbol{H}_{\ell,B}).$$

In the INR-ARQ scheme, the receiver attempts to decode at round $\ell$ based on the received signals collected in rounds $1, \ldots, \ell$. The entire channel after $\ell$ ARQ rounds is

$$\boldsymbol{Y}_{\overline{1,\ell}} = \boldsymbol{H}_{\overline{1,\ell}}\boldsymbol{X}_{\overline{1,\ell}} + \boldsymbol{W}_{\overline{1,\ell}} \tag{2}$$

where

$$\begin{aligned}
\boldsymbol{Y}_{\overline{1,\ell}} &= \left[\boldsymbol{Y}'_1, \ldots, \boldsymbol{Y}'_\ell\right]' \\
\boldsymbol{X}_{\overline{1,\ell}} &= \left[\boldsymbol{X}'_1, \ldots, \boldsymbol{X}'_\ell\right]' \\
\boldsymbol{H}_{\overline{1,\ell}} &= \mathrm{diag}\left(\sqrt{\frac{P_1}{N_t}}\boldsymbol{H}_1, \ldots, \sqrt{\frac{P_\ell}{N_t}}\boldsymbol{H}_\ell\right) \\
\boldsymbol{W}_{\overline{1,\ell}} &= \left[\boldsymbol{W}'_1, \ldots, \boldsymbol{W}'_\ell\right]'.
\end{aligned}$$

We consider transmission where the entries of $\boldsymbol{X}_\ell$ are equiprobably drawn from an input constellation $\mathcal{X} \subset \mathbb{C}$ of size $2^M$, and assume that the constellation $\mathcal{X}$ has unit average energy, i.e., entries $x \in \mathcal{X}$ of $\boldsymbol{X}_\ell$ satisfy $\mathbb{E}\left[|x|^2\right] = 1$. We further assume that the entries of $\boldsymbol{H}_{\ell,b}$ and $\boldsymbol{W}_\ell$ are independently drawn from a zero-mean unit-variance complex Gaussian distribution $\mathcal{N}_{\mathbb{C}}(0,1)$, and that $\boldsymbol{H}_{\ell,b}$ is available at the receiver. The average SNR at each receive antenna is then $P_\ell$.

Let $\overline{L}$ be the average number of transmission round per codeword, or equivalently the expected interrenewal time [6]. The average transmit power is

$$P_{\mathrm{av}} = \frac{\mathbb{E}_{\boldsymbol{H}_{\overline{1,L}}}\left[\sum_{\ell=1}^{L} P_\ell\right]}{\overline{L}} \tag{3}$$

where $P_\ell$ is adapted to $\boldsymbol{H}_{\overline{1,\ell-1}}$ through receiver feedback. For a system with long-term power constraint $P$, we study feedback-and-power-adaptation rules satisfying

$$P_{\mathrm{av}} \leq P. \tag{4}$$

## III. Preliminaries

### A. Accumulated Mutual Information

Assuming that the realized channel matrix at round $\ell$ is $\boldsymbol{H}_\ell$, the input-output mutual information of the MIMO channel in round $\ell$ is

$$I_\ell\left(\sqrt{\frac{P_\ell}{N_t}}\boldsymbol{H}_\ell\right) = \frac{1}{B}\sum_{b=1}^{B} I_{\mathcal{X}}\left(\sqrt{\frac{P_\ell}{N_t}}\boldsymbol{H}_{\ell,b}\right) \tag{5}$$

where $I_{\mathcal{X}}\left(\sqrt{\frac{P_\ell}{N_t}}\boldsymbol{H}_{\ell,b}\right)$ is the input-output mutual information [5], measured in bits per channel use (bpcu), of an AWGN MIMO channel with input constellation $\mathcal{X}$ and channel matrix $\sqrt{\frac{P_\ell}{N_t}}\boldsymbol{H}_{\ell,b}$. More specifically

$$I_{\mathcal{X}}(\boldsymbol{H}) = \mathbb{E}_{\boldsymbol{x},w}\left[\log_2 \frac{e^{-\|\boldsymbol{w}\|^2}}{\sum_{\boldsymbol{x}' \in \mathcal{X}^{N_t}} \frac{1}{2^M} e^{-\|\boldsymbol{w}-\boldsymbol{H}(\boldsymbol{x}-\boldsymbol{x}')\|^2}}\right] \tag{6}$$

where $\boldsymbol{x}$ is uniformly drawn from $\mathcal{X}^{N_t}$ and the entries of $\boldsymbol{w} \in \mathbb{C}^{N_r}$ are i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1)$. The average input-output mutual information after $\ell$ ARQ rounds is given by $\frac{1}{\ell}\sum_{l=1}^{\ell} I_l$ bpcu. Let

$$I_{\overline{1,\ell}} \triangleq \sum_{l=1}^{\ell} I_l \tag{7}$$

be the *accumulated mutual information* after $\ell$ ARQ rounds. We now propose the multibit feedback INR-ARQ transmission scheme based on the accumulated mutual information $I_{\overline{1,\ell}}$.

### B. Multilevel Feedback

We consider an INR-ARQ system with a delay constraint of $L$ ARQ rounds, where a feedback index $k \in \{0, \ldots, K-1\}$ is delivered after each transmission round through a zero-delay error-free feedback channel. Power and rate adaptation are performed based on receiver feedbacks. The overall system model is illustrated in Fig. 1.

*1) Transmitter:* Consider a code book $\mathcal{C}$ of rate $\frac{R}{L}$, $R \in (0, MN_t)$ bits per coded symbol, that maps a message $m \in \{1, \ldots, 2^{RBJ}\}$ to a codeword $\boldsymbol{x}(m) \in \mathcal{X}^{N_t BJL}$. At transmission round $\ell$, $N_t BJ$ of the coded symbols are formatted into $\boldsymbol{X}_\ell(m) \in \mathcal{X}^{BN_t \times J}$ and transmitted via the channel in (1) with power $P_\ell(\boldsymbol{k}_{\ell-1})$, where $\boldsymbol{k}_{\ell-1} = [k_1, \ldots, k_{\ell-1}]$ is the vector of feedback indices collected from rounds $1, \ldots, \ell-1$. The realized code rate of a single ARQ round is $R$ bpcu, and the realized code rate after $\ell$ ARQ rounds is $\frac{R}{\ell}$ bpcu. If feedback $k_\ell = K-1$ (denoting positive acknowledgment (ACK)) is received after $\ell$ transmission rounds, the transmission is successful and transmission of the next message starts. Otherwise, the transmitter continues with new transmission rounds until feedback index $K-1$ is received or until $L$ transmission rounds have elapsed.

*2) Receiver:* Upon receiving round $\ell$, the receiver attempts to decode the transmitted message from the received signals collected from rounds 1 to $\ell$. The receiver employs a decoder with error detection capabilities as described in [6]. The decoder outputs $\hat{m} \in \{1, \ldots, 2^{RBJ}\}$ if there exists a unique message $\hat{m}$ such that $\boldsymbol{X}_{\overline{1,\ell}}(\hat{m})$ and $\boldsymbol{Y}_{\overline{1,\ell}}$ are jointly typical conditioned on

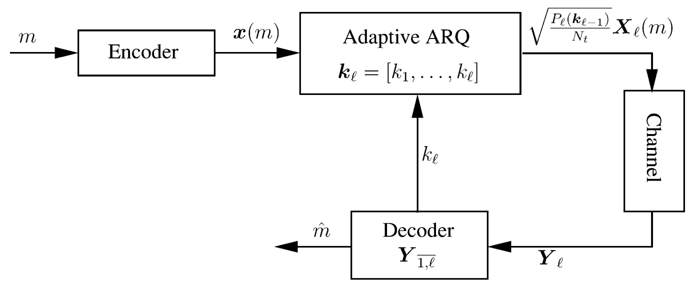


Fig. 1. INR-ARQ system with multibit feedback.

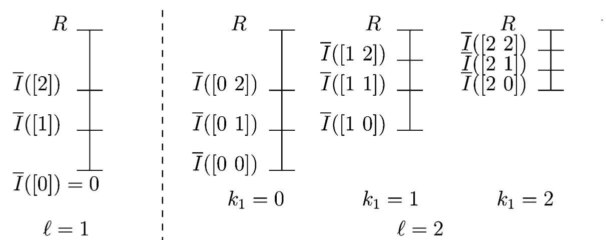

Fig. 2. Example of feedback thresholds.

$\boldsymbol{H}_{\overline{1,\ell}}$ [5]; then an ACK is delivered to the transmitter via feedback index $k_\ell = K-1$. Otherwise, a quantization of the *accumulated mutual information* $I_{\overline{1,\ell}}$ is delivered via feedback index $k_\ell$ satisfying $I_{\overline{1,\ell}} \in \left[\overline{I}([\boldsymbol{k}_{\ell-1}, k_\ell]), \overline{I}([\boldsymbol{k}_{\ell-1}, k_\ell+1])\right)$, with predefined quantization thresholds $\overline{I}(\boldsymbol{k}_\ell), \boldsymbol{k}_\ell \in \{0,\ldots,K-2\}^\ell$, and $\overline{I}([\boldsymbol{k}_{\ell-1}, K-1]) = \infty$ for $\ell = 1,\ldots,L-1$. An example of the feedback thresholds for the first two rounds of an ARQ system with $K = 4$ is illustrated in Fig. 2. Feedback index 3 is used to denote successful transmission. At the first ARQ round, the leftmost set of feedback thresholds is used; while at the second ARQ round, one of the three sets of feedback thresholds on the right is employed, depending on which feedback index was delivered in the first round. Noting that $I_{\overline{1,\ell+1}} \geq I_{\overline{1,\ell}}$, the feedback thresholds in round $\ell+1$ should be designed such that $\overline{I}(\boldsymbol{k}_\ell) = \overline{I}([\boldsymbol{k}_\ell, 0]) < \ldots < \overline{I}([\boldsymbol{k}_\ell, K-2])$. Thus, the set of quantization thresholds is completely defined by $\overline{I}(\boldsymbol{k}_{L-1})$ for practical purposes.

