Coding for the MIMO ARQ Block-Fading Channel with Imperfect Feedback and CSIR

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***Abstract*—We investigate the effects of imperfect channel knowledge and feedback in incremental-redundancy automatic-repeat request (INR-ARQ) coding systems over multiple-input multiple-output (MIMO) block-fading channels. We propose an ARQ decoder based on nearest neighbour decoding and evaluate the corresponding achievable rates. We then derive the optimal code diversity assuming that the feedback channel is modelled as a binary symmetric channel. Our main results show that the feedback link reliability must improve with the forward (transmission) signal-to-noise ratio (SNR) for the code to exploit the diversity offered by ARQ scheme. We also identify the conditions for achieving full diversity and for ARQ not helping in improving the system's diversity.**

I. Introduction

Incremental-redundancy automatic-repeat request (INR-ARQ) is a rate-adaptive technique that allows to increase the reliability of transmission over inherently unreliable channels like slowly-varying wireless channels [1]. This paper considers INR-ARQ over multiple-input multiple-output (MIMO) block-fading channels. The block-fading channel accurately models slowly-varying fading, and INR-ARQ is naturally suited to improve its performance. In a block-fading setup, coding schemes cannot achieve an arbitrarily small error probability since the outage probability, the probability that the channel is unable to support the desired data rate, limits the optimal error probability [2]. INR-ARQ scheme improves the error probability by retransmitting parts of the incorrectly decoded message until the message is correctly decoded or until the maximum transmission delay has been reached.

The code diversity performance of INR-ARQ over block-fading channels has been studied in a number of works. A notable result was the optimal rate-diversity-delay trade-off, firstly studied in [3] for Gaussian inputs under quasi-static fading channels and later in [4] for both Gaussian and discrete inputs under block-fading channels. Common assumptions of previous works are perfect channel state information at the receiver (CSIR) and perfect ARQ feedback [1], [3]–[5]. In practical implementations, unfortunately, these idealistic assumptions are difficult to guarantee.

This paper studies precisely the impact of imperfect CSIR and feedback on the diversity performance of INR-ARQ coding systems. In particular, we consider noisy CSIR and a simple binary symmetric channel (BSC) model for the feedback link. We analyse the performance of random coding schemes constructed over Gaussian and discrete signal constellations. We approach the problem by proposing an ARQ decoder based on nearest neighbour decoding that behaves as a typical-set threshold decoder up to round $(L-1)$ and as a maximum-metric decoder at L-th round. We then study the corresponding error probability. This error probability is fundamentally characterised using the generalised mutual information (GMI) [6], [7] for i.i.d. codebooks in the limit of large code length. We then derive the optimal signal-to-noise ratio (SNR) exponents. In particular, we show that the feedback link reliability must improve with the transmit SNR for the code to exploit the diversity offered by ARQ. We also identify the two extremes of imperfect feedback: the one that guarantees full diversity and the one that shows the inability of ARQ to improve the diversity performance.

Notation: X, x, and X denote random scalar, vector and matrix variables. x, $\boldsymbol{x}$, and $\boldsymbol{X}$ denote scalar, vector and matrix.

II. System Model

A. Channel Model

We consider an INR-ARQ coding scheme over a MIMO block-fading channel with L rounds and B fading blocks per round. The input-output relationship of the channel at ARQ round ℓ is

$$\boldsymbol{Y}_\ell = \sqrt{\frac{\mathsf{snr}}{n_{\rm t}}}\boldsymbol{H}_\ell \boldsymbol{X}_\ell + \boldsymbol{Z}_\ell, \quad \ell = 1, \ldots, L \tag{1}$$

where $\boldsymbol{Y}_\ell, \boldsymbol{Z}_\ell \in \mathbb{C}^{Bn_{\rm r} \times T}$ and $\boldsymbol{X}_\ell \in \mathcal{X}^{Bn_{\rm t} \times T}$ are the received, noise and transmitted signal matrices; $n_{\rm t}$, $n_{\rm r}$, T and $\mathcal{X}$ denote the number of transmit antennas, the number of receive antennas, the channel block length and the signal constellation, respectively. We assume that the entries of $\boldsymbol{H}_\ell$ and $\boldsymbol{Z}_\ell$ are i.i.d. complex circularly symmetric Gaussian random variables with zero mean and unit variance. The fading matrix, $\boldsymbol{H}_\ell \in \mathbb{C}^{Bn_{\rm r} \times Bn_{\rm t}}$, is defined by

