Outage Probability of the Free-Space Optical Channel with Doubly Stochastic Scintillation

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***Abstract*—We study the asymptotic outage probability of multiple-input multiple-output free-space optical communication with pulse-position modulation. In particular, we consider doubly stochastic scintillation models, lognormal-Rice and I-K distributions. First we consider the case when channel state information is available at the receiver only. Then we consider the case when it is also available at the transmitter.**

***Index Terms*—Optical communication, MIMO systems, scintillation, outage probability, information theory.**

I. INTRODUCTION

FREE-SPACE optical (FSO) communication offers an attractive alternative to radio frequency (RF) for transmitting data at very high rates. The main drawback of the FSO channel is the detrimental effect the atmosphere has on the propagating laser beam. One such effect is *scintillation*, caused by atmospheric turbulence, and refers to random fluctuations in the irradiance of the received optical laser beam (analogous to fading in RF systems) [1]. Compared to typical signalling rates, scintillation is a slow time-varying process with coherence time on the order of tens of milliseconds. Long deep fades can result in the loss of millions of consecutively transmitted data bits, resulting in system outage. The use of multiple lasers and multiple apertures, creating the well-known multiple-input multiple-output (MIMO) channel, has shown to significantly reduce the effects of scintillation (see [2] and references therein). In this letter we examine the *outage probability* [3] of the MIMO FSO channel. Previous works on this subject include [4]–[7]. Of direct relevance to this letter is our previous work in [8], which analysed the asymptotic behaviour of the outage probability of the MIMO FSO channel with pulse-position modulation (PPM) and non-ideal photodetection. This analysis considered three scintillation distributions: lognormal, exponential and gamma-gamma distributions. The first two model weak and strong turbulence conditions respectively [1]. The gamma-gamma distribution models scintillation over all turbulence conditions [9], for which a number of other distributions have also been proposed, including the lognormal-Rice (or Beckmann) [10] and I-K [11] distributions. These universal models are all based on the heuristic argument that scintillation is a *doubly stochastic* random process modelling small and large scale turbulence effects. All three have shown good agreement with simulations and experimental measurements [9]–[11].

In this letter we extend the analysis in [8] to doubly stochastic scintillation models, lognormal-Rice and I-K distributions. We derive signal-to-noise ratio (SNR) exponents when perfect channel state information (CSI) is known only at the receiver (CSIR case). Then we examine the case when perfect CSI is also known at the transmitter (CSIT) and power is optimally allocated subject to short- and long-term power constraints.

II. SYSTEM MODEL

We consider an $M \times N$ MIMO FSO system with M transmit lasers an N aperture receiver. Information data is first encoded by a binary code of rate R_c, and then modulated with Q-PPM, resulting in rate $R = R_c \log_2 Q$ (bits/channel use). Repetition transmission is employed such that the same PPM signal is transmitted by each of the M lasers [12]. We assume the distance between lasers and apertures is sufficient so that spatial correlation is negligible. At each aperture, the received optical signal is converted to an electrical signal via photodetection. Non-ideal photodetection is assumed such that the combined shot noise and thermal noise processes can be modelled as zero mean, signal independent additive white Gaussian noise (AWGN) (a common assumption n the literature [2], [13]). Since the scintillation is a slow time-varying process, we model the channel as a *block-fading channel* [3] such that the received signal at aperture n, $n = 1, \ldots, N$ can be written as

$$\boldsymbol{y}_b^n[\ell] = \left(\sum_{m=1}^{M} \tilde{h}_b^{m,n} \right) \sqrt{\tilde{p}_b}\, \boldsymbol{x}_b[\ell] + \tilde{\boldsymbol{z}}_b^n[\ell], \tag{1}$$

for $b = 1, \ldots, B, \ell = 1, \ldots, L$, where $\boldsymbol{y}_b^n[\ell], \tilde{\boldsymbol{z}}_b^n[\ell] \in \mathbb{R}^Q$ are the received and noise signals at block b, time instant ℓ and aperture n, $\boldsymbol{x}_b[\ell], \in \mathbb{R}^Q$ is the transmitted signal at block b and time instant ℓ, and $\tilde{h}_b^{m,n}$ denotes the scintillation fading coefficient between laser m and aperture n. Each transmitted symbol is drawn from a PPM alphabet, $\boldsymbol{x}_b[\ell] \in \mathcal{X}^{\rm ppm} \triangleq \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_Q\}$, where $\boldsymbol{e}_q$ is the canonical basis vector, i.e., it has all zeros except for a one in position q, the time slot where the pulse is transmitted. The noise samples of $\tilde{\boldsymbol{z}}_b^n[\ell]$ are independent realisations of a random variable $Z \sim \mathcal{N}(0, 1)$, and $\tilde{p}_b$ denotes the received electrical power of block b at each aperture in the absence of scintillation. The fading coefficients $\tilde{h}_b^{m,n}$ are independent realisations of a random variable $\tilde{H}$ with probability density function (pdf) $f_{\tilde{H}}(h)$. Equal gain combining (EGC) is assumed [14], such that the entire system is equivalent to a single-input single-output (SISO) channel,