*3) Power Constraint:* The probability of having feedback vector $\boldsymbol{k}_\ell$ at round $\ell$, denoted as $q(\boldsymbol{k}_\ell)$, is recursively expressed as

$$q(\boldsymbol{k}_0) = 1 \tag{8}$$

$$q([\boldsymbol{k}_{\ell-1}, k]) = \Pr\{k_\ell = k \mid \boldsymbol{k}_{\ell-1}\}\, q(\boldsymbol{k}_{\ell-1}), \tag{9}$$

$$\begin{aligned}&\Pr\left\{k_\ell = k \middle| \boldsymbol{k}_{\ell-1}\right\}\\ &\quad= \Pr\left\{I_{\overline{1,\ell-1}} + I_\ell \in \left[\overline{I}([\boldsymbol{k}_{\ell-1}, k]), \overline{I}([\boldsymbol{k}_{\ell-1}, k+1])\right) \middle| \boldsymbol{k}_{\ell-1}\right\}\end{aligned}$$

where $I_\ell$ is given by (5) with $P_\ell = P_\ell(\boldsymbol{k}_{\ell-1})$. Noting that $k_{\ell-1} = K-1$ denotes a successful decoding at round $\ell-1$, the power constraint in (4) can be written as

$$P_{\mathrm{av}} = \frac{P_1 + \sum_{\ell=2}^{L} \sum_{\boldsymbol{k}_{\ell-1} \in \{0,\ldots,K-2\}^{\ell-1}} q(\boldsymbol{k}_{\ell-1}) P_\ell(\boldsymbol{k}_{\ell-1})}{1 + \sum_{\ell=2}^{L} \sum_{\boldsymbol{k}_{\ell-1} \in \{0,\ldots,K-2\}^{\ell-1}} q(\boldsymbol{k}_{\ell-1})} \leq P. \tag{10}$$

### C. Information Outage

After $\ell$ ARQ rounds, the input-output mutual information is $\frac{I_{\overline{1,\ell}}}{\ell}$ and the realized code rate is $\frac{R_M N_t}{\ell} = \frac{R}{\ell}$ (bpcu). Hence, transmission is in outage at round $\ell$ if $I_{\overline{1,\ell}} < R$. The probability of having an outage at round $\ell$ is then given by

$$p(\ell) \triangleq \Pr\left\{I_{\overline{1,\ell}} < R\right\}. \tag{11}$$

With an optimal coding scheme, and in the limit of the number of channel uses $J \to \infty$, the codeword is correctly decoded whenever $I_{\overline{1,\ell}} > R$; otherwise, an error is detected [6]. Therefore, the outage probability $p(\ell)$ is an achievable lower bound on the word error probability at round $\ell$. For INR-ARQ transmission with delay constraint $L$, the overall outage probability is $p(L)$.

## IV. Asymptotic Analysis

Consider a power adaptation rule $P_\ell = P_\ell(\boldsymbol{k}_{\ell-1})$ satisfying the power constraint in (10). We prove that for large $P$, the optimal outage probability at round $\ell$ behaves like

$$p(\ell) \doteq P^{-d_\ell(R)} \tag{12}$$

where $d_\ell(R)$ is the outage diversity at round $\ell$ and the exponential equality ($\doteq$) indicates [12]

$$d_\ell(R) = \lim_{P\to\infty} \frac{-\log p(\ell)}{\log P}. \tag{13}$$

Subsequently, we determine the optimal rate-diversity-delay tradeoff $d_\ell(R)$ of ARQ systems with $K$ levels feedback and prove that the optimal outage diversity is achievable.

### A. MIMO Block-Fading Without ARQ

In order to characterize the outage diversity or achievable SNR-exponent for the MIMO INR-ARQ channel, we first study the corresponding limits for fixed-rate transmission over the MIMO block-fading channel. These results constitute the key ingredients in proving our main results for multibit ARQ.

*Theorem 1:* Consider fixed-rate transmission $(L = 1)$ with rate $R$ and power $P$ over the MIMO block-fading channel in (1) using constellation $\mathcal{X}$ of size $2^M$ and the transmission scheme described in Section III-B. Let $I = I_1\left(\sqrt{\frac{P}{N_t}}\boldsymbol{H}_1\right)$ be the realized input-output mutual information as defined in (5). For large $P$, we have that

$$\Pr\{I < R\} \doteq P^{-d(R)} \tag{14}$$

$$\Pr\{I \leq R\} \doteq P^{-\underline{d}(R)} \tag{15}$$

where $d(R)$ is bounded by $\underline{d}(R) \leq d(R) \leq \overline{d}(R)$, and

$$\overline{d}(R) \triangleq N_r \left(1 + \left\lfloor B\left(N_t - \frac{R}{M}\right)\right\rfloor\right) \quad (16)$$

$$\underline{d}(R) \triangleq N_r \left\lceil B\left(N_t - \frac{R}{M}\right)\right\rceil. \quad (17)$$

Furthermore, $\underline{d}(R)$ is the SNR-exponent achieved by using random codes with rate $R$, where the code symbols are drawn uniformly from $\mathcal{X}$.

*Proof:* See Appendix A.[1] ∎

To the best of our knowledge, this is the first rigorous proof for the outage diversity of a MIMO block-fading channel with a general discrete input constellation. The results of [13], [15] establish $\overline{d}(R)$ as an upper bound for the outage diversity for the quasi-static fading channel. Code design techniques in [15] show that $\overline{d}(R)$ can be achieved by specifically constructed input constellations. As a generalization, Theorem 1 shows that $\overline{d}(R)$ is the outage diversity for MIMO block-fading channels with any input constellation of size $2^M$ (except when $\frac{BR}{M}$ is an integer). Furthermore, Theorem 1 shows that $\overline{d}(R)$ is achievable by using random codes when $\frac{BR}{M}$ is noninteger, which is essential for analyzing the performance of INR-ARQ systems.

When $\frac{BR}{M} \in \mathbb{N}$, random codes do not achieve the upper bound on outage diversity. Therefore, we can only obtain a bound for $d(R)$. This leads to the corresponding bounds on outage diversity of the INR-ARQ system in the subsequent sections. One might see this as a potential limitation of i.i.d. random codes since there exist specific code constructions that can achieve the outage diversity $\overline{d}(R)$ [21], [25]. We conjecture that similar constructions would achieve the upper bounds on outage diversity of INR-ARQ systems subsequently presented, even for $\frac{BR}{M} \in \mathbb{N}$.

### B. Multibit MIMO ARQ

We now consider ARQ transmission over the block-fading channel in (1) using input constellation $\mathcal{X}$ as described in Section III-B1. Using Theorem 1, the optimal rate-diversity-delay tradeoff of the MIMO INR-ARQ scheme with multibit feedback is characterize as follows.

*Theorem 2:* Consider INR-ARQ transmission over the MIMO block-fading channel in (1) using constellation $\mathcal{X}$ of size $2^M$ and the transmission scheme described in Section III-B, where a codeword is considered successfully delivered at round $\ell$ if $I_{\overline{1,\ell}} \geq R$. Assume that the number of feedback levels is $K \geq \left\lceil \frac{BR}{M}\right\rceil + 1$. Subject to the power constraint in (10), the optimal rate-diversity-delay tradeoff is given by

$$d_\ell(R) = (1 + BN_tN_r)^{\ell-1}\left(\overline{d}(R) + 1\right) - 1 \quad (18)$$

when $\frac{BR}{M}$ is not an integer, where $\overline{d}(R)$ is given in Theorem 1.

*Proof:* See Appendix B for a proof. ∎

[1]A more general result of the theorem, which deals with power allocation for block-fading channels with mismatched channel state information, was derived in [24] after the submission of this paper. The proof given here is simpler and forms a basis for the result in [24].

Theorem 2 only gives the optimal outage diversity when $\frac{BR}{M}$ is noninteger. When $\frac{BR}{M}$ is an integer, the bounds for $d(R)$ in Theorem 1 does not coincide; thus, a definite value of $d_\ell(R)$ is not known. It can be shown that the optimal outage diversity is bounded by

$$\begin{aligned}(1 + BN_tN_r)^{\ell-1}\left(\underline{d}(R) + 1\right) - 1 &\leq d_\ell(R)\\ &\leq (1 + BN_tN_r)^{\ell-1}\left(\overline{d}(R) + 1\right) - 1. \quad (19)\end{aligned}$$

An intuitive explanation for the outage diversity gains offered by multibit feedback is given as follows. At round $\ell+1$, the feedback vector $\boldsymbol{k}_\ell$ provides the transmitter with the past channel realizations. This allows raising the transmit power in round $\ell+1$ by a factor of $\frac{1}{q(\boldsymbol{k}_\ell)}$ without violating the long-term power constraint. In the limit of large power constraint, the optimal transmit power in round $\ell+1$ satisfies

$$P_{\ell+1} \doteq P_\ell^{\overline{d}(\overline{I}(\boldsymbol{k}_\ell) - \overline{I}(\boldsymbol{k}_{\ell-1}))} \quad (20)$$

where $\overline{d}(R)$ is given by Theorem 1. Since only the exponent is significant in diversity analysis, the maximum outage diversity can be achieved if there are sufficient thresholds to feedback $\overline{d}(I)$ for $I \leq R$ ($I > R$ implies successful transmission). For MIMO block-fading channels with discrete input constellations, the rate-diversity tradeoff is a stair-case function; therefore, a finite number of feedback levels is sufficient to achieve the maximum outage diversity. Meanwhile, for systems with $\frac{BR}{M} < 1$, including systems with Gaussian input distribution, $\overline{d}(I)$ is a constant for $I < R$. Therefore, no gains in outage diversity can be obtained by multibit feedback. Conversely, in the multiplexing-diversity tradeoff analysis [12], the outage diversity is a continuous, decreasing function of the multiplexing gain, and thus, an infinite number of feedback levels is required to achieve the optimal diversity-multiplexing tradeoff in our ARQ setting.

*Remark 1:* The proof of Theorem 2 also gives the following guidelines to designing the feedback and power allocation rules.

- The optimal outage diversity of INR-ARQ systems is achievable with $\left\lceil \frac{BR}{M}\right\rceil + 1$ feedback levels, where the feedback thresholds of each round are fixed at $\hat{I}_t = \frac{Mt}{B}, t = 0, \ldots, \left\lfloor \frac{BR}{M}\right\rfloor$. Therefore, for systems with $K \geq \left\lceil \frac{BR}{M}\right\rceil + 1$, the optimal outage diversity is achievable if for $\ell = 1, \ldots, L$, $\{\hat{I}_t : R \geq \hat{I}_t \geq \overline{I}(\boldsymbol{k}_{\ell-1})\} \subseteq \{\overline{I}(\boldsymbol{k}_\ell), \boldsymbol{k}_\ell \in \{1, \ldots, K-1\}^\ell\}$.
- Furthermore, the outage probability in round $\ell+1$ is dominated by the events with $I_{\overline{1,\ell}} \in \left[0, \frac{M}{B}\right) \cup \left[\hat{I}_\tau, R\right)$, where $\tau = \left\lfloor \frac{BR}{M}\right\rfloor$. Therefore, after placing $\left\lceil \frac{BR}{M}\right\rceil + 1$ thresholds at $\hat{I}_t$, the remaining feedback thresholds (for systems with $K > \left\lceil \frac{BR}{M}\right\rceil + 1$) should give higher priority to quantizing the aforementioned region to improve outage performance.
- With the optimal feedback rule, the optimal outage diversity can be achieved with power allocation satisfying $q(\boldsymbol{k}_{\ell-1})P_\ell(\boldsymbol{k}_{\ell-1}) = \alpha P$ for all $\boldsymbol{k}_{\ell-1}, \ell = 1, \ldots, L$, where $\alpha$ is a constant chosen to satisfy the power constraint (10).