$$\boldsymbol{H}_\ell \triangleq \mathrm{diag}\left(\boldsymbol{H}_{\ell,1}, \ldots, \boldsymbol{H}_{\ell,B}\right) \tag{2}$$

where $\boldsymbol{H}_{\ell,b} \in \mathbb{C}^{n_{\rm r} \times n_{\rm t}}$ is the fading matrix for round ℓ and block b. We further assume that the fading process $\boldsymbol{H}_\ell$ follows the short-term static model for which the matrices $\boldsymbol{H}_{\ell,b}$ are i.i.d. from block to block and from round to round [3]. We write the channel outputs accumulated up to round ℓ as follows

$$\boldsymbol{Y}_{\bar{\ell}} = \sqrt{\frac{\mathsf{snr}}{n_{\rm t}}}\boldsymbol{H}_{\bar{\ell}}\boldsymbol{X}_{\bar{\ell}} + \boldsymbol{Z}_{\bar{\ell}} \tag{3}$$

where (noting that $(\cdot)'$ is non-conjugate transpose)

$$\boldsymbol{Y}_{\bar{\ell}} = [\boldsymbol{Y}_1', \ldots, \boldsymbol{Y}_\ell']', \quad \boldsymbol{X}_{\bar{\ell}} = [\boldsymbol{X}_1', \ldots, \boldsymbol{X}_\ell']' \tag{4}$$

$$\boldsymbol{Z}_{\bar{\ell}} = [\boldsymbol{Z}_1', \ldots, \boldsymbol{Z}_\ell']', \quad \boldsymbol{H}_{\bar{\ell}} = \mathrm{diag}\left(\boldsymbol{H}_1, \ldots, \boldsymbol{H}_\ell\right). \tag{5}$$

The receiver employs a channel estimation scheme yielding accurate yet imperfect channel estimates

$$\hat{\boldsymbol{H}}_\ell = \boldsymbol{H}_\ell + \boldsymbol{E}_\ell, \quad \ell = 1, \ldots, L \tag{6}$$

where $\hat{\boldsymbol{H}}_\ell \triangleq \mathrm{diag}\left(\hat{\boldsymbol{H}}_{\ell,1}, \ldots, \hat{\boldsymbol{H}}_{\ell,B}\right)$, $\hat{\boldsymbol{H}}_{\ell,b} \in \mathbb{C}^{n_{\rm r}\times n_{\rm t}}$ and $\boldsymbol{E}_\ell \triangleq \mathrm{diag}\left(\boldsymbol{E}_{\ell,1}, \ldots, \boldsymbol{E}_{\ell,B}\right)$, $\boldsymbol{E}_{\ell,b} \in \mathbb{C}^{n_{\rm r}\times n_{\rm t}}$ are the noisy channel estimate and the channel estimation error, respectively. In particular, the entries of $\boldsymbol{E}_{\ell,b}$ are modelled as i.i.d. Gaussian random variables with zero mean and variance equal to

$$\sigma_e^2 = \mathsf{snr}^{-d_e}, \quad \text{with } d_e > 0. \tag{7}$$

This model is widely used in pilot-based channel estimation for which the error variance is proportional to the reciprocal of the pilot SNR [8], [9]. We further incorporate the parameter d_e, denoting the channel estimation error diversity. The accumulated matrices $\hat{\boldsymbol{H}}_{\bar{\ell}}$ and $\boldsymbol{E}_{\bar{\ell}}$ are written as

$$\hat{\boldsymbol{H}}_{\bar{\ell}} = \mathrm{diag}\left(\hat{\boldsymbol{H}}_1, \ldots, \hat{\boldsymbol{H}}_\ell\right), \quad \boldsymbol{E}_{\bar{\ell}} = \mathrm{diag}\left(\boldsymbol{E}_1, \ldots, \boldsymbol{E}_\ell\right). \tag{8}$$

