Multiple-Access Channel with Independent Sources: Error Exponent Analysis

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***Abstract*—In this paper, an achievable error exponent for the multiple-access channel with two independent sources is derived. For each user, the source messages are partitioned into two classes and codebooks are generated by drawing codewords from an input distribution depending on the class index of the source message. The partitioning thresholds that maximize the achievable exponent are given by the solution of a system of equations. We also derive both lower and upper bounds for the achievable exponent in terms of Gallager's source and channel functions. Finally, a numerical example shows that using the proposed ensemble gives a noticeable gain in terms of exponent with respect to independent identically distributed codebooks.**

I. Introduction

For point-to-point communication, many studies show that joint source-channel coding might achieve a better error exponent than separate source-channel coding [1]–[4]. One strategy for joint source-channel coding is to assign source messages to disjoint classes, and to use codewords generated according to a distribution that depends on the class index. This random-coding ensemble achieves the sphere-packing exponent in those cases where it is tight [4].

Recent studies [5], [6] extended the same idea to the multiple-access channel (MAC) using a random-coding ensemble with independent message-dependent distributed codebooks. In [6], the joint source-channel coding problem over a MAC with correlated sources was considered, where codewords are generated by a symbol-wise conditional probability distribution that depends both on the instantaneous source symbol and on the empirical distribution of the source sequence. The achievable exponent derived in [6] was presented in the primal domain, i.e., as a multi-dimensional optimization problem over distributions that is generally difficult to analyze.

We study a simplified version of the problem posed in [6] in the dual domain, i.e., as a lower dimensional problem over parameters in terms of Gallager functions. We consider a two-user MAC with independent sources. For each user, source messages are assigned to two classes, and codewords are independently generated according to a distribution that depends on the class index of the source message. For such random-coding ensemble, we derive an achievable exponent in the dual domain, and show that this exponent is greater than that achieved using only one input distribution for each user.

This work has been funded in part by the European Research Council under grant 725411, and by the Spanish Ministry of Economy and Competitiveness under grant TEC2016-78434-C3-1-R.

II. System Model

A. Definitions and Notation

We consider two independent sources characterized by probability distributions P_{U_1}, P_{U_2} on alphabets $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. We use bold font to denote a sequence, such as the source sequences $\boldsymbol{u}_1 \in \mathcal{U}_1^n$ and $\boldsymbol{u}_2 \in \mathcal{U}_2^n$, and underlined font to represent a pair of quantities for users 1 and 2, such as $\underline{\gamma} = (\gamma_1, \gamma_2)$, $\underline{u} = (u_1, u_2)$, $\underline{\boldsymbol{u}} = (\boldsymbol{u}_1, \boldsymbol{u}_2)$ or $P_{\underline{U}}(\underline{u}) = P_{U_1,U_2}(u_1, u_2)$.

For user $\nu = 1, 2$, the source message $\boldsymbol{u}_\nu$ is mapped onto codeword $\boldsymbol{x}_\nu(\boldsymbol{u}_\nu)$, which also has length n and is drawn from the codebook $\mathcal{C}^\nu = \{\boldsymbol{x}_\nu(\boldsymbol{u}_\nu) \in \mathcal{X}_\nu^n \,:\, \boldsymbol{u}_\nu \in \mathcal{U}_\nu^n\}$. Both terminals send the codewords over a discrete memoryless multiple-access channel with transition probability $W(y|x_1, x_2)$, input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, and output alphabet $\mathcal{Y}$.

Given the received sequence $\boldsymbol{y}$, the decoder estimates the transmitted pair of messages $\underline{\boldsymbol{u}}$ based on the maximum a posteriori criterion, i.e.,

$$\hat{\underline{\boldsymbol{u}}} = \operatorname*{arg\,max}_{\underline{\boldsymbol{u}} \in \mathcal{U}_1^n \times \mathcal{U}_2^n} P_{\underline{\boldsymbol{U}}}^n(\underline{\boldsymbol{u}}) W^n\big(\boldsymbol{y}|\boldsymbol{x}_1(\boldsymbol{u}_1), \boldsymbol{x}_2(\boldsymbol{u}_2)\big). \tag{1}$$