We now prove that the rate-diversity-delay tradeoff $d_\ell(R)$ is achievable by using random codes, as given by the following theorem.

*Theorem 3:* Consider INR-ARQ transmission over the MIMO block-fading channel in (1) using constellation $\mathcal{X}$ of size $2^M$ and the transmission scheme described in Section III-B with power constraint $P$ given in (10). Assume that the number of feedback levels is $K \geq \lceil \frac{BR}{M} \rceil + 1$. With random-coding schemes and $J \to \infty$, for large $P$, the word error probability $P_e(\ell)$ at round $\ell$ satisfies $P_e(\ell) \doteq P^{-d_\ell^{(r)}(R)}$, where

$$d_\ell^{(r)}(R) = (1 + BN_tN_r)^{\ell-1} (\underline{d}(R) + 1) - 1 \qquad (21)$$

is the achievable SNR-exponent and $\underline{d}(\ell)$ is given in Theorem 1.

*Proof:* With a random coding scheme and $J \to \infty$, the codeword is correctly decoded with probability one at round $\ell$ if $I_{\overline{1,\ell}} > R$ [6], [26], in which case, the receiver feeds back an ACK (in contrast to the outage case, where an ACK is fed back if $I_{\overline{1,\ell}} \geq R$). The proof then follows similar arguments as the proof of Theorem 2, noting from Theorem 1 that $\Pr\{I_\ell \leq I\} \doteq P_\ell^{-\underline{d}(I)}$. ∎

Theorem 3 shows that the rate-diversity-delay tradeoff $d_\ell(R)$ stated in Theorem 2 is achievable with random codes using the transmission scheme described in Section III-B when $\frac{BR}{M}$ is not an integer; and then, the optimal rate-diversity-delay tradeoff is given by (21). Furthermore, the optimal outage diversity and SNR-exponent of INR-ARQ transmission with delay constraint $L$ is similarly characterized by $d_L(R)$ and $d_L^{(r)}(R)$ given in (18) and (21), respectively.

### C. One-Bit MIMO ARQ

In an INR-ARQ system with one-bit ACK/NACK feedback (classical INR-ARQ), the optimal rate-diversity-delay tradeoff is given by the following.

*Theorem 4:* Consider INR-ARQ transmission over the MIMO block-fading channel in (1) using constellation $\mathcal{X}$ of size $2^M$ and the transmission scheme described in Section III-B, where a codeword is considered successfully delivered at round $\ell$ if $I_{\overline{1,\ell}} \geq R$. Assume that the number of feedback levels is $K = 2$. Subject to the power constraint in (10), the optimal rate-diversity-delay tradeoff is given by

$$\hat{d}_1(R) = \overline{d}(R) \qquad (22)$$

$$\hat{d}_\ell(R) = BN_tN_r \left(\ell - 1 + \sum_{l=1}^{\ell-2} \hat{d}_l(R)\right) + (1 + \hat{d}_{\ell-1}(R))\hat{d}_1(R), \quad \ell \geq 2. \qquad (23)$$

for all $R$ such that $\hat{d}_1(R)$ is continuous. Furthermore, the rate-diversity-delay tradeoff $\hat{d}_\ell(R)$ is achievable when $\frac{BR}{M}$ is not an integer.

*Proof:* The proof follows the same arguments as that of Theorems 2 and 3, with only two feedback levels at 0 and $R$, respectively. ∎

Theorem 4 characterizes the optimal outage diversity for INR-ARQ systems with $K = 2$ when $\frac{BR}{M}$ is noninteger. When $\frac{BR}{M}$ is integer, the outage diversity at round $\ell$ is upper bounded by $\hat{d}_\ell(R)$ given in (23). A lower bound on the outage diversity is given by the recursive formula in (23) with $\hat{d}_1(R) = \underline{d}(R)$.

### D. Numerical Results

We numerically compare the optimal rate-diversity-delay tradeoff of INR-ARQ systems with $K \geq \lceil \frac{BR}{M} \rceil + 1$, and with $K = 2$ as well as the optimal tradeoff of an INR-ARQ system with constant transmit power. The optimal rate-diversity-delay tradeoff $d_L(R)$ and $\hat{d}_L(R)$ for INR-ARQ transmission with $L = 1, 2, 3$ over the MIMO block-fading channel with $N_t = N_r = B = 2$ are illustrated in Fig. 3(a).

For an INR-ARQ system with delay constraint $L$ and constant transmit power (short-term power constraint), the outage probability $p(L)$ is the same as that obtained by transmission with rate $\frac{R}{L}$ over a block-fading channel with $BL$ fading blocks [21]. From Theorem 1, the optimal outage diversity $\underline{d}_L(R)$ is given by[2]

$$\underline{d}_L(R) = N_r \left(1 + \left\lfloor BL\left(N_t - \frac{R}{LM}\right)\right\rfloor\right) \qquad (24)$$

and is achievable by random codes for all rates $R$ such that $\underline{d}_L(R)$ is continuous. The rate-diversity-delay tradeoff of the INR-ARQ system with constant transmit power is plotted in Fig. 3(b). Fig. 3 shows an order-of-magnitude improvement in outage diversity of INR-ARQ when a long-term power constraint is allowed. Furthermore, significant gains in outage diversity are provided by multibit feedback, especially at transmission rates $R$ close to $N_tM$. Since high $R$ is particularly relevant in ARQ systems, the result suggests that multibit feedback will give significant gains in practical implementations.

## V. Power Adaptation and Feedback Design

The design of optimal feedback and transmission rules for an ARQ system with multibit feedback includes joint optimization of the overall set of quantization thresholds $\{\overline{I}(\boldsymbol{k}_{L-1}), \boldsymbol{k}_{L-1} \in \{0, \ldots, K-2\}^{L-1}\}$ and the corresponding power adaptive rule $P_\ell(\boldsymbol{k}_{\ell-1})$. The optimal feedback and power adaptation rule is obtained by minimizing

$$\sum_{\boldsymbol{k}_{L-1}} q(\boldsymbol{k}_{L-1}) p(L \,|\, \boldsymbol{k}_{L-1}) \qquad (25)$$

subject to the power constraint in (10). To the best of our knowledge, the optimization problem is not analytically tractable. We, therefore, propose to partition the design problem into two steps.

Step 1: At round $\ell$, determine a set of feedback thresholds $\overline{I}([\boldsymbol{k}_{\ell-1}, k])$ for every feedback vector $\boldsymbol{k}_{\ell-1} \in \{1, \ldots, K-2\}^{\ell-1}$.

Step 2: Given the set of feedback thresholds in Step 1, determine the corresponding transmit power rule, minimizing the outage probability.

The above procedure sub-optimally partitions the joint optimization problem into two sequential problems. Moreover, in the following, each individual problem is also sub-optimally solved. Nevertheless, this design procedure leads to a practically implementable algorithm that achieves the optimal diversity derived in the previous section.

[2]The rate-diversity-delay tradeoff of [21] is larger than that given in (24) since it is obtained with rotations, which increase the constellation size, complexity and peak-to-average power ratio.

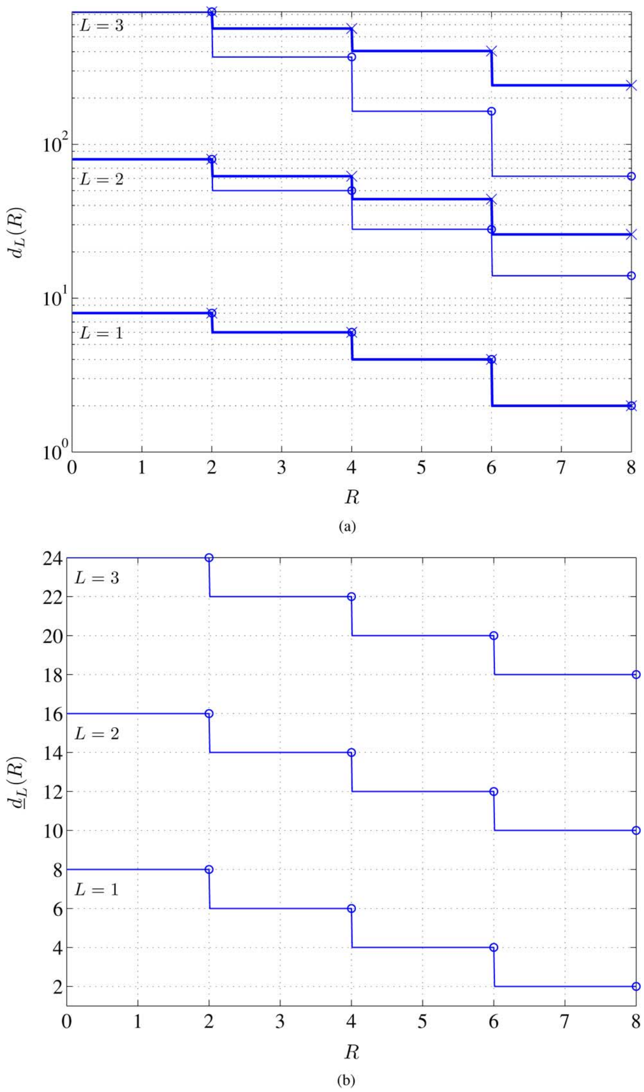


Fig. 3. Optimal rate-diversity-delay tradeoff of ARQ transmission with (a) long-term power constraint and (b) and constant power. 16-QAM is used over a MIMO block-fading channel with $N_t = N_r = 2, B = 2, L = 1, 2, 3$. Thick and thin lines in (a) represent the optimal tradeoffs $d_L(R)$ achieved by multibit feedback ($K \geq \lceil BR/M \rceil + 1$) and $\hat{d}_L(R)$ achieved by one-bit feedback ($K = 2$), respectively. Crosses and circles correspond to the rate points where the SNR-exponent of random codes does not achieve the optimal diversity.