B. Encoder

A message $m \in \{1, \ldots, 2^{BTR_1}\}$ is mapped into a codeword of the code $\mathcal{C}$ with rate $R_L \triangleq \frac{R_1}{L}$, where $R_1 = \frac{1}{BT}\log_2|\mathcal{C}|$ is the coding rate for the first ARQ round. The resulting codeword is divided into LB coding blocks, $\boldsymbol{X}_{\ell,b} \in \mathcal{X}^{n_{\rm t}\times T}$. Each symbol is drawn i.i.d. from an input signal set; here we focus on Gaussian and discrete inputs. The codewords are assumed to satisfy the average unit energy $\frac{1}{BT}\mathbb{E}\left[\|\mathsf{X}_\ell\|_F^2\right] \le n_{\rm t}$.

The transmission is started by sending the first ARQ round. Then, the transmitter receives through a feedback channel a one-bit *positive* acknowledgement or *negative* acknowledgement. If the positive acknowledgement is received, the transmitter understands that the message has been successfully delivered and starts the transmission of the next message. Instead, if the negative acknowledgement is received, the transmitter sends the next ARQ round corresponding to the current message. This process continues until the positive acknowledgement is received or until the maximum number of rounds has been reached.

C. Decoder

For rounds $\ell = 1, \ldots, L-1$, the ARQ scheme requires a decoder with error detection capability. In practice, this can be accomplished using parity check bits or by using advanced stopping criteria in iterative decoders. For perfect CSIR case, an ARQ decoder structure has been proposed in [1], [3] composed of typical-set decoding for rounds $\ell = 1, \ldots, L-1$ and maximum-likelihood decoding for the last round. In the case of imperfect CSIR, we cannot use that decoder and hence we consider a decoder with the following metric

$$Q(\boldsymbol{X}_{\bar{\ell}}(m), \boldsymbol{Y}_{\bar{\ell}}|\boldsymbol{H}_{\bar{\ell}}, \boldsymbol{E}_{\bar{\ell}}) = e^{-\|\boldsymbol{Y}_{\bar{\ell}} - \sqrt{\frac{\mathsf{snr}}{n_{\rm t}}}\hat{\boldsymbol{H}}_{\bar{\ell}}\boldsymbol{X}_{\bar{\ell}}(m)\|_F^2}. \tag{9}$$

For rounds $\ell = 1, \ldots, L-1$, we use a modified threshold decoder of [10, Lemma 6.9]. Consider the following typical-set with any $s \ge 0$ and $\delta > 0$

$$\mathcal{T}_\delta(\ell) = \Bigg\{\boldsymbol{X}_{\bar{\ell}}(m) \in \mathcal{X}^{n_{\rm t}\times \ell BT}, \boldsymbol{Y}_{\bar{\ell}} \in \mathbb{C}^{n_{\rm r}\times \ell BT} : \tag{10}$$

$$\frac{Q^s(\boldsymbol{X}_{\bar{\ell}}(m), \boldsymbol{Y}_{\bar{\ell}}|\boldsymbol{H}_{\bar{\ell}}, \boldsymbol{E}_{\bar{\ell}})}{\mathbb{E}\left[Q^s(\mathsf{X}_{\bar{\ell}}', \mathsf{Y}_{\bar{\ell}}|\boldsymbol{H}_{\bar{\ell}}, \boldsymbol{E}_{\bar{\ell}})|\mathsf{Y}_{\bar{\ell}} = \boldsymbol{Y}_{\bar{\ell}}\right]} \ge \frac{|\mathcal{C}|}{\delta}\Bigg\}$$

where $\mathsf{X}_{\bar{\ell}}'$ and $\mathsf{Y}_{\bar{\ell}}$ are random matrices with same dimension as $\boldsymbol{X}_{\bar{\ell}}$ and $\boldsymbol{Y}_{\bar{\ell}}$, respectively. Let $\phi(\cdot)$ be the threshold decoder based on this typical set. Conditioned on fixed channel and estimation error realisations: $\boldsymbol{H}_{\bar{\ell}}$ and $\boldsymbol{E}_{\bar{\ell}}$, the decoder outputs the following