An error occurs if $\hat{\underline{\boldsymbol{u}}} \neq \underline{\boldsymbol{u}}$. Using the convention that scalar random variables are denoted by capital letters, and capital bold font letters denote random vectors, the error probability for a given pair of codebooks $(\mathcal{C}^1, \mathcal{C}^2)$ is given by

$$\epsilon^n(\mathcal{C}^1, \mathcal{C}^2) \triangleq \mathbb{P}\left[(\hat{\boldsymbol{U}}_1, \hat{\boldsymbol{U}}_2) \neq (\boldsymbol{U}_1, \boldsymbol{U}_2)\right]. \tag{2}$$

The pair of sources (U_1, U_2) is transmissible over the channel if there exists a sequence of pairs of codebooks $(\mathcal{C}_n^1, \mathcal{C}_n^2)$ such that $\lim_{n\to\infty} \epsilon^n(\mathcal{C}_n^1, \mathcal{C}_n^2) = 0$. An exponent $E(P_{\underline{U}}, W)$ is achievable if there exists a sequence of codebooks such that

$$\liminf_{n\to\infty} -\frac{1}{n} \log \epsilon^n(\mathcal{C}_n^1, \mathcal{C}_n^2) \geq E(P_{\underline{U}}, W). \tag{3}$$

In order to show the existence of such sequences of codebooks, we use random-coding arguments, i.e., we find a sequence of ensembles whose error probability averaged over the ensemble, denoted as $\bar{\epsilon}^n$, tends to zero.

B. Message-Dependent Random Coding

For user $\nu = 1, 2$, we fix a threshold $0 \le \gamma_\nu \le 1$ to partition the source-message set $\mathcal{U}_\nu^n$ into two classes $\mathcal{A}_\nu^1$ and $\mathcal{A}_\nu^2$ defined as

$$\mathcal{A}_\nu^1 = \{\boldsymbol{u}_\nu \in \mathcal{U}_\nu^n \,:\, P^n_{\boldsymbol{U}_\nu}(\boldsymbol{u}_\nu) \ge \gamma_\nu^n\}, \tag{4}$$

$$\mathcal{A}_\nu^2 = \{\boldsymbol{u}_\nu \in \mathcal{U}_\nu^n \,:\, P^n_{\boldsymbol{U}_\nu}(\boldsymbol{u}_\nu) < \gamma_\nu^n\}. \tag{5}$$

For every message $\boldsymbol{u}_\nu \in \mathcal{A}_\nu^i$, we randomly generate a codeword $\boldsymbol{x}_\nu(\boldsymbol{u}_\nu)$ according to the probability distribution $Q_{\nu,i}(\boldsymbol{x}_\nu) = \prod_{\ell=1}^n Q_{\nu,i}(x_{\nu,\ell})$, where $Q_{\nu,i}$, for $i = 1, 2$, is a probability distribution that depends on the class of $\boldsymbol{u}_\nu$.

We use the symbol $\tau \in \{\{1\}, \{2\}, \{1, 2\}\}$ to denote the error event type of the error probability (2), i.e., respectively $(\hat{\boldsymbol{u}}_1, \boldsymbol{u}_2) \ne (\boldsymbol{u}_1, \boldsymbol{u}_2)$, $(\boldsymbol{u}_1, \hat{\boldsymbol{u}}_2) \ne (\boldsymbol{u}_1, \boldsymbol{u}_2)$ and $(\hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_2) \ne (\boldsymbol{u}_1, \boldsymbol{u}_2)$. We denote the complement of τ as τ^c among the subsets of $\{1, 2\}$. For example, $\tau^c = \{2\}$ for $\tau = \{1\}$ and $\tau^c = \emptyset$ for $\tau = \{1, 2\}$. In order to simplify some expressions, it will prove convenient to adopt the following notational convention for an arbitrary variable u

$$u_\tau = \begin{cases} \emptyset & \tau = \emptyset \\ u_1 & \tau = \{1\} \\ u_2 & \tau = \{2\} \\ \underline{u} & \tau = \{1, 2\}. \end{cases} \tag{6}$$