### A. Selecting the Set of Feedback Thresholds

From the observations in Remark 1, we propose the following choice of feedback thresholds. Consider the feedback levels at round $\ell$ for a given feedback vector $\boldsymbol{k}_{\ell-1}$. Let $\tau \triangleq \left\lfloor \frac{BR}{M} \right\rfloor$, $\hat{I}_t = \frac{Mt}{B}$ and $t' \triangleq \left\lfloor \frac{B\overline{I}(\boldsymbol{k}_{\ell-1})}{M} \right\rfloor$. The feedback thresholds in round $\ell$, given $\boldsymbol{k}_{\ell-1}$ is then determined as follows.

1) Place a threshold at $\overline{I}([\boldsymbol{k}_{\ell-1}, 0]) = \overline{I}(\boldsymbol{k}_{\ell-1})$, and at $\overline{I}([\boldsymbol{k}_{\ell-1}, K-1]) = R$;
2) Place $\tau - t'$ thresholds at $\hat{I}_t, t = t'+1, \ldots, \tau$;
3) Place the remaining $K-2-\tau+t'$ thresholds sequentially within

$$\left(\hat{I}_\tau, R\right), \left(\overline{I}(\boldsymbol{k}_{\ell-1}), \hat{I}_{t'+1}\right), \left(\hat{I}_{\tau-1}, \hat{I}_\tau\right), \left(\hat{I}_{t'+1}, \hat{I}_{t'+2}\right), \ldots$$

until no more thresholds are left to place, and such that the thresholds uniformly partition each region.

The procedure for choosing the thresholds $\overline{I}(\boldsymbol{k}_\ell)$, given the feedback vector $\boldsymbol{k}_{\ell-1}$, is illustrated in Fig. 4. More particularly, the feedback thresholds for INR-ARQ transmission over the block-fading channel with $N_t = N_r = 1$, $B = 2$, $K = 4$, $L = 2$, and $R = 3.5$ using 16-QAM constellations are illustrated in Fig. 2, where $\overline{I}(\boldsymbol{k}_{\ell-1}) = \overline{I}([\boldsymbol{k}_{\ell-1}, 0])$, and the values of $\overline{I}(\boldsymbol{k}_2)$ are reported in Table II.

### B. Power Adaptation

The sub-optimal power adaptation rule is obtained from the following simplifications.

- We consider a power constraint more stringent than the constraint in (10),

$$\sum_{\boldsymbol{k}_\ell \in \{0,\ldots,K-1\}^\ell} q(\boldsymbol{k}_\ell) P_{\ell+1}(\boldsymbol{k}_\ell) \leq \frac{P}{L} \tag{26}$$

for $\boldsymbol{k}_\ell \in \{0, \ldots, K-1\}^\ell, \ell = 0, \ldots, L-1$, where $q(\boldsymbol{k}_0) = 1$ by definition.
- When feedback $\boldsymbol{k}_{\ell-1}$ is received, we have that $I_{\overline{1,\ell-1}} \geq \overline{I}(\boldsymbol{k}_{\ell-1})$. Then, the feedback probability is approximated from (9) by replacing $I_{\overline{1,\ell-1}}$ with $\overline{I}(\boldsymbol{k}_{\ell-1})$; and the outage probability can be upper bounded as

$$p(\ell \,|\, \boldsymbol{k}_{\ell-1}) \leq \hat{p}(\ell \,|\, \boldsymbol{k}_{\ell-1}) \triangleq \Pr\left\{ I_\ell + \overline{I}(\boldsymbol{k}_{\ell-1}) < R \right\} \tag{27}$$

where $I_\ell$ is given by (5) with $P_\ell = P_\ell(\boldsymbol{k}_{\ell-1})$.
- To further simplify the problem, we consider minimizing $\hat{p}(\ell), \ell = 1, \ldots, L$ sequentially.

Based on the simplifications, the corresponding power adaptation rule $P_\ell(\boldsymbol{k}_{\ell-1})$ is obtained by solving

$$\begin{cases} \text{Minimize} & \sum_{\boldsymbol{k}_{\ell-1}} q(\boldsymbol{k}_{\ell-1}) \hat{p}(\ell \,|\, \boldsymbol{k}_{\ell-1}) \\ \text{Subject to} & \sum_{\boldsymbol{k}_{\ell-1}} q(\boldsymbol{k}_{\ell-1}) P_\ell(\boldsymbol{k}_{\ell-1}) \leq \frac{P}{L}. \end{cases} \tag{28}$$

The optimization problem is separable and, thus, can be solved via a branch-and-bound simplex algorithm using piece-wise linear approximation [27]. For single-input multiple-output (SIMO) channels, the probabilities $q(\boldsymbol{k}_{\ell-1})$ and $\hat{p}(\ell \,|\, \boldsymbol{k}_{\ell-1})$ in (28) can be approximated numerically by shifting the outage probability bounds in [28] according to the gap between the bound and the corresponding simulation curve at high SNR. For MIMO channels, solving (28) requires tabulating the probabilities $q(\boldsymbol{k}_{\ell-1})$ and $\hat{p}(\ell \,|\, \boldsymbol{k}_{\ell-1})$, which can be obtained from Monte-Carlo simulations.

It follows from (27) that the objective function in (28) upper bounds the outage probability for a given power allocation rule. The bound is useful to evaluate the system performance, especially at low outage probability where Monte Carlo simulation is too computational demanding.

### C. Numerical Results

First consider SISO ($N_t = N_r = 1$) INR-ARQ transmission with $L = 2$ at rate $R = 3.5$ over the block-fading channel in (1) with $B = 2$ using 16-QAM input constellations. The outage

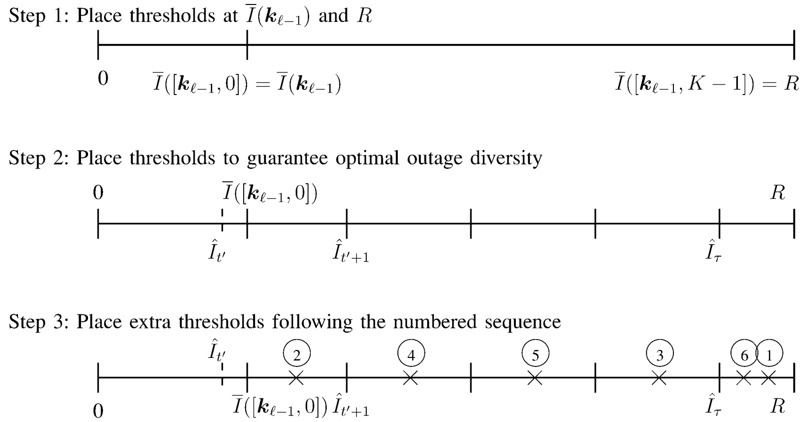


Fig. 4. Example of feedback threshold design ($K = 12$).

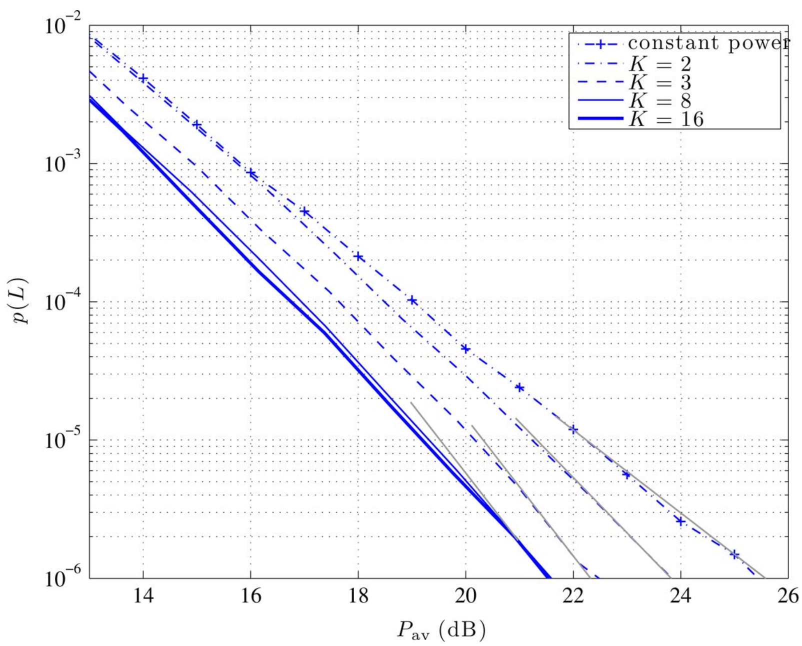


Fig. 5. Outage performance of ARQ transmission schemes for a 16-QAM input block-fading channel with $L = 2, N_t = N_r = 1, B = 2, R = 3.5$. The gray lines represent the optimal asymptotic slope.

TABLE II
FEEDBACK THRESHOLDS FOR $N_t = N_r = 1, B = 2, L = 2, R = 3.5$

| | $k_2 = 0$ | $k_2 = 1$ | $k_2 = 2$ |
|---|---|---|---|
| $k_1 = 0$ | 0 | 2 | 2.75 |
| $k_1 = 1$ | 2 | 2.5 | 3.0 |
| $k_1 = 2$ | 2.75 | 3.0 | 3.25 |

performance of systems with $K = 2, 3, 8, 16$ is illustrated in Fig. 5. The simulation shows that the slope of the outage curves approach the optimal outage diversity, which is 3 for constant power, 4 for $K = 2$ and 5 for $K \geq 3$. Due to the diversity gain, we observe significant gains in outage performance when power allocation is allowed, especially in systems with multibit feedback. Particularly, power allocation for the system with $K = 2$ provides approximately 2 dB gain in power at outage probability $10^{-6}$; while an additional 2 dB gain is observed when $K \geq 8$.

The outage performance of MIMO INR-ARQ transmission over the block-fading channel in (1) with $N_t = 2, N_r = 1, B = 1, R = 7.5$ using 16-QAM input constellations is illustrated in Figs. 6. Similar to the SISO case, Fig. 6 shows that systems with power allocation significantly outperform that with constant transmit power. A 6 dB gain in power is observed at outage probability $10^{-3}$ when power allocation is employed in an ARQ systems with one bit feedback. Further performance gains are obtained with multibit. For $K \geq 8$, an additional 1.5 dB gain in power is observed at outage probability $10^{-3}$.

In both cases, the simulation results suggest that increasing $K$ beyond 8 does not substantially improve the outage performance; and thus, the figures show that even for $K = 3$, the suboptimal choice of feedback thresholds in Section V-A performs within 1 dB of systems with large $K$ and optimal thresholds.