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i.e.

$$\boldsymbol{y}_b[\ell] = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \boldsymbol{y}_b^n[\ell] = \sqrt{p_b} h_b \boldsymbol{x}_b[\ell] + \boldsymbol{z}_b[\ell], \quad (2)$$

where $\boldsymbol{z}_b[\ell] = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \tilde{\boldsymbol{z}}_b^n[\ell] \sim \mathcal{N}(0,1)$, and h_b, a realisation of the random variable H, is defined as the normalised combined fading coefficient, i.e. $h_b = \frac{c}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \tilde{h}_b^{m,n}$, where $c = 1/(\mathbb{E}[\tilde{H}]\sqrt{1+\sigma_I^2/(MN)})$ is a constant to ensure $\mathbb{E}[H^2] = 1$, and $\sigma_I^2 \triangleq \mathbb{E}[\tilde{H}^2]/(\mathbb{E}[\tilde{H}])^2 - 1$ is defined as the *scintillation index*. Thus, the total instantaneous received electrical power at block b is $p_b = M^2 N \tilde{p}_b / c^2$, and the total average received SNR is $\mathsf{snr} \triangleq \mathbb{E}[h_b^2 p_b] = \mathbb{E}[p_b]$.

III. Scintillation Distributions

In this letter we consider lognormal-Rice and I-K distributed scintillation. Like the gamma-gamma case analysed in [8], they are doubly stochastic, yet as we shall see, both yield significantly different asymptotic outage behaviours.

The lognormal-Rice distribution arises from the product of two independent random variables, i.e. $\tilde{H} = XY$, where X is lognormal and $\sqrt{Y}$ is Rice distributed. The resulting pdf of $\tilde{H}$ can be written in integral form as shown in (3), where r is referred to as the *coherence parameter*, and σ is standard deviation of $\log(X)$ [10]. The lognormal-Rice includes the lognormal, lognormal-exponential and exponential distributions as special cases.

The I-K distribution arises from a compound statistical model whereby the conditional irradiance distribution is assumed to be the modified Rice-Nakagami pdf. When averaged over gamma statistics, the unconditional pdf has the form shown in (4), where α is the effective number of scatterers, and ρ is also referred to as the coherence parameter [11]. The I-K distribution contains the exponential and K distributions as special cases.

IV. Main Results

The channel capacity in the strict Shannon sense of the channel described by (2) under the quasi-static assumption is zero. The codeword error probability of any coding scheme can be lower bounded by the outage probability [3], $P_{\text{out}}(\mathsf{snr}, R) = \Pr(I(\boldsymbol{p}, \boldsymbol{h}) < R)$, where R is the transmission rate and (from [15]), $I(\boldsymbol{p}, \boldsymbol{h}) = \frac{1}{B} \sum_{b=1}^{B} I^{\text{awgn}}(p_b h_b^2)$, is the instantaneous mutual information for a given power allocation $\boldsymbol{p} = (p_1, \ldots, p_b)$ and vector channel realisation $\boldsymbol{h} \triangleq (h_1, \ldots, h_B)$. We denote $I^{\text{awgn}}(\gamma)$ as the mutual information for the AWGN channel with SNR γ, with PPM [13]. Note that the results we present in this letter will be the same for any Q-ary signal set, as long as $\lim_{\gamma \to \infty} I^{\text{awgn}}(\gamma) = \log_2(Q)$ and $\lim_{\mathsf{snr} \to 0} I^{\text{awgn}}(\mathsf{snr}) = 0$ bits, where $I^{\text{awgn}}(\gamma)$ is now the input-output mutual information for the AWGN non-fading channel of the particular Q-ary signal set.

