A Lower Bound on the Error Exponent of Random Gilbert-Varshamov Codes

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We consider transmission over a discrete memoryless channel (DMC) $W(y|x)$ with finite alphabets $\mathcal{X}$ and $\mathcal{Y}$. We consider an (n, M_n)-codebook $\mathcal{M}_n = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{M_n}\}$ with rate $R_n = \frac{1}{n}\log M_n$. The type-dependent maximum-metric decoder estimates the transmitted message as

$$\hat{m} = \arg\max_{\boldsymbol{x}_i \in \mathcal{M}_n} q(\hat{P}_{\boldsymbol{x}_i, \boldsymbol{y}}), \tag{1}$$

where $\hat{P}_{\boldsymbol{x},\boldsymbol{y}}$ is the joint empirical distribution [1, Ch. 2] of the pair $(\boldsymbol{x}, \boldsymbol{y})$ and the metric $q : \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \to \mathbb{R}$ is continuous. Maximum-likelihood (ML) decoding is a special case of (1), but the decoder may in general be *mismatched* [2], [3].

We construct the code $\mathcal{M}_n$ such that any two distinct codewords $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{M}_n$ satisfy $d(\boldsymbol{x}, \boldsymbol{x}') > \Delta$ for a given distance function $d(\cdot,\cdot)$ and $\Delta \in \mathbb{R}$. This guarantees that the minimum distance of the codebook exceeds Δ. Similar constructions are used to prove the Gilbert-Varshamov bound in Hamming spaces [4], [5]. Our construction depends on an input distribution $P \in \mathcal{P}(\mathcal{X})$, and we let P_n denote an arbitrary type [1, Ch. 2] whose entries are $\frac{1}{n}$-close to P. The set of sequences with type P_n is denoted by $\mathcal{T}(P_n)$.

Fixing n, M_n, an input distribution $P \in \mathcal{P}(\mathcal{X})$, a distance function $d(\cdot,\cdot)$, and constants $\delta > 0, \Delta \in \mathbb{R}$, the construction is described by the following steps:

1) The first codeword, $\boldsymbol{x}_1$, is drawn uniformly over $\mathcal{T}_1(P_n)$, given by $\mathcal{T}_1(P_n) = \mathcal{T}(P_n)$;
2) The second codeword $\boldsymbol{x}_2$ is uniformly drawn from
$$\mathcal{T}_2(P_n, \boldsymbol{x}_1) = \{\bar{\boldsymbol{x}} \in \mathcal{T}(P_n) \,:\, d(\bar{\boldsymbol{x}}, \boldsymbol{x}_1) > \Delta\} \tag{2}$$
the set of sequences of composition P_n whose distance to $\boldsymbol{x}_1$ exceeds Δ;
3) The i-th codeword $\boldsymbol{x}_i$ is drawn uniformly from
$$\mathcal{T}_i(P_n, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}) = \{\bar{\boldsymbol{x}} \in \mathcal{T}(P_n) \,:\, d(\bar{\boldsymbol{x}}, \boldsymbol{x}_j) > \Delta,\, j = 1 \ldots, i-1\} \tag{3}$$

In order to ensure that the above procedure generates the desired number of codewords $M_n = \mathrm{e}^{nR_n}$ (i.e., the sets $\mathcal{T}_i$ are non-empty for $i = 1, \ldots, M_n$), set Δ and δ such that

$$\mathrm{e}^{n(R_n+\delta)}\mathrm{vol}_{\boldsymbol{x}}(\Delta) \le |\mathcal{T}(P_n)| \tag{4}$$

This work was supported in part by the Israel Science Foundation under grant 631/17, the European Research Council under Grant 725411, and the Spanish Ministry of Economy and Competitiveness under Grant TEC2016-78434-C3-1-R.

where $\mathrm{vol}_{\boldsymbol{x}}(\Delta) = |\{\bar{\boldsymbol{x}} \in \mathcal{T}(P_n) \,:\, d(\bar{\boldsymbol{x}}, \boldsymbol{x}) \le \Delta\}|$ is the volume of a ball of radius Δ according to distance $d(\cdot,\cdot)$ centered at $\boldsymbol{x} \in \mathcal{T}(P_n)$. If the distance d is symmetric and type-dependent, $\mathrm{vol}_{\boldsymbol{x}}(\Delta)$ does not depend on $\boldsymbol{x} \in \mathcal{T}(P_n)$.

Our main result is as follows, namely, a single-letter lower bound for the error exponent of the RGV construction.

Theorem 1. *For any $P \in \mathcal{P}(\mathcal{X})$, $\delta > 0$, $\Delta \in \mathbb{R}$, type-dependent distance function d, and $R > 0$ satisfying*

$$R \le \min_{P_{X\widetilde{X}} \,:\, d(P_{X\widetilde{X}}) \le \Delta,\, P_X = P_{\widetilde{X}} = P} I(X;\widetilde{X}) - 3\delta, \tag{5}$$

the RGV construction with parameters $(n, R, P, d, \Delta, \delta)$ and decoding metric $q(\cdot)$ over the DMC W achieves the following error exponent

$$E_{\mathrm{RGV}}(R, P, W, q, d, \Delta) = \min_{V \in \mathcal{T}_{d,q,P}(\Delta)} D(V_{Y|X}\|W|P) + \left|I(\widetilde{X}; Y, X) - R\right|_+, \tag{6}$$

and

$$\mathcal{T}_{d,q,P}(\Delta) \triangleq \Big\{V_{X\widetilde{X}Y} \in \mathcal{P}(\mathcal{X}\times\mathcal{X}\times\mathcal{Y}) \,:\, V_X = V_{\widetilde{X}} = P,\; q(V_{\widetilde{X}Y}) \ge q(V_{XY}),\, d(P_{X\widetilde{X}}) \ge \Delta\Big\}. \tag{7}$$

The following corollary shows that when the distance function $d(\cdot,\cdot)$ is optimized, and Δ is chosen appropriately, the exponent in Theorem 1 recovers the exponent of [6], denoted by $E_q(R, P, W)$, known to be at least as large as the maximum of the random-coding and expurgated exponents.

Corollary 1. *Setting $d(P_{X\widetilde{X}}) = -I(X;\widetilde{X})$, $\Delta = -(R + 3\delta)$ gives that for sufficiently small $\delta > 0$ and $\epsilon > 0$*

$$E_{\mathrm{RGV}}(R, P, W, q, d, \Delta) \ge E_q(R, P, W) - \epsilon. \tag{8}$$

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