The simulation ranges in Figs. 5 and 6 do not reveal the outage diversity of the ARQ systems. We employ the bounds obtained from solving (28) to numerically illustrate the outage diversity

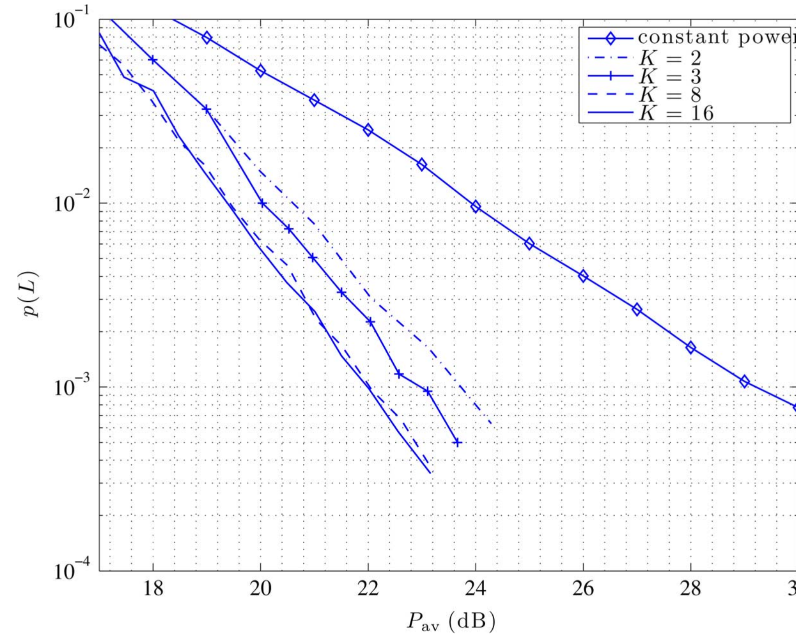


Fig. 6. Outage performance of ARQ transmission using the 16-QAM input constellation over the block-fading channel with $L = 2, N_t = 2, N_r = 1, B = 1, R = 7.5$. Systems with constant transmit power, and systems employing power adaptation with $K = 2, 3, 8, 16$ are considered.

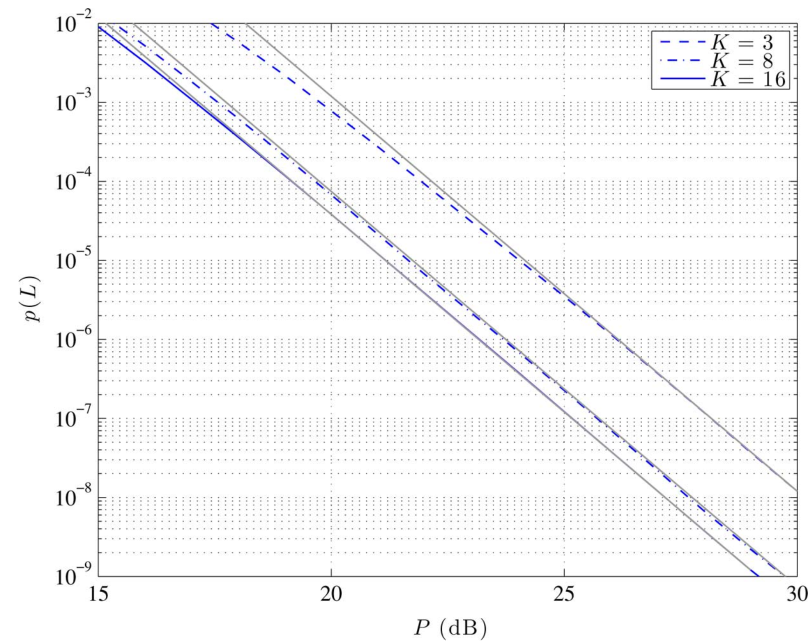


Fig. 7. Upper bound on outage performance of ARQ transmission schemes using 16-QAM constellation over the block-fading channel with $L = 2, N_t = N_r = 1, B = 2, R = 3.5$ and $K = 3, 8, 16$. The gray lines represent the asymptotic bound with diversity 5.

achieved by multibit feedback. Figs. 7 and 8 plot the bounds of the outage performance presented in Figs. 5 and 6. The plots show that in both the SISO and MIMO systems, outage diversity 5 is achieved when $K \geq 3$, as predicted by Theorem 2.

## VI. Conclusion

We have studied the outage performance of MIMO block-fading channels with and without employing the INR-ARQ strategy. An information-theoretic multibit feedback INR-ARQ scheme is proposed based on the accumulative mutual information, which potentially improves the performance of INR-ARQ transmission with minimal extra overhead requirement compared to classical INR-ARQ. The study on power adaptation has revealed large gains in outage diversity provided by multibit feedback in INR-ARQ systems with a long-term power constraint. More generally, the multibit feedback INR-ARQ based on accumulated mutual information may prove useful in obtaining the fundamental limit of multibit feedback INR-ARQ systems. Furthermore, since the proposed scheme is a generalization to that in [17] and [19], it promises

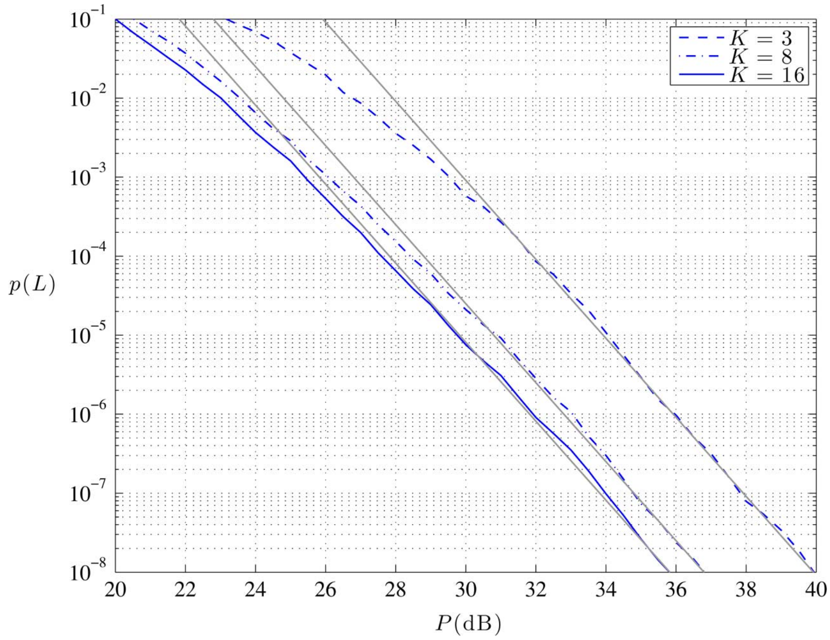


Fig. 8. Upper bound on outage performance of ARQ transmission using 16-QAM input constellations over the block-fading channel with $L = 2, N_t = 2, N_r = 1, B = 1, R = 7.5$ and $K = 3, 8, 16$. The gray lines represent the asymptotic bound with diversity 5.

further gain from the throughput performance obtained in [17], [19].

## APPENDIX A
## *PROOF OF THEOREM 1

We first assume a genie-aided receiver that perfectly eliminates the interference between the transmit antennas. This results in $N_t$ parallel SIMO block-fading channels, each with $N_r$ receive antennas. Let $I^{\mathrm{ga}}$ be the realized input-output mutual information of the genie-aided channel, then $I^{\mathrm{ga}} \geq I$. Furthermore, from the analysis in [26], [28], [29], we have that

$$\Pr\{I^{\mathrm{ga}} < R\} \doteq P^{-\overline{d}(R)}. \tag{29}$$

Therefore

$$\Pr\{I < R\} \dot{\geq} P^{-\overline{d}(R)}. \tag{30}$$

The proof is thus completed by proving that

$$\Pr\{I \leq R\} \doteq P^{-\underline{d}(R)}. \tag{31}$$

Following the arguments in [26], [28], [29], we have that

$$\Pr\{I^{\mathrm{ga}} \leq R\} \doteq P^{-\underline{d}(R)} \tag{32}$$

and therefore

$$\Pr\{I \leq R\} \dot{\geq} P^{-\underline{d}(R)}. \tag{33}$$

We now prove that $\Pr\{I \leq R\} \dot{\leq} P^{-\underline{d}(R)}$. Considering transmission over the block-fading channel in (1) with random codes of rate $R$, where the $JBN_t$ coded symbols in $\boldsymbol{x}$ are drawn uniformly random from the constellation $\mathcal{X}$. Let ${P_e}^{(r)}$ be the word error probability achieved by random coding. We have from the random-coding achievability and the strong converse theorem [3]–[5] that for a channel realization $\boldsymbol{H}$

$${P_e}^{(r)}(\boldsymbol{H}) = \begin{cases} 1, & \text{if } I < R \\ 0, & \text{if } I > R \end{cases} \tag{34}$$

when $J \to \infty$. Therefore, the word error probability of random codes satisfies

$${P_e}^{(r)} = \Pr\{I \leq R\}. \tag{35}$$

We now prove that ${P_e}^{(r)} \dot{\leq} P_\ell^{-\underline{d}(I)}$. Consider encoding and transmitting a message $m$ as a random codeword $\boldsymbol{X}$. Assuming that the channel realization is $\boldsymbol{H}$, the pairwise error probability between $\boldsymbol{X}$ and $\boldsymbol{X}'$ is bounded by [25]

$$P_{\mathrm{PEP}}\left(\boldsymbol{X} \to \boldsymbol{X}' \,|\, \boldsymbol{H}\right) \leq \exp\left(-\frac{1}{4} g^2(\boldsymbol{X}, \boldsymbol{X}', \boldsymbol{H})\right) \tag{36}$$

where, by letting $\hat{P} = \frac{P_\ell}{N_t}$

$$g^2(\boldsymbol{X}, \boldsymbol{X}', \boldsymbol{H}) = \sum_{b=1}^{B} \sum_{j=1}^{J} \sum_{r=1}^{N_r} \left| \sum_{t=1}^{N_t} \sqrt{\hat{P}} h_{b,t,r} (\boldsymbol{X}_{b,t,j} - \boldsymbol{X}'_{b,t,j}) \right|^2. \tag{37}$$

Here, $h_{b,t,r}$ is the channel gain from transmit antenna $t$ to receive antenna $r$ in block $b$, and $X_{b,t,j}$ is the coded symbol transmitted by antenna $t$ at time instant $j$ of block $b$. Let us write

