# A Closed-Form Approximation for the Error Probability of BPSK Fading Channels

Alfonso Martinez, Albert Guillén i Fàbregas, and Giuseppe Caire

***Abstract*— This letter presents a simple closed-form expression to evaluate the error probability of binary fully-interleaved fading channels. The proposed expression does not require a numerical Laplace transform inversion, numerical integration or similar techniques, and captures the role of the relevant system parameters in the overall error performance. The expression has the same asymptotic behavior as the Bhattacharyya (Chernoff)-union bound but closes the gap with the simulation results. Its precision is numerically validated for coded and uncoded transmission over generic Nakagami-$m$ fading channels.**

***Index Terms*— BPSK modulation, AWGN channel, fading channel, error probability, saddlepoint approximation.**

## I. Introduction and Main Result

THE computation of error probabilities in fading channels suffers from the absence of a simple formula akin to the $Q(\cdot)$ function in pure additive white Gaussian noise (AWGN) channels. The Chernoff bound [1] can be expressed in closed form but it is loose. The exact probability may be computed by inverting a Laplace transform [2], but the numerical procedure is typically rather involved. Alternatively, one can use Craig's expression of the $Q(\cdot)$ function [3], which usually requires numerical integration. In a recent paper [4] we proposed the use of the saddlepoint approximation to evaluate the error probability of bit-interleaved coded modulation (BICM) [5]. This computation can be seen as an approximation to the numerical inversion of the Laplace transform. In this paper, we particularize the analysis for the case of binary transmission (BPSK) and show that the approximation admits a simple closed-form expression. For fully-interleaved Nakagami fading with parameter $m$ [1], average signal-to-noise ratio $\mathrm{SNR}$, and a diversity scheme with $D$ identical branches, the error probability $\mathrm{PEP}(d, \mathrm{SNR})$ between two codewords at Hamming distance $d$, can be closely approximated by

$$\mathrm{PEP}(d, \mathrm{SNR}) \simeq \frac{1}{2\sqrt{\pi d\, \mathrm{SNR}}}\left(1+\frac{\mathrm{SNR}}{mD}\right)^{-mdD+\frac{1}{2}}. \quad (1)$$

Manuscript received November 2, 2005; revised May 3, 2006; accepted September 20, 2006. The associate editor coordinating the review of this paper and approving it for publication was F. Daneshgaran.

A. Martinez is with the Department of Electrical Engineering, Technische Universiteit Eindhoven, Postbus 513, 5600 MB Eindhoven, The Netherlands (e-mail: alfonso.martinez@ieee.org).

A. Guillén i Fàbregas was with the Institute for Telecommunications Research, University of South Australia. He is now with the Department of Engineering, University of Cambridge, Trumpington Street, Cambridge, CB2 1PZ, UK. (e-mail: guillen@ieee.org). The work of A. Guillén i Fàbregas has been supported by the Australian Research Council under grants DP0558861 and RN0459498.

G. Caire is with the Electrical Engineering Department, University of Southern California, 3740 McClintock Ave., Los Angeles, CA 90089, USA (e-mail: caire@usc.edu).

Digital Object Identifier 10.1109/TWC.2007.05851.

As this pairwise error probability is the key ingredient in the union bound to the bit or codeword error rates, the formula also approximates the latter. It is worth recalling that Nakagami fading subsumes as special cases the Rayleigh, Rice, and unfaded AWGN channels. The approximation is therefore valid for these cases as well.

## II. Error Probability Analysis

### A. Channel Model

We study coded modulation over binary-input (BPSK) Gaussian noise channels. The discrete-time received signal can be expressed as

$$y_k = \sqrt{\mathrm{SNR}}\, h_k\, x_k + z_k, \quad k = 1, \dots, N \quad (2)$$

where $y_k$ is the $k$-th received sample, $h_k$ the $k$-th fading attenuation, $x_k$ the transmitted signal at time $k$, and $z_k$ the $k$-th noise sample, assumed to be i. i. d. $\sim \mathcal{N}(0, \frac{1}{2})$. The average received signal-to-noise ratio is $\mathrm{SNR}$. All variables are real.