For the CSIR case, we employ uniform power allocation, i.e. $p_1 = \ldots = p_B = \mathsf{snr}$. For codewords transmitted over B blocks, obtaining a closed form expression for the outage probability is intractable. Even for $B = 1$, the complicated pdfs (3) and (4) prohibit us from determining the exact distribution of H and therefore the outage probability. Instead, as we shall see, obtaining the asymptotic behaviour of the outage probability is substantially simpler. Towards this end, and following the footsteps of [16], [17], we derive the *SNR exponent*, defined as

$$d \triangleq -\lim_{\mathsf{snr} \to \infty} \frac{\log P_{\text{out}}(\mathsf{snr}, R)}{\log \mathsf{snr}}. \quad (5)$$

The SNR exponent is thus the asymptotic slope of the outage probability as a function of SNR in a log-log scale.

Theorem 4.1: The optimal SNR exponent for a MIMO FSO communications system modelled by (2) with lognormal-Rice and I-K scintillation is respectively given by

$$d^{\text{LN-R}} = \frac{MN}{2} \left(1 + \lfloor B(1 - R_c) \rfloor\right) \quad (6)$$

$$d^{\text{I-K}} = \frac{MN}{2} \alpha \left(1 + \lfloor B(1 - R_c) \rfloor\right) \quad (7)$$

where $R_c = R/\log_2(Q)$ is the rate of the binary code.

Sketch of the proof: The proof follows the same steps as the proof for the exponential, lognormal and gamma-gamma cases [8]. We describe here the steps of the proof for I-K scintillation. We first define the normalised (with respect to SNR) fading coefficients [16], $\zeta_b^{m,n} = -\frac{2 \log \tilde{h}_b^{m,n}}{\log \mathsf{snr}}$ and find their corresponding distribution from the distribution of the coefficients $\tilde{h}_b^{m,n}$. Then, it is not difficult to show that $I^{\text{awgn}}(\mathsf{snr} h_b^2) = 0$ if all $\zeta_b^{m,n} > 1$ and $I^{\text{awgn}}(\mathsf{snr} h_b^2) = \log_2 Q$ if at least one $\zeta_b^{m,n} < 1$. Then, the asymptotic outage event is given by

$$\mathcal{A} = \left\{ \boldsymbol{\zeta} \in \mathbb{R}^{BMN} : \sum_{b=1}^{B} \mathbb{1}\{\boldsymbol{\zeta}_b \succ \mathbf{1}\} > B(1 - R_c) \right\} \quad (8)$$

where $\mathbb{1}\{\cdot\}$ denotes the indicator function, $\mathbf{1}$ is a $1 \times MN$ vector of 1's, and $\succ$ denotes component-wise inequality. It can be shown that the distribution of the vector $\boldsymbol{\zeta}$ asymptotically behaves as

$$f(\boldsymbol{\zeta}) \doteq \exp\left(-\log \mathsf{snr} \alpha \frac{MN}{2} \sum_{b=1}^{B} \sum_{m=1}^{M} \sum_{n=1}^{N} \zeta_b^{m,n} \right). \quad (9)$$

$$f_{\tilde{H}}(h) = \frac{(1+r)e^{-r}}{\sqrt{2\pi}\sigma} \int_0^{\infty} \frac{1}{z^2} I_0\left(2\sqrt{\frac{(1+r)rh}{z}}\right) \exp\left(-\frac{(1+r)h}{z} - \frac{1}{2\sigma^2}\left(\log z + \frac{1}{2}\sigma^2\right)^2 \right) dz \quad (3)$$

$$f_{\tilde{H}}(h) = \begin{cases} 2\alpha(1+\rho)\left[\frac{(1+\rho)h}{\rho}\right]^{\frac{\alpha-1}{2}} \mathrm{K}_{\alpha-1}(2\sqrt{\alpha\rho})\, \mathrm{I}_{\alpha-1}(2\sqrt{\alpha h(1+\rho)}) & h < \frac{\rho}{1+\rho} \\ 2\alpha(1+\rho)\left[\frac{(1+\rho)h}{\rho}\right]^{\frac{\alpha-1}{2}} \mathrm{I}_{\alpha-1}(2\sqrt{\alpha\rho})\, \mathrm{K}_{\alpha-1}(2\sqrt{\alpha h(1+\rho)}) & h > \frac{\rho}{1+\rho} \end{cases} \quad (4)$$