$h_{b,t,r} = |h_{b,t,r}|e^{i\theta_{b,t,r}}$, where $i = \sqrt{-1}$. Further define a matrix of normalized fading gains $\boldsymbol{\alpha} \in \mathbb{R}^{B\times N_t\times N_r}$ where $\alpha_{b,t,r} \triangleq -\frac{\log(|h_{b,t,r}|^2)}{\log(\hat{P})}$, then

$$g^2(\boldsymbol{X},\boldsymbol{X}',\boldsymbol{H}) = \sum_{b=1}^{B}\sum_{j=1}^{J}\sum_{r=1}^{N_r}\left|\sum_{t=1}^{N_t}\hat{P}^{\frac{1-\alpha_{b,t,r}}{2}}e^{i\theta_{b,t,r}}(\boldsymbol{X}_{b,t,j}-\boldsymbol{X}'_{b,t,j})\right|^2. \quad (38)$$

By averaging (36) over the random coding ensemble, the pairwise error probability of random codes is

$$P_{\rm PEP}^{(r)}(\boldsymbol{X}\to\boldsymbol{X}'\,|\,\boldsymbol{H}) \le \prod_{b=1}^{B}\left\{\frac{1}{2^{2MN_t}}\sum_{\mathbf{x}\in\mathcal{X}^{N_t}}\sum_{\mathbf{x}'\in\mathcal{X}^{N_t}}\exp\left(-\frac{1}{4}\times\sum_{r=1}^{N_r}\left|\sum_{t=1}^{N_t}\hat{P}^{\frac{1-\alpha_{b,t,r}}{2}}e^{i\theta_{b,t,r}}(x_t-x'_t)\right|^2\right)\right\}^J \quad (39)$$
$$\le \exp\left(BMJ\log(2)\left(-2N_t+\frac{1}{BM}T(\hat{P},\boldsymbol{\alpha})\right)\right) \quad (40)$$

where $x_t$ is the $t$th entry of vector $\mathbf{x}$ and

$$T(\hat{P},\boldsymbol{\alpha}) \triangleq \sum_{b=1}^{B}\log_2\left(\sum_{\mathbf{x}\in\mathcal{X}^{N_t}}\sum_{\mathbf{x}'\in\mathcal{X}^{N_t}}\exp\left(-\frac{1}{4}\times\sum_{r=1}^{N_r}\left|\sum_{t=1}^{N_t}\hat{P}^{\frac{1-\alpha_{b,t,r}}{2}}e^{i\theta_{b,t,r}}(x_t-x'_t)\right|^2\right)\right). \quad (41)$$

By summing over the $2^{BRJ}-1$ possible error events, the union bound on the word error probability is given by

$$P_e{}^{(r)}(\boldsymbol{H}) \le \min\left\{1, \exp\left(BMJ\log(2)\times\left(-2N_t+\frac{R}{M}+\frac{1}{BM}T(\hat{P},\boldsymbol{\alpha})\right)\right)\right\}. \quad (42)$$

For any $\epsilon > 0$, denote $\mathcal{S}_b^{(\epsilon)} \triangleq \bigcup_{r=1}^{N_r}\mathcal{S}_{b,r}^{(\epsilon)}$, and $\kappa_b \triangleq |\mathcal{S}_b^{(\epsilon)}|$, where

$$\mathcal{S}_{b,r}^{(\epsilon)} \triangleq \{t:\alpha_{b,t,r}\le 1-\epsilon, t=1,\ldots,N_t\}. \quad (43)$$

Then, for any given $r \in \{1,\ldots,N_r\}$, and letting $\alpha_{b,r} = \max\{\alpha_{b,t,r}, t\in\mathcal{S}_{b,r}^{(\epsilon)}\}$, we can write

$$\lim_{\hat{P}\to\infty}\sum_{t=1}^{N_t}\hat{P}^{\frac{1-\alpha_{b,t,r}}{2}}e^{i\theta_{b,t,r}}(x_t-x'_t) \ge_{\hat{P}\to\infty}\sum_{\substack{t\in\mathcal{S}_{b,r}^{(\epsilon)}\\ x_t\neq x'_t}}\hat{P}^{\frac{1-\alpha_{b,t,r}}{2}}e^{i\theta_{b,t,r}}(x_t-x'_t) \quad (44)$$
$$\ge \lim_{\hat{P}\to\infty}\hat{P}^{\frac{1-\alpha_{b,r}}{2}}\sum_{\substack{t\in\mathcal{S}_{b,r}^{(\epsilon)}\\ x_t\neq x'_t}}e^{i\theta_{b,t,r}}(x_t-x'_t). \quad (45)$$

Since the $\theta_{b,t,r}$'s are uniformly drawn from $[-\pi,\pi]$, we have that

$$\lim_{\hat{P}\to\infty}\sum_{t=1}^{N_t}\hat{P}^{\frac{1-\alpha_{b,t,r}}{2}}e^{i\theta_{b,t,r}}(x_t-x'_t)=\infty \quad (46)$$

with probability 1 if there exists $t\in\mathcal{S}_{b,r}^{(\epsilon)}$ such that $x_t\neq x'_t$. Noting that $\kappa_b = |\mathcal{S}_b^{(\epsilon)}|$, it follows from (41) that

$$\lim_{\hat{P}\to\infty}T(\hat{P},\boldsymbol{\alpha}) = \sum_{b=1}^{B}\lim_{\hat{P}\to\infty}\log_2\left(\sum_{\mathbf{x}\in\mathcal{X}^{N_t}}\sum_{\substack{\mathbf{x}'\in\mathcal{X}^{N_t}\\ x'_t=x_t,\forall t\in\mathcal{S}_b^{(\epsilon)}}}\exp\left(-\frac{1}{4}\times\sum_{r=1}^{N_r}\left|\sum_{t=1}^{N_t}\hat{P}^{\frac{1-\alpha_{b,t,r}}{2}}e^{i\theta_{b,t,r}}(x_t-x'_t)\right|^2\right)\right)$$
$$\le\sum_{b=1}^{B}\log_2\left(2^{MN_t}2^{M(N_t-\kappa_b)}\right) = \sum_{b=1}^{B}M(2N_t-\kappa_b). \quad (47)$$

Thus, the error probability in (42) is asymptotically upper-bounded by

$$\lim_{\hat{P}\to\infty}P_e{}^{(r)}(\boldsymbol{H}) \le \min\left\{1,\exp\left(-BMJ\log(2)\times\left(\frac{1}{B}\sum_{b=1}^{B}\kappa_b-\frac{R}{M}\right)\right)\right\}. \quad (48)$$

Let $\mathcal{B}^{(\epsilon)} \triangleq \left\{\boldsymbol{\alpha}\in\mathbb{R}^{B\times N_t\times N_r}:\sum_{b=1}^{B}\kappa_b\le\frac{BR}{M}\right\}$ be the outage set. By averaging over the fading matrix and letting $J\to\infty$, the error probability is bounded by

$$P_e{}^{(r)} \le \int_{\boldsymbol{\alpha}\in\mathcal{B}^{(\epsilon)}} f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})d\boldsymbol{\alpha} \quad (49)$$

where $f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})$ is the joint pdf of the random vector $\boldsymbol{\alpha}$. Following the analysis in [29], and letting $J\to\infty$, the SNR-exponent for the case of using random codes is lower bounded by

$$\inf_{\boldsymbol{\alpha}\in\mathcal{B}^{(\epsilon)}\cap\mathbb{R}_+^{BN_r\times BN_t}}\left\{\sum_{b=1}^{B}\sum_{t=1}^{N_t}\sum_{r=1}^{N_r}\alpha_{b,t,r}\right\} = N_r\left(BN_t-\left\lfloor\frac{BR}{M}\right\rfloor\right)(1-\epsilon) \quad (50)$$
$$= N_r\left\lceil B\left(N_t-\frac{R}{M}\right)\right\rceil(1-\epsilon). \quad (51)$$

Thus, by letting $\epsilon\downarrow 0$, the outage diversity $\underline{d}(R)$ is achievable using random codes. Therefore, we have from (35) that

$$\Pr\{I\le R\}\dot{\le}\hat{P}^{-\underline{d}(R)}\doteq P^{-\underline{d}(R)}. \quad (52)$$

Thus, (31) is obtained from (33).

## Appendix B
## Proof of Theorem 2

A sketch of the proof is given as follows. We first lower-bound the outage diversity by considering a sub-optimal ARQ system with $\underline{K} = \lceil \frac{BR}{M} \rceil + 1$ feedback levels, where the quantization thresholds are placed at $\overline{I}([\boldsymbol{k}_{\ell-1}, k_\ell]) = \frac{k_\ell M}{B}, k_\ell = 0, \ldots, \lfloor \frac{BR}{M} \rfloor$. Using Theorem 1, we prove by induction that the outage diversity of the sub-optimal ARQ system at round $\ell$ is $d_\ell(R)$.

Conversely, consider an optimal INR-ARQ system with $K \geq \lceil \frac{BR}{M} \rceil + 1$ feedback levels. The outage performance of the system can be improved by adding $\lfloor \frac{BR}{M} \rfloor + 1$ extra quantization thresholds (and corresponding feedback indices) at $\frac{tM}{B}, t = 0, \ldots, \lfloor \frac{BR}{M} \rfloor$. Using Theorem 1, we prove by induction that the outage diversity at round $\ell$ of the improved systems (with $K + \lfloor \frac{BR}{M} \rfloor + 1$ feedback levels) is also given by $d_\ell(R)$. Therefore, $d_\ell(R)$ is the optimal outage diversity at round $\ell$ for an ARQ system with $K \geq \lceil \frac{BR}{M} \rceil + 1$ feedback levels.

### A. Lower Bound on the Optimal Outage Diversity

To get a lower bound to the outage diversity, consider an ARQ system with $K = \lceil \frac{BR}{M} \rceil + 1$ feedback levels, where the following (sub-optimal) set of feedback thresholds is employed,

$$\overline{I}(\boldsymbol{k}_\ell) = \begin{cases} \hat{I}_{k_\ell}, & 0 \leq k_\ell < K-1 \\ R, & k_\ell = K-1 \end{cases} \tag{53}$$

with $\hat{I}_t = \frac{tM}{B}$. In this case, feedback index $k_\ell = t$ is delivered at round $\ell$ if $I_{\overline{1,\ell}} \in [\hat{I}_t, \hat{I}_{t+1})$, regardless of the realized feedback indices of the previous rounds. At round $\ell$, the transmit power is sub-optimally adapted to the feedback index $k_{\ell-1}$ as $P_\ell = P_\ell(k_{\ell-1})$, where

$$P_\ell(k_{\ell-1}) = \begin{cases} \dfrac{P}{KL \Pr\left\{I_{\overline{1,\ell-1}} \in \left[\hat{I}_{k_{\ell-1}}, \hat{I}_{k_{\ell-1}+1}\right)\right\}}, & k_{\ell-1} < K-1 \\ 0, & \text{otherwise.} \end{cases} \tag{54}$$

The power adaptation rule in (54) satisfies the power constraint in (10). We now derive the outage diversity achieved by the aforementioned system.