The fading coefficients satisfy $h_k \geq 0$, and perfect channel state information (CSI) at the receiver is assumed. We shall consider a general Nakagami-$m$ fading [1] with $m \in (0, +\infty)$. With no real loss of generality, we consider a memoryless channel, where the coefficients $h_k$ are i. i. d. that is, fading is fully-interleaved. The coefficients $h_k$ are independently drawn from a common density function $p_H(h_k)$,

$$p_H(h_k) = \frac{2m^m h_k^{2m-1}}{\Gamma(m)} e^{-mh_k^2} \quad (3)$$

and the squared fading coefficient $\chi_k = |h_k|^2$ has the density

$$p_\chi(\chi_k) = \frac{m^m \chi_k^{m-1}}{\Gamma(m)} e^{-m\chi_k} \quad (4)$$

We recover the unfaded AWGN with $m \to +\infty$, the Rayleigh fading by letting $m = 1$ and the Rician fading with parameter $\mathcal{K}$ by setting $m = (\mathcal{K}+1)^2/(2\mathcal{K}+1)$.

The codewords $\mathbf{x} = (x_1, \dots, x_N)$ are obtained by mapping the codewords $\mathbf{c} = (c_1, \dots, c_N)$ of the code $\mathcal{C}$, each of dimension $K$ information bits and length $N$, with the labelling rule $0 \to -1, 1 \to +1$. The corresponding transmission rate is $R = \frac{K}{N}$ bits per channel use.

[1] Even though the case $m \geq 0.5$ is usually considered in the literature [1], [3], the distribution is well-defined for $0 < m < 0.5$. In general, reliable transmission is possible for $m > 0$.

### B. Error Probability Under ML Decoding

The error probability of linear binary codes for maximum likelihood decoding (ML) is accurately given by the union bound in a region above the cut-off rate [6]. The codeword error probability $\Pr(e)$ is very closely upper bounded by

$$\Pr(e) \le \sum_d A_d \, \mathrm{PEP}(d, \mathrm{SNR}), \tag{5}$$

where $A_d$ denotes the number of codewords in $\mathcal{C}$ with Hamming weight $d$, and $\mathrm{PEP}(d, \mathrm{SNR})$ is the pairwise error probability (PEP) for two codewords differing in $d$ bits[2]. A pairwise error event takes place when a candidate codeword has a larger a posteriori probability than the transmitted codeword. For a memoryless channel and a binary code, this probability can be written as the tail probability of a sum of random variables

$$\mathrm{PEP}(d, \mathrm{SNR}) = \Pr\left(\sum_{j=1}^{d} \Lambda_j > 0\right), \tag{6}$$

where the variables $\Lambda_j$ are the a posteriori log-likelihood ratio, that is, the ratio of the a posteriori likelihoods of the bit $j$-th taking the values $\bar{c}$ and $c$, having transmitted bit $c$. Their numerical value, denoted by $\lambda_j$, is given by

$$\lambda_j = \log \frac{\Pr(\hat{c}_j = \bar{c} | y_j)}{\Pr(\hat{c}_j = c | y_j)}, \tag{7}$$

and depends on all the random elements in the channel, that is the noise and fading realizations $z_j$ and $h_j$ respectively; they are independent and identically distributed. The index $j$ is a dummy variable, which need not coincide with the index $k$ in the codeword. For the sake of notational simplicity, we drop the time index $j$ in the following.

Conditioned on a fading realization[3] $h$, the ratio $\Lambda$ is Gaussian [4] $\mathcal{N}(-4\chi\,\mathrm{SNR}, 8\chi\,\mathrm{SNR})$, where $\chi = |h|^2$. Note that it does not depend on the transmitted bit $c$, and the fading coefficient only depends through $\chi = |h|^2$. In estimates of tail probabilities, the cumulant transform $\kappa(s)$ (or cumulant generating function) is a more convenient representation than the density [7], [8]. The transform is given by

$$\kappa(s) \triangleq \log \mathrm{E}\big[e^{s\Lambda}\big], \tag{8}$$

with $s$ a complex number.