For types of error $\tau = \{1\}$ and $\tau = \{2\}$, we denote $WQ_{\tau^c,i}$ as a point-to-point channel with input and output alphabets $\mathcal{X}_\tau$ and $\mathcal{X}_{\tau^c} \times \mathcal{Y}$, respectively, and transition probability $W(y|x_1, x_2)Q_{\tau^c,i}(x_{\tau^c})$. For $\tau = \{1, 2\}$, the input distribution Q_{τ,i_τ} is the product distribution $Q_{1,i_1}(x_1)Q_{2,i_2}(x_2)$ over the alphabet $\mathcal{X}_1 \times \mathcal{X}_2$, and $WQ_{\tau^c,i} = W$.

C. Single User Communication

For point to point communication, using i.i.d random coding to transmit a discrete memoryless source P_U, $u \in \mathcal{U}$ over the discrete memoryless channel W with input and output alphabets $\mathcal{X}$ and $\mathcal{Y}$, leads to Gallager's source and channel functions [1]

$$E_s(\rho, P_U) = \log\left(\sum_u P_U(u)^{\frac{1}{1+\rho}}\right)^{1+\rho}, \tag{7}$$

$$E_0(\rho, Q, W) = -\log \sum_y \left(\sum_x Q(x) W(y|x)^{\frac{1}{1+\rho}}\right)^{1+\rho}, \tag{8}$$

where Q denotes the input distribution.

In [4], message-dependent random coding was studied for single-user communication using a threshold $\gamma \in [0, 1]$ to partition the source messages into two classes. The derivation of the achievable exponent in [4] involves the following source exponent functions [4, Lemma 1]

$$E_{s,1}(\rho, P_U, \gamma) = \begin{cases} E_s(\rho, P_U) & \frac{1}{1+\rho} \ge \frac{1}{1+\rho_\gamma}, \\ E_s(\rho_\gamma, P_U) + E_s'(\rho_\gamma)(\rho - \rho_\gamma) & \frac{1}{1+\rho} < \frac{1}{1+\rho_\gamma}, \end{cases} \tag{9}$$

and

$$E_{s,2}(\rho, P_U, \gamma) = \begin{cases} E_s(\rho, P_U) & \frac{1}{1+\rho} < \frac{1}{1+\rho_\gamma}, \\ E_s(\rho_\gamma, P_U) + E_s'(\rho_\gamma)(\rho - \rho_\gamma) & \frac{1}{1+\rho} \ge \frac{1}{1+\rho_\gamma}. \end{cases} \tag{10}$$

In (9) and (10), the parameter ρ_γ is the solution of the implicit equation

$$\frac{\sum_u P_U(u)^{\frac{1}{1+\rho}} \log P_U(u)}{\sum_u P_U(u)^{\frac{1}{1+\rho}}} = \log(\gamma), \tag{11}$$

when $\min_u P_U(u) \le \gamma \le \max_u P_U(u)$ is satisfied. We observe that $E_{s,1}(\rho, \cdot)$ follows the Gallager $E_s(\rho, \cdot)$ function for an interval of ρ, while it is the straight line tangent to $E_s(\rho, \cdot)$ beyond that interval, and similarly for $E_{s,2}(\rho, \cdot)$.

When $\gamma \in [0, \min_u P_U(u))$, we have that $\rho_\gamma = -1_-$ and hence $E_{s,1}(\rho, \cdot) = E_s(\rho, \cdot)$ and $E_{s,2}(\rho, \cdot) = -\infty$. Otherwise, when $\gamma \in (\max_u P_U(u), 1]$, we have that $\rho_\gamma = -1_+$ and hence $E_{s,1}(\rho, \cdot) = -\infty$ and $E_{s,2}(\rho, \cdot) = E_s(\rho, \cdot)$. In our analysis, it suffices to consider $\gamma = 0$ or $\gamma = 1$ to represent the cases where $E_{s,1}(\rho, \cdot)$ or $E_{s,2}(\rho, \cdot)$ are infinity. For such cases, we have

$$E_{s,1}(\rho, P_U, 0) = E_s(\rho, P_U), \quad E_{s,2}(\rho, P_U, 0) = -\infty, \tag{12}$$

$$E_{s,1}(\rho, P_U, 1) = -\infty, \quad E_{s,2}(\rho, P_U, 1) = E_s(\rho, P_U). \tag{13}$$

III. Main Results

We now derive an achievable exponent for the MAC with independent sources using the random-coding ensemble introduced in Sec. II-B in terms of the exponent functions defined in (7)–(10). We also derive simpler lower and upper bounds to the achievable exponent in Sec. III-A and III-B, respectively.