- $\phi(\boldsymbol{Y}_{\bar{\ell}}) = \hat{m}$, $\hat{m} \in \{1, \ldots, 2^{BTR_1}\}$ if $\boldsymbol{X}_{\bar{\ell}}(\hat{m})$ is the unique jointly typical sequence with $\boldsymbol{Y}_{\bar{\ell}}$ in $\mathcal{T}_\delta(\ell)$. Positive acknowledgement is then generated by the receiver.
- $\phi(\boldsymbol{Y}_{\bar{\ell}}) = 0$, otherwise. Negative acknowledgement is then generated by the receiver.

At the last round L, the decoder outputs the message $\hat{m}$ with

$$\hat{m} = \arg\max_{m\in\{1,\ldots 2^{BTR_1}\}} Q(\boldsymbol{X}_{\bar{L}}(m), \boldsymbol{Y}_{\bar{L}}|\boldsymbol{H}_{\bar{L}}, \boldsymbol{E}_{\bar{L}}). \tag{11}$$

D. Feedback Channel

We model the feedback channel as a BSC with crossover probability $p_{\rm fb}$. The motivation behind this model is that the acknowledgement signal sent by the receiver may be interpreted incorrectly by the transmitter due to the unreliable medium. We assume that the crossover probability is such that

$$p_{\rm fb} = \min\left\{p_0, \frac{p_0}{\mathsf{snr}^{d_{\rm fb}}}\right\} \tag{12}$$

where $0 \le p_0 \le \frac{1}{2}$ and $d_{\rm fb}$ is the feedback diversity. This models a feedback channel whose quality increases with the forward link SNR. The perfect feedback assumption [3], [4], [11] is a special case of this BSC feedback with $p_{\rm fb} = 0$ ($p_0 = 0$ or $d_{\rm fb} \to \infty$ for $\mathsf{snr} > 1$). The acknowledgement signal generated by the receiver and decoded by the transmitter are denoted by ACKr and ACKt, respectively; 0 and 1 denote the negative and positive acknowledgements, respectively.

III. Information-Theoretic Preliminaries

The generalised mutual information (GMI) is the largest rate for reliable communication using long i.i.d. codebooks when the decoder is mismatched [12]–[14]. At ARQ round ℓ, the *accumulated GMI* is shown to be [15]

$$I_{\bar{\ell}}^{\rm gmi} = \sup_{s\ge 0}\frac{1}{B}\sum_{l=1}^{\ell}\sum_{b=1}^{B} I_{l,b}^{\rm gmi}(\mathsf{snr}, \boldsymbol{H}_{l,b}, \hat{\boldsymbol{H}}_{l,b}, s) \tag{13}$$

where

$$I_{l,b}^{\rm gmi}(\mathsf{snr}, \boldsymbol{H}_{l,b}, \hat{\boldsymbol{H}}_{l,b}, s) = \tag{14}$$

$$\mathbb{E}\left[\log_2\frac{Q^s(\mathsf{x}, \mathsf{y}|\boldsymbol{H}_{l,b}, \boldsymbol{E}_{l,b})}{\mathbb{E}\left[Q^s(\mathsf{x}', \mathsf{y}|\boldsymbol{H}_{l,b}, \boldsymbol{E}_{l,b})|\,\mathsf{y}, \boldsymbol{H}_{l,b}, \boldsymbol{E}_{l,b}\right]}\,\middle|\, \boldsymbol{H}_{l,b}, \boldsymbol{E}_{l,b}\right].$$

and x, x' and y are random column complex-vectors with n_{t}, n_{t} and n_{r} elements, respectively. Due to the imperfect CSIR and the mismatched metric (9), the accumulated GMI plays the role of the *accumulated mutual information* [4], [11] to characterise the communication outages in the block-fading channel. At ARQ round ℓ, the effective coding rate becomes $R_\ell = \frac{R_1}{\ell}$ and the generalised outage probability at round ℓ [15], which quantifies the communication outage after ℓ-th round, is given as

$$P_{\mathrm{gout}}^{\ell}(R_1) \triangleq \Pr\left\{\frac{1}{\ell} I_{\bar{\ell}}^{\mathrm{gmi}} < \frac{R_1}{\ell}\right\} = \Pr\{I_{\bar{\ell}}^{\mathrm{gmi}} < R_1\}. \quad (15)$$