The expectation is performed over the random elements in the channel, noise $z$ and fading $h$. The expectation can be done in two steps, first fixing the fading coefficients,

$$\kappa(s) = \log \mathrm{E}\,\mathrm{E}\big[e^{s\Lambda}|\chi\big]. \tag{9}$$

For fixed $\chi$, as $\Lambda$ is a Gaussian random variable, the expression $\mathrm{E}\big[e^{s\Lambda}|\chi\big]$ is the moment generating function of a Gaussian random variable, given by $e^{\mu s + \frac{1}{2}\sigma^2 s^2}$ [7]. Therefore,

$$\begin{aligned} \kappa(s) &= \log \mathrm{E}\big[e^{-4s\chi\,\mathrm{SNR} + 4s^2\chi\,\mathrm{SNR}}\big] && (10)\\ &= \log \int_0^{+\infty} \frac{m^m t^{m-1}}{\Gamma(m)} e^{-mt}\, e^{-4st\,\mathrm{SNR} + 4s^2 t\,\mathrm{SNR}}\, dt, && (11) \end{aligned}$$

where we have explicitly written the density of $\chi$.

Setting $\alpha = m + 4s\,\mathrm{SNR} - 4s^2\,\mathrm{SNR}$, and using the definition of the Gamma function (see for instance [9]),

$$\int_0^{\infty} \alpha^m t^{m-1} e^{-\alpha t}\, dt = \Gamma(m), \tag{12}$$

we have

$$\kappa(s) = \log \frac{m^m}{\alpha^m} = \log\left(1 + \frac{4s\,\mathrm{SNR}}{m} - \frac{4s^2\,\mathrm{SNR}}{m}\right)^{-m}. \tag{13}$$

The saddlepoint $\hat{s}$ is the value for which $\kappa'(\hat{s}) = 0$. In this case, it is easy to verify that $\hat{s} = 1/2$. For later use, we write down the first few derivatives at the saddlepoint,

$$\kappa(\hat{s}) = -m \log\left(1 + \frac{\mathrm{SNR}}{m}\right) \tag{14}$$

$$\kappa'(\hat{s}) = 0 \tag{15}$$

$$\kappa''(\hat{s}) = \frac{8\,\mathrm{SNR}}{1 + \frac{\mathrm{SNR}}{m}} \tag{16}$$

$$\kappa'''(\hat{s}) = 0 \tag{17}$$

$$\kappa^4(\hat{s}) = \frac{192\,\mathrm{SNR}^2}{m\left(1 + \frac{\mathrm{SNR}}{m}\right)^2}. \tag{18}$$

For independent realizations of the fading and noise coefficients, the cumulant transform for the sum $\sum_{j=1}^{d} \Lambda_j$ is simply given by the sum of the transforms, that is $d\kappa(s)$.

### C. Effect of Diversity

The cumulant transform can also be easily computed for scenarios with receiver diversity, where the signal comes along several branches. For notational simplicity, we assume maximal ratio combining, or "coherent" combining of $D$ identical diversity branches. Conditioned on a realization of the fading coefficients in the $D$ receiver branches, $\underline{h} = (h_1 \dots, h_D)$, and assuming that the total average received signal-to-noise ratio is SNR, each $\Lambda$ is normally distributed, $\Lambda \sim \mathcal{N}(-4D^{-1}\sum_d \chi_d\,\mathrm{SNR}, 8D^{-1}\sum_d \chi_d\,\mathrm{SNR})$, where $\chi_d = |h_d|^2$. Let the corresponding vector be denoted by $\underline{\chi} = (\chi_1, \dots, \chi_D)$.