Proposition 1. *For the two-user MAC with transition probability W, source probability distributions $P_{\underline{U}}$ and class distributions $\{Q_{\nu,1}, Q_{\nu,2}\}$ with user index $\nu = 1, 2$, an achievable exponent $E(P_{\underline{U}}, W)$ is given by*

$$E(P_{\underline{U}}, W) = \max_{\gamma_1,\gamma_2 \in [0,1]} \min_{\tau \in \{\{1\},\{2\},\{1,2\}\}} \min_{i_\tau, i_{\tau^c} = 1,2} F_{\tau, i_\tau, i_{\tau^c}}(\gamma_1, \gamma_2), \tag{14}$$

where

$$F_{\tau, i_\tau, i_{\tau^c}}(\gamma_1, \gamma_2) = \max_{\rho \in [0,1]} E_0(\rho, Q_{\tau, i_\tau}, WQ_{\tau^c, i_{\tau^c}}) - E_{s, i_\tau}(\rho, P_{U_\tau}, \gamma_\tau) - E_{s, i_{\tau^c}}(0, P_{U_{\tau^c}}, \gamma_{\tau^c}). \tag{15}$$

In (15)*, the functions $E_0(\cdot)$, $E_{s,1}(\cdot)$ and $E_{s,2}(\cdot)$ are respectively given by* (8)*,* (9) *and* (10)*, and we define $E_{s,i_{\{1,2\}}}(\rho, P_{\underline{U}}, \underline{\gamma}) = E_{s,i_1}(\rho, P_{U_1}, \gamma_1) + E_{s,i_2}(\rho, P_{U_2}, \gamma_2)$.*

Proof: See [7, Appendix A]. ∎

We remark that the optimal assignment of input distributions to source classes is considered in (14). Since we considered two source-message classes $\mathcal{A}_\nu^1$, $\mathcal{A}_\nu^2$ and two input distributions $Q_{\nu,1}, Q_{\nu,2}$ for each user $\nu = 1, 2$, there are four possible assignments.

The derived achievable exponent (14) contains a maximization over γ_1 and γ_2, the thresholds that determine how source messages are partitioned into classes. Rearranging the minimizations over τ, i_τ and i_{τ^c}, defining $f_{i_1,i_2}(\gamma_1,\gamma_2)$ as

$$f_{i_1,i_2}(\gamma_1,\gamma_2) = \min_{\tau\in\{\{1\},\{2\},\{1,2\}\}} F_{\tau,i_\tau,i_{\tau^c}}(\gamma_1,\gamma_2), \tag{16}$$

where $F_{\tau,i_\tau,i_{\tau^c}}(\gamma_1,\gamma_2)$ is given in (15), the achievable exponent (14) can be written as

$$E(P_{\underline{U}},W) = \max_{\gamma_1,\gamma_2\in[0,1]} \min_{i_1,i_2=1,2} f_{i_1,i_2}(\gamma_1,\gamma_2). \tag{17}$$

We note that regardless the values of i_2, $f_{1,i_2}(\underline{\gamma})$ is non-decreasing with respect to γ_1 and $f_{2,i_2}(\underline{\gamma})$ is non-increasing with respect to γ_1. Similarly, regardless the values of i_1, $f_{i_1,1}(\underline{\gamma})$ is non-decreasing with respect to γ_2 and $f_{i_1,2}(\underline{\gamma})$ is non-increasing with respect to γ_2. As a result, we derive a system of equations to compute the optimal thresholds $\gamma_1^\star$ and $\gamma_2^\star$.