For $I \in \left(\hat{I}_t, \hat{I}_{t+1}\right)$, we have from Theorem 1 that

$$\Pr\{I_1 < I\} \doteq \Pr\left\{I_1 \in \left[\hat{I}_t, \hat{I}_{t+1}\right)\right\} \doteq P^{-\delta_1(t)} \tag{55}$$

where $\delta_1(t) \triangleq d(\hat{I}_{t+1}) = N_r(BN_t - t)$.

For $t = 0, \ldots, BN_t - 1$ and a given $I \in \left(\hat{I}_t, \hat{I}_{t+1}\right)$, we now prove by induction that for $\ell = 1, \ldots, L$

$$\Pr\{I_{\overline{1,\ell}} < I\} \doteq \Pr\left\{I_{\overline{1,\ell}} \in \left[\hat{I}_t, \hat{I}_{t+1}\right)\right\} \doteq P^{-\delta_\ell(t)} \tag{56}$$

where $\delta_\ell(t) = d_\ell\left(\hat{I}_{t+1}\right)$ is given in (18).

Equation (55) shows that (56) is correct at round 1. Assume now that (56) is correct at round $\ell$. From (54) we have that

$$P_{\ell+1}(t) = \frac{P}{KL \Pr\left\{I_{\overline{1,\ell}} \in \left[\hat{I}_t, \hat{I}_{t+1}\right)\right\}} \doteq P^{1+\delta_\ell(t)}. \tag{57}$$

Therefore, for $I \in \left(\hat{I}_t, \hat{I}_{t+1}\right)$

$$\begin{aligned} \Pr\left\{I_{\overline{1,\ell+1}} < I\right\} &= \sum_{j=0}^{t} \Pr\left\{I_{\overline{1,\ell}} \in \left[\hat{I}_j, \hat{I}_j + I - \hat{I}_t\right)\right\} \\ &\quad \times \Pr\left\{I_{\ell+1} < I - I_{\overline{1,\ell}} \middle| I_{\overline{1,\ell}} \in \left[\hat{I}_j, \hat{I}_j + I - \hat{I}_t\right)\right\} \\ &\quad + \sum_{j=0}^{t} \Pr\left\{I_{\overline{1,\ell}} \in \left[\hat{I}_j + I - \hat{I}_t, \hat{I}_{j+1}\right)\right\} \\ &\quad \times \Pr\left\{I_{\ell+1} < I - I_{\overline{1,\ell}} \middle| I_{\overline{1,\ell}} \in \left[\hat{I}_j + I - \hat{I}_t, \hat{I}_{j+1}\right)\right\}. \end{aligned} \tag{58}$$

Given $I_{\overline{1,\ell}} \in \left[\hat{I}_j, \hat{I}_j + I - \hat{I}_t\right)$ and $I \in \left(\hat{I}_t, \hat{I}_{t+1}\right)$, we have that $I - I_{\overline{1,\ell}} \in \left(\hat{I}_{t-j}, \hat{I}_{t-j+1}\right)$. Therefore, by applying Theorem 1, and noting the transmit power in (57), we have that

$$\Pr\left\{I_{\ell+1} < I - I_{\overline{1,\ell}} \middle| I_{\overline{1,\ell}} \in \left[\hat{I}_j, \hat{I}_j + I - \hat{I}_t\right)\right\} \doteq P^{-(1+\delta_\ell(j))N_r(BN_t - t + j)}. \tag{59}$$

Since (56) is assumed at round $\ell$, the first summation dominates in (58). Thus, from (59), we have that

$$\Pr\left\{I_{\overline{1,\ell+1}} < I\right\} \doteq \sum_{j=0}^{t} P^{-\delta_\ell(j) - [1+\delta_\ell(j)]N_r(BN_t - t + j)}. \tag{60}$$

The asymptotic exponent in (60) is given by

$$\begin{aligned} &\min_{j=0,\ldots,t} \delta_\ell(j) + [1+\delta_\ell(j)]\, N_r(BN_t - t + j) && (61) \\ &= \min_{j=0,\ldots,t} -1 + (1 + BN_rN_t)^{\ell-1} \\ &\quad \times [1 + N_r(BN_t - j)]\,[1 + N_r(BN_t - t + j)] && (62) \\ &= -1 + (1 + BN_tN_r)^{\ell}\,[1 + N_r(BN_t - t)] && (63) \\ &= \delta_{\ell+1}(t) && (64) \end{aligned}$$

where (62) follows from assumption $\delta_\ell(j) = d_\ell(\hat{I}_{j+1})$ in (56), and (63) follows since the minimum in (62) is achieved with either $j = 0$ or $j = t$. Therefore, from (60)

$$\Pr\left\{I_{\overline{1,\ell+1}} < I\right\} \doteq P^{-\delta_{\ell+1}(t)} \tag{65}$$

where $\delta_{\ell+1}(t) = d_{\ell+1}(\hat{I}_{t+1})$ in (18). Thus, (56) is correct for $\ell = 1, \ldots, L$ by induction. Consequently, for any $R \in \left(\hat{I}_\tau, \hat{I}_{\tau+1}\right)$, we have that

$$\Pr\left\{I_{\overline{1,\ell}} < R\right\} \doteq P^{-\delta_\ell(\tau)} = P^{-d_\ell(\hat{I}_{\tau+1})} \tag{66}$$

and thus, the diversity in (18) is achieved by the ARQ system with $\tau+2=\left\lceil\frac{BR}{M}\right\rceil+1$ feedback levels.

Noting when $\Pr\left\{I_{\overline{1,\ell+1}}<R\right\}\doteq P^{-\delta_\ell(\tau)}$, the outage probability at round $\ell$ is dominated by the events with $j=0$ and $j=\tau$ in (60), which correspond to the events with $I_{\overline{1,\ell}}\in\left[0,\hat{I}_1\right)\cup\left[\hat{I}_\tau,R\right)$. The observation is useful for designing the feedback thresholds for the system, as summarized in Remark 1.

## *B. Upper Bound on the Optimal Outage Diversity*

Conversely, we derive an upper bound to the outage diversity achieved by a system with optimal feedback threshold $\overline{I}(\boldsymbol{k}_\ell)$ with $K$ levels per transmission round. We first assume that $R\in\left(\hat{I}_\tau,\hat{I}_{\tau+1}\right)$ for some $\tau\in\{0,\ldots,BN_t-1\}$. Consider improving the performance of the system by employing a feedback threshold set $I^\dagger\left(\boldsymbol{k}_\ell\right)$ with $\overline{K}=K+\tau+1$ feedback levels per ARQ round by adding $\tau+1$ levels to the optimal feedback threshold set $\{\overline{I}(\boldsymbol{k}_\ell)\}$. The extra $\tau+1$ levels are located at $\hat{I}_t=\frac{tM}{B}, t=0,\ldots,\tau$.

Let $\mathcal{A}_{\boldsymbol{k}_{\ell-1}}(k)\triangleq\left[I^\dagger\left([\boldsymbol{k}_{\ell-1},k]\right),I^\dagger\left([\boldsymbol{k}_{\ell-1},k+1]\right)\right),\ell=1,\ldots,L, k=0,\ldots,\overline{K}-2$, and further let $\overline{\mathcal{A}}_{\boldsymbol{k}\ell-1}(k)\triangleq\left(I^\dagger\left([\boldsymbol{k}_{\ell-1},k]\right),I^\dagger\left([\boldsymbol{k}_{\ell-1},k+1]\right)\right)$. Then, given that the feedback vector at round $\ell-1$ is $\boldsymbol{k}_{\ell-1}$, the receiver delivers feedback index $\overline{K}-1$ if $I_{\overline{1,\ell}}\geq I^\dagger\left([\boldsymbol{k}_{\ell-1},\overline{K}-1]\right)=R$; otherwise, it delivers index $k_\ell$, where $k_\ell$ is chosen such that $I_{\overline{1,\ell}}\in\mathcal{A}_{\boldsymbol{k}_{\ell-1}}(k_\ell)$.

From the power constraint (10), the optimal power allocation rule is upper-bounded by

$$\overline{P}_\ell(\boldsymbol{k}_{\ell-1})=\begin{cases}P, & \ell=1\\ \frac{P}{\Pr\left\{I_{\overline{1,\ell-1}}\in\mathcal{A}_{\boldsymbol{k}_{\ell-2}}(k_{\ell-1})\right\}}, & k_{\ell-1}<K-1\\ 0, & \text{otherwise.}\end{cases}\tag{67}$$

Meanwhile, the power adaptation rule

$$\underline{P}_\ell(\boldsymbol{k}_{\ell-1})=\begin{cases}\frac{P}{L}, & \ell=1\\ \frac{P}{\overline{K}L\Pr\left\{I_{\overline{1,\ell-1}}\in\mathcal{A}_{\boldsymbol{k}_{\ell-2}}(k_{\ell-1})\right\}}, & k_{\ell-1}<K-1\\ 0, & \text{otherwise.}\end{cases}\tag{68}$$

satisfies the power constraint in (10). Therefore, the optimal power allocation rule asymptotically satisfies $P_\ell(\boldsymbol{k}_{\ell-1})\doteq\overline{P}_\ell(\boldsymbol{k}_{\ell-1})$ given in (67).

For $t=0,\ldots,\tau$, let $\mathcal{S}_{\boldsymbol{k}_{\ell-1}}(t)=\left\{k\in\{1,\ldots,\overline{K}-2\}:\mathcal{A}_{\boldsymbol{k}_{\ell-1}}(k)\subseteq\left[\hat{I}_t,\hat{I}_{t+1}\right)\right\}$.
Since $\hat{I}_t$, for $t=1,\ldots,\tau$, belongs to the set of thresholds $\{I^\dagger\left([\boldsymbol{k}_{\ell-1},k_\ell]\right),k_\ell=0,\ldots,\overline{K}-1\}$, $\overline{\mathcal{A}}_{\boldsymbol{k}\ell-1}(k)\subseteq\left(\hat{I}_t,\hat{I}_t+1\right)$ for some $t\in\{1,\ldots,\tau\}$. Applying Theorem 1, for any $I\in\left(\hat{I}_t,\hat{I}_{t+1}\right)$ and $k\in\mathcal{S}_{\boldsymbol{k}_0}(t)$, we have that

$$\Pr\{I_1<I\}\doteq P^{-N_r(BN_t-t)}\doteq P^{-\delta_1(t)}\tag{69}$$

$$\Pr\{I_1\in\mathcal{A}_{\boldsymbol{k}_0}(k)\}\doteq\Pr\left\{I<I^\dagger([k+1])\right\}\doteq P^{-\delta_1(t)}\tag{70}$$

where $\delta_1(t)=d_1(\hat{I}_{t+1})$ given in (18).