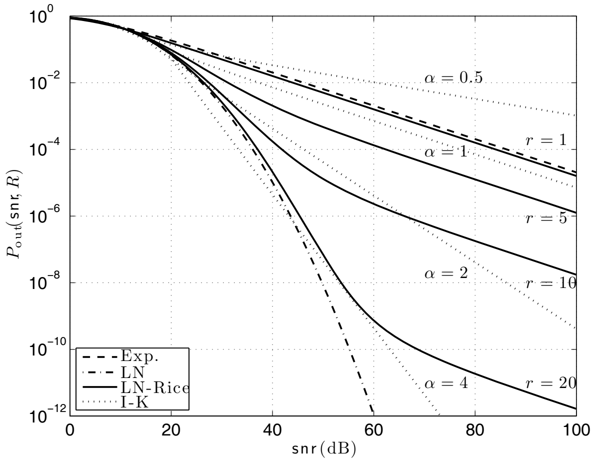

Fig. 1. Outage probability with CSIR for lognormal-Rice (solid), exponential (dashed), lognormal (dot-dashed) and I-K scintillation (dotted) with $M = N = B = 1$, $Q = 2$, $R_c = 1/2$ and $\sigma_I^2 = 1$.

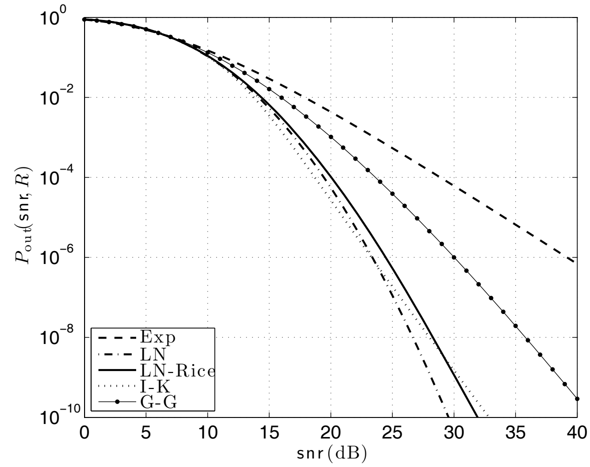

Fig. 2. Outage probability with CSIR for lognormal-Rice (solid), exponential (dashed), lognormal (dot-dashed), I-K scintillation (dotted) and gamma-gamma (solid with markers) with $MN = 4$, $B = 1$, $Q = 2$, $R_c = 1/2$, $\sigma_I^2 = 1$, $r = 10$, $\alpha = 2$.

Now, since $-\lim_{\mathsf{snr}\to\infty} \log P_{\text{out}}(\mathsf{snr}, R) = -\lim_{\mathsf{snr}\to\infty} \log \int_{\mathcal{A}} f(\boldsymbol{\zeta}) d\boldsymbol{\zeta}$, we can use Varadhan's lemma [18], and write that

$$d = \alpha \frac{MN}{2} \inf_{\mathcal{A}} \left\{ \sum_{b=1}^{B} \sum_{m=1}^{M} \sum_{n=1}^{N} (\zeta_b^{m,n})^2 \right\} \tag{10}$$

which immediately yields the desired result. For lognormal-Rice scintillation, we define two normalised coefficients, for the lognormal and Rice random variables, respectively. The application of the same steps (outage event definition, asymptotic behaviour of normalised coefficients density and Varadhan's lemma) to the new normalised coefficients yields the result. The optimality follows from random coding arguments [17]. ■

From (6) and (7) we immediately see the benefits of spatial and block diversity on the system. In particular, each exponent is proportional to: the number of lasers times the number of apertures, reflecting the spatial diversity; a channel related parameter that is dependent on the scintillation distribution; and the Singleton bound, which is the optimal rate-diversity tradeoff for Rayleigh-faded block fading channels [17], [19]. Interestingly, we see that both exponents are independent of their respective coherence parameters r and ρ. For the lognormal-Rice case, the exponent is the same as the asymptotic strong turbulence case, corresponding to exponential distributed scintillation (derived in [8]). The lognormal component does not influence the asymptotic slope of the outage probability curve. Only the Rice component, believed to be caused by small eddy cells [10], affects the SNR exponent. The Rice component introduces an error floor with the same slope as the exponential case. The SNR at which the error floor begins to dominate depends on the coherence parameter r (see Fig. 1). For small r, the floor totally dominates performance, and the outage behaviour is much like the exponential case. As r increases, the floor dominates at increasingly high SNR values and at low SNR the outage behaves much like the pure lognormal case (analysed in [8]). The exponent of the I-K case shows much different behaviour to the lognormal-Rice. Here, the exponent is proportional to α, which corresponds to the effective number of scatterers [11]. Thus the more scatterers the steeper the outage probability curve (see Fig. 1). Results for the MIMO case with $MN = 4$ are shown in Fig. 2. For comparison the figure also includes the outage performance for gamma-gamma scintillation (using the same parameters as in [8]). Note that for $B = 1$, then