Proposition 2. *The optimal $\gamma_1^\star$ and $\gamma_2^\star$ maximizing* (14) *satisfy*

$$\begin{cases} \min\limits_{i_2=1,2} f_{1,i_2}(\gamma_1^\star,\gamma_2^\star) = \min\limits_{i_2=1,2} f_{2,i_2}(\gamma_1^\star,\gamma_2^\star), \\ \min\limits_{i_1=1,2} f_{i_1,1}(\gamma_1^\star,\gamma_2^\star) = \min\limits_{i_1=1,2} f_{i_1,2}(\gamma_1^\star,\gamma_2^\star). \end{cases} \tag{18}$$

When (18) *has no solutions, then $\gamma_\nu^\star \in \{0,1\}$. In particular, if $f_{1,i_2}(0,\gamma_2) > f_{2,i_2}(0,\gamma_2)$ then $\gamma_1^\star = 0$, otherwise $\gamma_1^\star = 1$; and if $f_{i_1,1}(\gamma_1,0) > f_{i_1,2}(\gamma_1,0)$, we have $\gamma_2^\star = 0$, otherwise $\gamma_2^\star = 1$.*

Proof: See [7, Appendix B].. ■

We note that the optimal $\gamma_1^\star$ and $\gamma_2^\star$ are the points where the minimum of all non-decreasing functions with respect to γ_ν are equal with the minimum of all non-increasing functions with respect to γ_ν, for both $\nu = 1,2$. Even though $\gamma_1^\star$ and $\gamma_2^\star$ can be computed through equation (18), the final expression of the achievable exponent (14) is still coupled with $\gamma_1^\star$ and $\gamma_2^\star$. In the sequel, we alternatively study both lower and an upper bounds that do not depend on γ_1 and γ_2.

A. A Lower Bound for the Achievable Exponent

In order to find a lower bound for the achievable exponent presented in (14), we use properties (12) and (13). Firstly, we maximize over $\gamma_\nu \in \{0,1\}$ rather than $\gamma_\nu \in [0,1]$, for $\nu = 1,2$, to lower bound (14). Let $d(\gamma_1,\gamma_2)$ be

$$d(\gamma_1,\gamma_2) = \min_{i_1,i_2} f_{i_1,i_2}(\gamma_1,\gamma_2). \tag{19}$$

Then,

$$E(P_{\underline{U}},W) = \max_{\gamma_1,\gamma_2\in[0,1]} d(\gamma_1,\gamma_2) \geq \max_{\gamma_1,\gamma_2\in\{0,1\}} d(\gamma_1,\gamma_2). \tag{20}$$

On the other hand,

$$\max_{\gamma_1,\gamma_2\in\{0,1\}} d(\gamma_1,\gamma_2) = \max\{d(0,0), d(0,1), d(1,0), d(1,1)\}. \tag{21}$$

Taking into account properties (12) and (13), we note that $f_{i_1,i_2}(\gamma_1,\gamma_2)$, for $\gamma_1,\gamma_2 \in \{0,1\}$, is either infinity, or the Gallager's source-channel exponent, i.e.,

$$\max_{\rho\in[0,1]} E_0(\rho, Q_{\tau,i_\tau}, WQ_{\tau^c,i_{\tau^c}}) - E_s(\rho, P_{U_\tau}). \tag{22}$$

For example, $f_{i_1,i_2}(0,1)$ equals equation (22) for $i_1 = 1$ and $i_2 = 2$, and $f_{i_1,i_2}(0,1) = \infty$ for the rest of combinations of i_1 and i_2. Thus, $d(0,1) = \min_\tau \max_{\rho\in[0,1]} E_0(\rho, Q_{\tau,i_\tau}, WQ_{\tau^c,i_{\tau^c}}) - E_s(\rho, P_{U_\tau})$ for $i_1 = 1$ and $i_2 = 2$. Similarly, $d(1,0) = \min_\tau \max_{\rho\in[0,1]} E_0(\rho, Q_{\tau,i_\tau}, WQ_{\tau^c,i_{\tau^c}}) - E_s(\rho, P_{U_\tau})$ for $i_1 = 2$ and $i_2 = 1$, and so on. Hence, we obtain the following lower bound