Reproducing the analysis in Eqs. (9)-(10), the cumulant transform is given by

$$\begin{aligned} \kappa(s) &= \log \mathrm{E}\,\mathrm{E}\big[e^{s\Lambda}|\underline{\chi}\big] && (19)\\ &= \log \mathrm{E}\big[e^{-4sD^{-1}\sum_d \chi_d\,\mathrm{SNR} + 4s^2 D^{-1}\sum_d \chi_d\,\mathrm{SNR}}\big] && (20)\\ &= \sum_{d=1}^{D} \log \mathrm{E}\big[e^{-4sD^{-1}\chi_d\,\mathrm{SNR} + 4s^2 D^{-1}\chi_d\,\mathrm{SNR}}\big]. && (21) \end{aligned}$$

Recall that $\Lambda$ is a sum of $D$ independent Gaussian random variables, with mean $-4D^{-1}\chi_d\,\mathrm{SNR}$ and variance

[2] Similarly, the bit-error probability $P_b$ is given by the right-hand side of Eq. (5) with $A_d$ replaced by $\tilde{A}_d = \sum_i \frac{i}{K} A_{i,d}$, $A_{i,d}$ being the number of codewords in $\mathcal{C}$ with output Hamming weight $d$ and input weight $i$.

[3] In absence of fading, $\chi = 1$, and $\Lambda \sim \mathcal{N}(-4\,\mathrm{SNR}, 8\,\mathrm{SNR})$, Eq. (6) implies that $\mathrm{PEP}(d, \mathrm{SNR})$ is equal to the probability that a Gaussian random variable of mean $-4d\,\mathrm{SNR}$ and variance $8d\,\mathrm{SNR}$ be larger than zero, that is, the well-known formula $\mathrm{PEP}(d, \mathrm{SNR}) = \int_0^{\infty} \frac{1}{\sqrt{16\pi d\,\mathrm{SNR}}} e^{-\frac{(t + 4d\,\mathrm{SNR})^2}{16d\,\mathrm{SNR}}}\, dt = Q\big(\sqrt{2d\,\mathrm{SNR}}\big)$.

$8D^{-1}\chi_d \,\mathrm{SNR}$, and the cumulant transform of a sum of independent random variables is the sum of the transforms.

Let us now evaluate Eq. (13), assuming all $\chi_d$ are identically distributed. We may repeat the procedure leading to Eq. (13). First, we explicitly write the density of the fading coeffient $\chi_d$, we then use identity Eq. (12), and finally obtain

$$\kappa(s) = -mD \log\left(1 + \frac{4s\,\mathrm{SNR}}{mD} - \frac{4s^2\,\mathrm{SNR}}{mD}\right). \qquad (22)$$

This is equivalent to a Nakagami fading with parameter $\tilde{m} = mD$. In the limit $D \to \infty$ it is equal to that of unfaded AWGN.

### D. Saddlepoint Approximation

In [4] we presented a derivation of the saddlepoint approximation and an estimate of the approximation error to the PEP. In general, the approximation is an asymptotic series in inverse powers of $d\kappa''(\hat{s})$. Keeping the first two terms, the PEP is

$$\mathrm{PEP}(d, \mathrm{SNR}) \simeq \beta \frac{e^{d\kappa(\hat{s})}}{\sqrt{2\pi d\kappa''(\hat{s})}\,\hat{s}}. \qquad (23)$$

where $\beta$ is a correction term given by

$$\beta = 1 + \frac{1}{d\kappa''(\hat{s})}\left(-\frac{1}{\hat{s}^2} + \frac{\kappa^{(4)}(\hat{s})}{8\kappa''(\hat{s})}\right). \qquad (24)$$

where we used that $\kappa'''(\hat{s}) = 0$. After using the appropriate values of the derivatives of $\kappa(s)$, and some algebraic simplifications, the PEP becomes

$$\mathrm{PEP}(d, \mathrm{SNR}) \simeq \frac{\beta}{2\sqrt{\pi d\,\mathrm{SNR}}}\left(1 + \frac{\mathrm{SNR}}{m}\right)^{-md+\frac{1}{2}} \qquad (25)$$

and

$$\beta = 1 - \frac{1}{2d\,\mathrm{SNR}} - \frac{1}{8dm}. \qquad (26)$$

Including the correction term $\beta$ has negligible impact in practical calculations and from now on, we take $\beta = 1$, which corresponds to the classical saddlepoint approximation.