$$P_{\text{out}}(\mathsf{snr}, R) = F_H\left(\sqrt{\frac{\mathsf{snr}_R^{\text{awgn}}}{\mathsf{snr}}}\right), \tag{11}$$

where $F_H(h)$ denotes the cdf of H, and $\mathsf{snr}_R^{\text{awgn}} \triangleq I^{\text{awgn},-1}(R)$ denotes the SNR value at which the mutual information is equal to R. To generate the curves given in Fig. 1, we have computed F_H via numerical convolution.

For the CSIT case, the transmitter finds the optimal power allocation that minimises the outage probability subject to short- and long-term power constraints, i.e. $\frac{1}{B}\sum_{b=1}^{B} p_b \le P$ and $\mathbb{E}\left[\frac{1}{B}\sum_{b=1}^{B} p_b\right] \le P$ respectively. In particular, we use results from [20], as done in [8]. The SNR exponent under a short-term constraint is the same as the CSIR case [20]. For the long-term power constraint, since the lognormal-Rice has the same SNR exponent as the exponential case, it follows from [8, Corollary 6.1] that *delay-limited capacity* [21] is positive only when $MN > 2(1+\lfloor B(1-R_c)\rfloor)^{-1}$. Whereas for I-K scintillation, we require $MN > 2\alpha^{-1}(1+\lfloor B(1-R_c)\rfloor)^{-1}$. Otherwise delay-limited capacity is zero, i.e. there exists no threshold SNR at which $P_{\text{out}} \to 0$. Fig. 3 illustrates the outage behaviour of the $B = 1$ CSIT case under the long-term power constraint. We see that with $M = N = 1$, the delay-limited capacity is zero for the lognormal-Rice case, as it has an error floor with the same slope as the exponential case. For the I-K case with $\alpha = 2$ and $M = N = 1$, delay-limited capacity is also zero (since we require $MN > 1$ in this case). Increasing MN to 4, we see that the outage curves are vertical in all cases, as predicted by our analysis, illustrating the large power savings possible with power control.

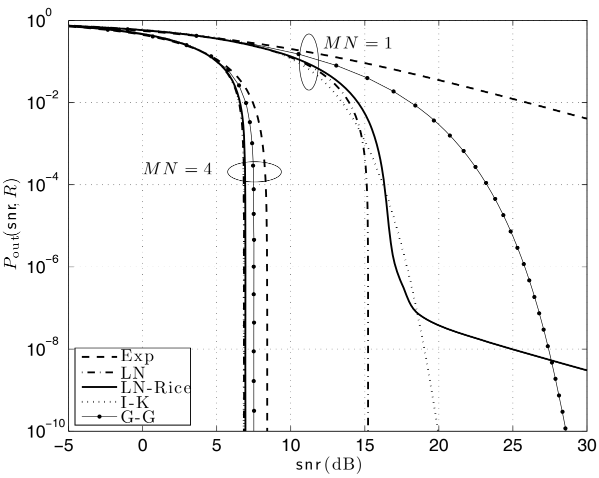

Fig. 3. Outage probability with CSIT for lognormal-Rice (solid), exponential (dashed), lognormal (dot-dashed), I-K scintillation (dotted) and gamma-gamma (solid with markers) with $B = 1$, $Q = 2$, $R_c = 1/2$, $r = 10$, $\alpha = 2$ and $\sigma_I^2 = 1$.

V. Conclusion

In this letter we analysed the asymptotic outage probability behaviour of the FSO MIMO channel under the assumption of PPM in the presence of doubly stochastic lognormal-Rice and I-K distributed scintillation. For the CSIR case, we showed that the SNR exponent for the lognormal-Rice case is the same as that of the exponential case (which corresponds to very strong turbulence). For the I-K case, the SNR exponent is proportional to the effective number of scatters α. Both of these results are different from the gamma-gamma case in [8], which illustrates the importance of correct modelling using these universal scintillation distributions.

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