$$E(P_{\underline{U}},W) \geq E_{\mathrm{L}}(P_{\underline{U}},W), \tag{23}$$

where

$$E_{\mathrm{L}}(P_{\underline{U}},W) = \max_{i_1\in\{1,2\}} \max_{i_2\in\{1,2\}} \min_{\tau\in\{\{1\},\{2\},\{1,2\}\}} F^{\mathrm{L}}_{\tau,i_\tau,i_{\tau^c}}, \tag{24}$$

with

$$F^{\mathrm{L}}_{\tau,i_\tau,i_{\tau^c}} = \max_{\rho\in[0,1]} E_0(\rho, Q_{\tau,i_\tau}, WQ_{\tau^c,i_{\tau^c}}) - E_s(\rho, P_{U_\tau}). \tag{25}$$

We note that for $\tau = \{1\}$ and $\tau = \{2\}$, $F^{\mathrm{L}}_{\tau,i_\tau,i_{\tau^c}}$ in (25) is the error exponent of the point-to-point channel $WQ_{\tau^c,i_{\tau^c}}$ for an i.i.d. random-coding ensemble with distribution $Q_{\tau,i}$. For $\tau = \{1,2\}$, we have $WQ_{\tau^c,i_{\tau^c}} = W$ and $E_s(\rho, P_{U_\tau}) = E_s(\rho, P_{U_1}) + E_s(\rho, P_{U_2})$, so that (25) is the error exponent of the point-to-point channel W for an i.i.d. random-coding ensemble with distribution $Q_{1,i_1}Q_{2,i_2}$. Hence, the lower bound (24) selects the best assignment of input distributions over all four combinations through i_1 and i_2.

B. An Upper Bound for the Achievable Exponent

Now, we derive an upper bound for (14) inspired by the tools used in [4] for single user communication. Let $E_0(\rho,\mathcal{Q},W) = \max_{Q\in\mathcal{Q}} E_0(\rho,Q,W)$, where $\mathcal{Q}$ is a set of distributions. We denote $\bar{E}_0(\rho,\mathcal{Q},W)$ as the concave hull of $E_0(\rho,\mathcal{Q},W)$, defined as the point-wise supremum over all convex combinations of any two values of the function $E_0(\rho,\mathcal{Q},W)$, i.e.,

$$\bar{E}_0(\rho,\mathcal{Q},W) \triangleq \sup_{\substack{\rho_1,\rho_2,\theta\in[0,1]: \\ \theta\rho_1+(1-\theta)\rho_2=\rho}} \Big\{\theta E_0(\rho_1,\mathcal{Q},W) + (1-\theta)E_0(\rho_2,\mathcal{Q},W)\Big\}. \tag{26}$$

In [4], it is proved that joint source-channel random coding where source messages are assigned to different classes and codewords are generated according to a distribution that depends on the class index of source message, achieves the following exponent

$$\max_{\rho\in[0,1]} \bar{E}_0(\rho,\mathcal{Q},W) - E_s(\rho,P_U), \tag{27}$$

which coincides with the sphere-packing exponent [2, Lemma 2] whenever it is tight.

For the MAC with independent sources, we use the max-min inequality [8] to upper-bound (14) by swapping the maximization over γ_1,γ_2 with the minimization over τ. Then, for a given τ, we use Lemma 2 in [7, Appendix C] to obtain the following result.

Proposition 3. *The achievable exponent* (14) *is upper bounded as*

$$E(P_{\underline{U}}, W) \le E_{\mathrm{U}}(P_{\underline{U}}, W), \tag{28}$$

where

$$E_{\mathrm{U}}(P_{\underline{U}}, W) = \min_{\tau \in \{\{1\},\{2\},\{1,2\}\}} F_{\tau}^{\mathrm{U}}, \tag{29}$$

where

$$F_{\tau}^{\mathrm{U}} = \max_{i_{\tau^c}=1,2} \max_{\rho\in[0,1]} \bar{E}_0(\rho, \{Q_{\tau,1}, Q_{\tau,2}\}, WQ_{\tau^c,i_{\tau^c}}) - E_s(\rho, P_{U_\tau}). \tag{30}$$

We recall that for $\tau = \{1,2\}$*, we have* $\{Q_{\tau,1}, Q_{\tau,2}\} = \{Q_{1,1}, Q_{2,1}, Q_{1,2}, Q_{2,2}\}$ *and* $E_s(\rho, P_{U_\tau}) = E_s(\rho, P_{U_1}) + E_s(\rho, P_{U_2})$.