For Rayleigh fading ($m = 1$), Eq (25) improves on Chernoff's bound [1] (which coincides with Bhattacharyya's bound),

$$\mathrm{PEP}(d, \mathrm{SNR}) \le e^{d\kappa(\hat{s})} = (1 + \mathrm{SNR})^{-d}. \qquad (27)$$

Similarly, under diversity with $D$ identical branches, the PEP can be approximated by

$$\mathrm{PEP}(d, \mathrm{SNR}) \simeq \frac{1}{2\sqrt{\pi d\,\mathrm{SNR}}}\left(1 + \frac{\mathrm{SNR}}{mD}\right)^{-mdD+\frac{1}{2}}. \qquad (28)$$

It should be noted that for $m \to \infty$ or $D \to \infty$, i. e. AWGN, the approximation gives the first two terms in the classical expansion of the $Q(\cdot)$ function into an exponential,

$$\mathrm{PEP}(d, \mathrm{SNR}) \simeq \frac{1}{2\sqrt{\pi d\,\mathrm{SNR}}} e^{-d\,\mathrm{SNR}}\left(1 - \frac{1}{2d\,\mathrm{SNR}}\right) \qquad (29)$$

$$\simeq Q\left(\sqrt{2d\,\mathrm{SNR}}\right). \qquad (30)$$

This suggests that the saddlepoint approximation generalizes the classical expansion of the $Q(\cdot)$ function from AWGN to fading channels.

Finally, it is worthwhile noting that the saddlepoint method is an approximation to direct integration in the complex plane, used in [2] to exactly compute the error probability.

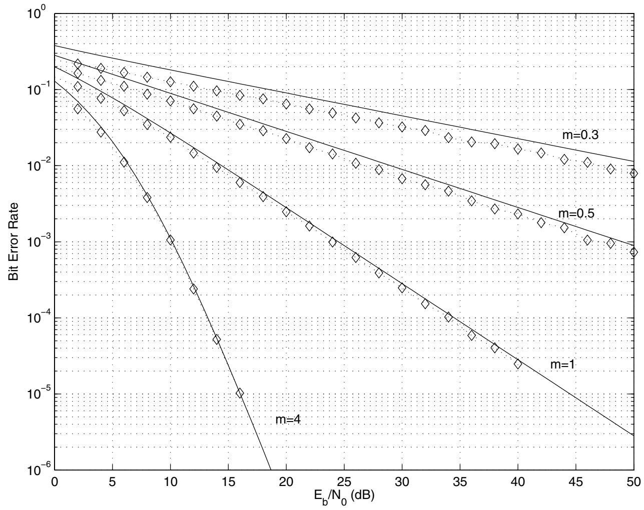


Fig. 1. Comparison of simulation and saddlepoint approximation for uncoded BPSK in Nakagami fading of parameter $m = 0.3, 0.5, 1, 4$.

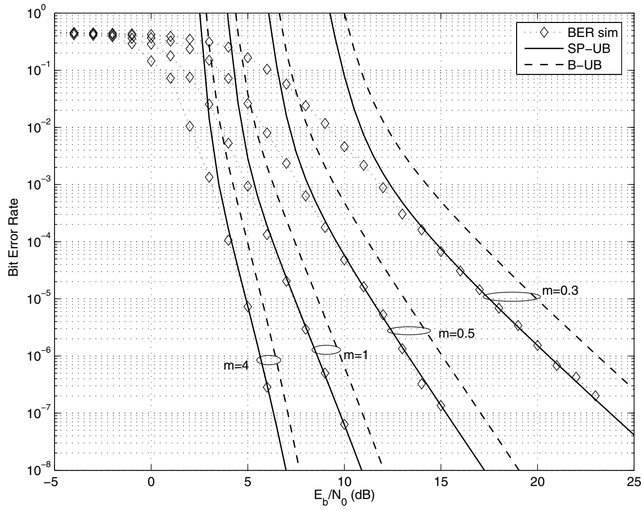


Fig. 2. Comparison of simulation, Bhattacharyya union bound and saddlepoint approximation for the 64-state rate 1/2 convolutional code in Nakagami fading of parameter $m = 0.3, 0.5, 1, 4$.