For the induction proof, assume that when $I\in\left(\hat{I}_t,\hat{I}_{t+1}\right)$ and $k\in\mathcal{S}_{\boldsymbol{k}_{\ell-1}}(t)$, we have

$$\Pr\left\{I_{\overline{1,\ell}}<I\right\}\doteq\Pr\left\{I_{\overline{1,\ell}}\in\mathcal{A}_{\boldsymbol{k}_{\ell-1}}(k)\right\}\doteq P^{-\delta_\ell(t)}\tag{71}$$

where $\delta_\ell(t)=d_\ell(\hat{I}_{t+1})$ given in (18). The assumption is correct for $\ell=1$. We prove that (71) is also valid at round $\ell+1$. In fact, considering $I\in\left(\hat{I}_t,\hat{I}_{t+1}\right)$, we have

$$\begin{aligned}\Pr\left\{I_{\overline{1,\ell+1}}<I\right\}&=\sum_{j=0}^{t}\Pr\left\{I_{\overline{1,\ell}}\in\left[\hat{I}_j,\hat{I}_j+I-\hat{I}_t\right)\right\}\\&\quad\times\Pr\left\{I_{\ell+1}<I-I_{\overline{1,\ell}}\middle|I_{\overline{1,\ell}}\in\left[\hat{I}_j,\hat{I}_j+I-\hat{I}_t\right)\right\}\\&\quad+\sum_{j=0}^{t}\Pr\left\{I_{\overline{1,\ell}}\in\left[\hat{I}_j+I-\hat{I}_t,\hat{I}_{j+1}\right)\right\}\\&\quad\times\Pr\left\{I_{\ell+1}<I-I_{\overline{1,\ell}}\middle|I_{\overline{1,\ell}}\in\left[\hat{I}_j+I-\hat{I}_t,\hat{I}_{j+1}\right)\right\}.\end{aligned}$$

From assumption (71) and power allocation rule (67), when $I_{\overline{1,\ell}}\in\mathcal{A}_{\boldsymbol{k}_{\ell-1}}(k_\ell)$, the transmit power in round $\ell+1$ is $P_{\ell+1}\doteq\frac{P}{\Pr\left\{I_{\overline{1,\ell}}\in\mathcal{A}_{\boldsymbol{k}_{\ell-1}}(k_\ell)\right\}}\doteq P^{1+\delta_\ell(j)}$ for all $k_\ell\in\mathcal{S}_{\boldsymbol{k}_\ell}(j)$. Therefore, when $I_{\overline{1,\ell}}\in\left[\hat{I}_j,\hat{I}_{j+1}\right)$, $P_{\ell+1}\doteq P^{1+\delta_\ell(j)}$. Thus, with similar arguments that are used to derive (59), we have that

$$\Pr\left\{I_{\overline{1,\ell+1}}<I\right\}\doteq\sum_{j=0}^{t}P^{-(\delta_\ell(j)+(1+\delta_\ell(j))N_r(BN_t-t+j))}\tag{72}$$

as given in (60). Therefore, following the steps used to derive (65), we have that

$$\Pr\left\{I_{\overline{1,\ell+1}}<I\right\}\doteq P^{-\delta_{\ell+1}(t)}\tag{73}$$

for $I\in\left(\hat{I}_t,\hat{I}_{t+1}\right)$. It follows that

$$\begin{aligned}\Pr\left\{I_{\overline{1,\ell+1}}\in\mathcal{A}_{\boldsymbol{k}_\ell}(k)\right\}&\doteq\Pr\left\{I_{\overline{1,\ell+1}}<I^\dagger\left([\boldsymbol{k}_\ell,k]\right)\right\}\\&\doteq P^{-\delta_{\ell+1}(t)}\end{aligned}\tag{74}$$

for all $k\in\mathcal{S}_{\boldsymbol{k}_\ell}(t)$. The results in (73) and (74) prove that assumption (71) is valid at round $\ell+1$, and thus, by mathematical induction, (71) is valid for $\ell=1,\ldots,L$.

Since $R\in\left(\hat{I}_\tau,\hat{I}_{\tau+1}\right)$

$$\Pr\left\{I_{\overline{1,\ell}}<R\right\}\doteq P^{-\delta_\ell(\tau)}\doteq P^{-d_\ell(\hat{I}_{\tau+1})}\tag{75}$$

which proves that the outage diversity of the system with $\overline{K}$-level feedback is the same as that given in (18).

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**Khoa D. Nguyen** (S'06–M'10) was born in Vietnam in 1982. He received the Bachelor of Engineering degree (electrical and electronics engineering) from the University of Melbourne, Australia, in December 2005 and the Ph.D. degree from the Institute for Telecommunications Research, University of South Australia, in March 2010.

Since 2009, Dr. Nguyen has been a Research Fellow at the Institute for Telecommunications Research. He was a summer research scholar at the Australian National University in 2004 and held a visiting appointment at the University of Cambridge, Cambridge, U.K., in 2007. His research interests are in the areas of information theory and communication theory, coding theory, and adaptation and signal processing for wireless applications.

**Lars K. Rasmussen** (S'92–M'93–SM'01) was born on March 8, 1965 in Copenhagen, Denmark. He got his M.Eng. in 1989 from the Technical University of Denmark (Lyngby, Denmark) and his Ph.D. degree from Georgia Institute of Technology (Atlanta, GA) in 1993. From 1993 to 1995, he was a Research Fellow at the Institute for Telecommunications Research (ITR), University of South Australia (Adelaide, Australia). From 1995 to 1998, he was a Senior Member of Technical Staff with the Centre for Wireless Communications at the National University of Singapore (Singapore). From 1999 to 2002, he was an Associate Professor at Chalmers University of Technology (Gothenburg, Sweden), where he maintained a part-time appointment until 2005. From 2002 to 2008, he held a position as Research Professor at ITR, University of South Australia (Adelaide, Australia), where he was the Convenor of the Australian Research Council (ARC) Communications Research Network (ACoRN), and a co-founder of Cohda Wireless Pty Ltd. He has held visiting positions at University of Pretoria (Pretoria, South Africa), Southern Poro Communications (Sydney, Australia), and Aalborg University (Aalborg, Denmark). He now holds a position as Professor in Communications Theory, School of Electrical Engineering, and the ACCESS Linnaeus Center at the KTH Royal Institute of Technology (Stockholm, Sweden). He is a Senior Member of the IEEE, a member of the IEEE Information Theory and Communications Societies and served as Chairman for the Australian Chapter of the IEEE Information Theory Society 2004–2005, and has been a board member of the joint IEEE Communications Society and IEEE Vehicular Technology Chapter in Sweden since 2010. He is an associate editor for IEEE TRANSACTIONS ON COMMUNICATIONS and was a guest editor for IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS (2007). He was also a member of the organizing committees for the IEEE 2004 International Symposium on Spread Spectrum Systems and Applications (Sydney, Australia), and the IEEE 2005 International Symposium on Information Theory (Adelaide, Australia), as well as the TPC co-chair of the Communications Theory Symposium at the IEEE Global Communications Conference (Globecom) 2009. His research interests include transmission strategies and coding schemes for wireless communications, coding for delay-constrained applications, ad hoc wireless networks, cooperative communications, communications and control, communications and positioning, and vehicular communication systems.

**Albert Guillén i Fàbregas** (S'01–M'05–SM'09) was born in Barcelona, Catalunya, Spain, in 1974. He received the Telecommunications Engineering Degree and the Electronics Engineering Degree from Universitat Politècnica de Catalunya, Barcelona, Catalunya, Spain, and the Politecnico di Torino, Torino, Italy, respectively, in 1999, and the Ph.D. in communication systems from Ecole Polytechnique Fédérale de Lausanne (EPFL), Lausanne, Switzerland, in 2004.

From August 1998 to March 1999, he conducted his Final Research Project at the New Jersey Institute of Technology, Newark, NJ. He was with Telecom Italia Laboratories, Italy, from November 1999 to June 2000 and with the European Space Agency (ESA), Noordwijk, The Netherlands, from September 2000 to May 2001. During his doctoral studies, from 2001 to 2004, he was a Research and Teaching Assistant at Institut Eurcom, Sophia-Antipolis, France. From June 2003 to July 2004, he was a Visiting Scholar at EPFL. From September 2004 to November 2006, he was a Research Fellow at the Institute for Telecommunications Research, University of South Australia, Mawson Lakes, Australia. Since 2007, he has been a Lecturer in the Department of Engineering, University of Cambridge, Cambridge, U.K., where he is also a Fellow of Trinity Hall. He has held visiting appointments at Centrum Wiskunde & Informatica, Amsterdam, The Netherlands; Ecole Nationale Supérieure des Télécommunications, Paris, France; Texas A&M University, Doha, Qatar; Universitat Pompeu Fabra, Barcelona, Spain; and the University of South Australia, Australia. His research interests are in communication theory, information theory, coding theory, digital modulation, and signal processing techniques with wireless applications.

Dr. Guillén i Fàbregas is currently an Editor of the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS. He received a pre-doctoral Research Fellowship of the Spanish Ministry of Education to join ESA. He received the Young Authors Award of the 2004 European Signal Processing Conference EUSIPCO 2004, Vienna, Austria and the 2004 Nokia Best Doctoral Thesis Award in Mobile Internet and 3rd Generation Mobile Solutions from the Spanish Institution of Telecommunications Engineers. He is also a member of the ARC Communications Research Network (ACoRN) and a Junior Member of the Isaac Newton Institute for Mathematical Sciences.

**Nick Letzepis** (M'03) received the Bachelor's Degree in electrical and electronic engineering with First Class Honours from Flinder's University of South Australia in 1998, and the Ph.D. Degree in telecommunications from the Institute for Telecommunications Research, University of South Australia, 2006. From 1998 to 2003, he worked as a Research Engineer for Dspace Pty., Ltd., specializing in the research and development of digital satellite communication systems. From 2003 to 2010 he was with the ITR, first as a Ph.D. candidate (2003–2006) and then as a Research Fellow in the Coding and Information theory group (2006–2010). Since 2010, he has been in the Command Control Communications and Intelligence Division of the Defence Science and Technology Organization as a Senior Communications Research Engineer. His research experience encompasses a broad range of fields, including: coding and information theory, communications theory and signal processing. More specifically, his research interests include: digital satellite communications, multi-antenna communications, multi-user communications, multi-carrier communications (OFDM), spread spectrum, vehicle-to-vehicle communications, iterative decoding, block-fading channels, and free-space optical communications.