Proof: [7, Appendix C]. ∎

From equation (29), we observe that the upper bound is the minimum of three terms depending on $\tau \in \{\{1\},\{2\},\{1,2\}\}$. For $\tau \in \{\{1\},\{2\}\}$, we know that the message of user τ^c is decoded correctly so that user τ is virtually sent either over channel $WQ_{\tau^c,1}$ or $WQ_{\tau^c,2}$. Hence, the objective function of (29) is the single-user exponent for source P_{U_τ} and point-to-point channel $WQ_{\tau^c,i_{\tau^c}}$ where codewords are generated according to two assigned input distributions $\{Q_{\tau,1}, Q_{\tau,2}\}$ depending on class index of source messages. As a result, we note that the maximization over $i_{\tau^c} = 1,2$ is equivalent to choose the best channel (either $WQ_{\tau^c,1}$ or $WQ_{\tau^c,2}$) in terms of error exponent.

C. Numerical Example

Here we provide a numerical example comparing the achievable exponent, the lower bound and the upper bound given in (14), (24) and (29), respectively. We consider two independent discrete memoryless sources with alphabet $\mathcal{U}_\nu = \{1,2\}$ for $\nu = 1,2$ where $P_{U_1}(1) = 0.028$ and $P_{U_2}(1) = 0.01155$. We also consider a discrete memoryless multiple-access channel with $\mathcal{X}_1 = \mathcal{X}_2 = \{1,2,\dots,6\}$ and $|\mathcal{Y}| = 4$. The transition probability of this channel, denoted as W, is given by

$$W = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \end{pmatrix}, \tag{31}$$

Table I

VALUES OF $F_{\tau,i_\tau,i_{\tau^c}}(\gamma_1^\star, \gamma_2^\star)$ IN (15) WITH OPTIMAL THRESHOLDS $\gamma_1^\star = 0.8159$ $\gamma_2^\star = 0.7057$, FOR TYPES OF ERROR τ, AND USER CLASSES i_τ AND i_{τ^c}.

	(i_1,i_2)			
	(1,1)	(2,1)	(1,2)	(2,2)
$\tau=\{1\}$	0.2566	0.1721	0.1057	0.1103
$\tau=\{2\}$	0.2597	0.1057	0.2526	0.2087
$\tau=\{1,2\}$	0.1057	0.1073	0.1127	0.1180

where

$$W_1 = \begin{pmatrix} 1-3k_1 & k_1 & k_1 & k_1 \\ k_1 & 1-3k_1 & k_1 & k_1 \\ k_1 & k_1 & 1-3k_1 & k_1 \\ k_1 & k_1 & k_1 & 1-3k_1 \\ 0.5-k_2 & 0.5-k_2 & k_2 & k_2 \\ k_2 & k_2 & 0.5-k_2 & 0.5-k_2 \end{pmatrix}, \tag{32}$$

for $k_1 = 0.056$ and $k_2 = 0.01$. W_2 and W_3 are 6×4 matrices whose rows are all the copy of 5th and 6th row of matrix W_1, respectively. Let the m-th row of matrix W_1 is denoted by $W_1(m)$. W_4, W_5 and W_6 are respectively given by

$$W_4 = \begin{pmatrix} W_1(2) \\ W_1(3) \\ W_1(4) \\ W_1(1) \\ W_1(6) \\ W_1(5) \end{pmatrix} \quad W_5 = \begin{pmatrix} W_1(3) \\ W_1(4) \\ W_1(1) \\ W_1(2) \\ W_1(5) \\ W_1(6) \end{pmatrix} \quad W_6 = \begin{pmatrix} W_1(4) \\ W_1(1) \\ W_1(2) \\ W_1(3) \\ W_1(6) \\ W_1(5) \end{pmatrix}. \tag{33}$$