## III. Numerical Results and Discussion

In this section we show some numerical results that illustrate the accuracy of the proposed methods as well as its asymptotic behavior. In particular, we show the following: the Bhattacharyya union bound (B-UB), the saddlepoint approximation (25) union bound (SP-UB) using $\beta = 1$, and the simulation of the bit-error rate (BER sim) for both convolutional and turbo-like code ensembles.

Figure 1 shows the bit-error probability simulation and saddlepoint approximation for uncoded BPSK in Nakagami fading with parameter $m = 0.3, 0.5, 1$ and $4$. We observe that both curves are very close.

Figure 2 shows the bit-error probability simulation and bounds for the 64-state rate 1/2 convolutional code in Nakagami fading with parameter $m = 0.3, 0.5, 1$ and $4$. As we see, the closed-form saddlepoint union bound yields an accurate estimation of the error probability.

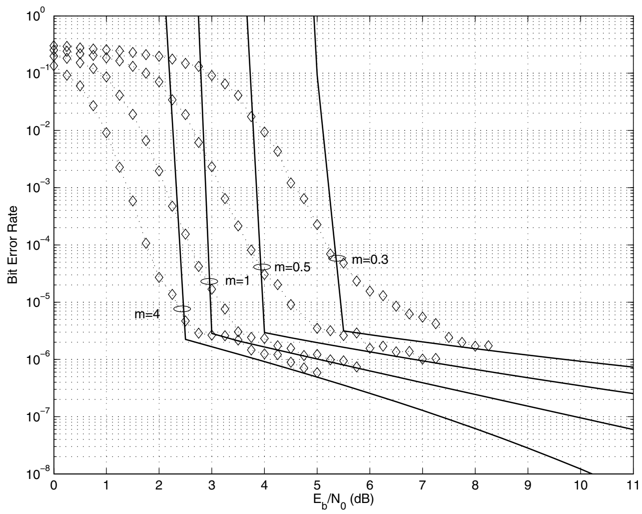


Fig. 3. Comparison of simulation and saddlepoint approximation for the average over rate 1/4 repeat-accumulate codes in Nakagami fading of parameter $m = 0.3, 0.5, 1, 4$.

Figure 3 shows the corresponding curves[4] for the average over the class of regular repeat-accumulate codes of rate 1/4 with $K = 512$. For every block of information bits a different interleaver is randomly generated. The simulation points correspond to an iterative decoder with 20 iterations. The saddlepoint union bound gives an accurate estimation of the error floor region. We have used the closed-form of the code spectrum computed by Divsalar *et al.* [10]. Furthermore, it does also provide an accurate approximation to the "knee" of the error curve, i. e., the transition between the waterfall and the error floor regions.

In all the cases the saddlepoint approximation gives an extremely accurate result at a fraction of the complexity required by alternative computation methods [3], such as the (exact) formula for the uncoded case (a Gauss hypergeometric function), or numerical integration of Craig's form of the $Q(\cdot)$ function. Furthermore, as opposed to the numerical integration method, the saddlepoint approximation is useful in an engineering sense, as it highlights the role of all relevant system parameters in the overall error probability.

## IV. Conclusions

In this correspondence we have presented a simple method to compute a tight closed-form approximation to the error probability of binary transmission over fully-interleaved fading channels. This probability corresponds in a natural way to the tail probability of a sum of independent random variables, which is computed by a saddlepoint approximation. In contrast to numerical integration methods, the proposed saddlepoint approximation yields a simple expression that highlights the design tradeoffs among the different system parameters.

We have verified the validity of the approximation for uncoded and coded (convolutional and turbo-like code ensembles) transmission with various fading parameters.

The general underlying method allows for straightforward extensions to other fading models, for instance, with correlation among successive fading realizations.

[4]In order to remove clutter from the plot, the Bhathacharyya Union Bound is not depicted. As seen in Fig. 3, it is not tight, compared to the saddlepoint approximation.

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