We observe that W is a 36×4 matrix where the transition probability $W(y|x_1,x_2)$ is placed at the row $x_1 + 6(x_2-1)$ of matrix W, for $(x_1,x_2) \in \{1,2,...,6\}\times\{1,2,...,6\}$. Recalling that each source has two classes and that four input distributions generate codewords, there are four possible assignments of input distributions to classes. Among all possible permutations, we select the one that gives the highest exponent. Here, for user $\nu = 1,2$, we consider the set of input distributions $\big\{[0\ 0\ 0\ 0\ 0.5\ 0.5],\ [0.25\ 0.25\ 0.25\ 0.25\ 0\ 0]\big\}$. For the channel given in (31), the optimal assignment is

$$Q_{\nu,1} = [0\ 0\ 0\ 0\ 0.5\ 0.5], \tag{34}$$

$$Q_{\nu,2} = [0.25\ 0.25\ 0.25\ 0.25\ 0\ 0], \tag{35}$$

for both $\nu = 1,2$. Since we consider two input distributions for each user, the function $\max_{\rho\in[0,1]} E_0(\rho, Q_{\tau,i_\tau}, WQ_{\tau^c,i_{\tau^c}})$ is not concave in ρ [4]. For this example, from (18), we numerically compute the optimal $\gamma_1^\star$ and $\gamma_2^\star$ maximizing (14) leading to $\gamma_1^\star = 0.8159$ and $\gamma_2^\star = 0.7057$.

Tables I, II and III respectively show the objective functions $F_{\tau,i_\tau,i_{\tau^c}}(\gamma_1,\gamma_2)$, $F_{\tau,i_\tau,i_{\tau^c}}^{\mathrm{L}}$, and F_τ^{U} given in (15), (25) and (30), involved in the derivation of the achievable exponent (14), lower bound (24) and upper bound (29). The shaded elements in Tables I and III respectively are the exponent and the upper bound. Additionally, the shaded elements in Table II are the i.i.d. exponent for different input distributions

Table II
VALUES OF $F^{\mathrm{L}}_{\tau,i_\tau,i_{\tau^c}}$ IN (25) FOR TYPES OF ERROR τ, AND INPUT DISTRIBUTION Q_{1,i_1}, Q_{2,i_2}.

	$Q_{1,1}$,$Q_{2,1}$	$Q_{1,2}$,$Q_{2,1}$	$Q_{1,1}$,$Q_{2,2}$	$Q_{1,2}$,$Q_{2,2}$
$\tau = \{1\}$	0.1723	0.1721	0.0251	0.0342
$\tau = \{2\}$	0.2526	0.0989	0.2526	0.2019
$\tau = \{1,2\}$	0.0900	0.1073	0.0900	0.0984

Table III
VALUES OF F^{U}_{τ} IN (30) FOR TYPES OF ERROR τ.

$\tau = \{1\}$	$\tau = \{2\}$	$\tau = \{1,2\}$
0.1734	0.2526	0.1073

assignments. Solving equations (14), (24), (29) using the partial optimizations in Tables I, II and III, we respectively obtain

$$E(P_{\underline{U}}, W) = 0.1057, \tag{36}$$

$$E_{\mathrm{L}}(P_{\underline{U}}, W) = 0.0989, \tag{37}$$

$$E_{\mathrm{U}}(P_{\underline{U}}, W) = 0.1073. \tag{38}$$

We observe that the percentage difference between the achievable exponent $E(P_{\underline{U}}, W)$ and the lower bound $E_{\mathrm{L}}(P_{\underline{U}}, W)$ is 6.875%. For a given set of two distributions for each user, the lower bound $E_{\mathrm{L}}(P_{\underline{U}}, W)$ corresponds to the i.i.d. random-coding error exponent when each user uses only one input distribution. In [4], a similar comparison is made for point-to-point communication where the exponent achieved by an ensemble with two distributions is 0.75% higher than the one achieved by the i.i.d. ensemble. Hence, our example illustrates that using message-dependent random coding with two class distributions may lead to higher error exponent gain in the MAC than in point-to-point communication, compared to i.i.d. random coding.

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