Consider an ARQ round ℓ, $\ell \in \{1,\dots,L-1\}$. We can write the whole error event ξ_ℓ as the union of the detected error event $\mathcal{D}_\ell$ and undetected error event $\mathcal{D}_\ell^c$. The correct decoding event is subsequently denoted as ξ_ℓ^c. Using similar steps to [10, Lemma 6.9], the average error probability can be upper-bounded as

$$\Pr\{\mathcal{E}_\ell|\boldsymbol{H}_{\bar{\ell}},\boldsymbol{E}_{\bar{\ell}}\} \le 2^{-\ell BT\gamma} + \inf_{s\ge 0} \Pr\left\{\frac{1}{\ell BT}\log_2\frac{Q^s(\mathsf{X},\mathsf{Y}|\boldsymbol{H}_{\bar{\ell}},\boldsymbol{E}_{\bar{\ell}})}{\mathbb{E}\left[Q^s(\mathsf{X}',\mathsf{Y}|\boldsymbol{H}_{\bar{\ell}},\boldsymbol{E}_{\bar{\ell}})|\mathsf{Y}\right]} < \frac{R_1}{\ell}+\gamma\right\} \quad (16)$$

for any $\gamma > 0$ where X and X' are random matrices of the same dimension as $\boldsymbol{X}_{\bar{\ell}}$ and Y is a random matrix of the same dimension as $\boldsymbol{Y}_{\bar{\ell}}$. On the other hand, the converse shows that for $\epsilon > 0$ and sufficiently large T, the error probability of random codes with i.i.d. codebooks is lower-bounded by [14]

$$\Pr\{\xi_\ell|\boldsymbol{H}_{\bar{\ell}},\boldsymbol{E}_{\bar{\ell}}\} \ge 1 - e^{-e^{-\ell BT\left(\frac{1}{\ell}I_{\bar{\ell}}^{\mathrm{gmi}}+\epsilon-\frac{R_1}{\ell}\right)}+e^{-\ell BT\left(\frac{1}{\ell}I_{\bar{\ell}}^{\mathrm{gmi}}+\epsilon\right)}}. \quad (17)$$

At round ℓ, if the accumulated GMI, $I_{\bar{\ell}}^{\mathrm{gmi}}$ is greater than data rate, R_1, the achievability result implies that the error probability vanishes for sufficiently large T. Conversely, if the accumulated GMI is less than data rate, then the converse result implies that the error probability for i.i.d. codes with large block length tends to one. Using similar arguments as in [1, Appendix A] and [10, Lemma 6.9], the undetected error probability can be upper-bounded as [16]

$$\Pr\{\mathcal{D}_\ell^c\} < 2^{-\ell BT\gamma}. \quad (18)$$

This implies that we are able to detect errors with high probability as long as T is sufficiently large.

At round L, the decoder selects the message whose codeword maximises (9). As shown in [15], this maximum metric decoder has similar achievability and converse properties as the typical-set decoder for $\ell = 1,\dots,L-1$ described in the preceding paragraph as the block length tends to infinity, $T \to \infty$.

IV. Imperfect Feedback

The imperfect feedback affects the ARQ system performance, and in particular, throughput, latency and diversity. The performance is characterised by the error events. As in the perfect CSIR case [3], [4], the average error probability with imperfect CSIR and perfect feedback ($p_{\mathrm{fb}} = 0$) can be expressed as

$$P_e = P_L^{\mathrm{NN}} + \sum_{\ell=1}^{L-1}\Pr\{\mathcal{D}_\ell^c\} \quad (19)$$

where P_L^{NN} is the probability of decoding error using a nearest neighbour decoder (9) at round L. With imperfect feedback, however, there are additional error events that need to be accounted for. In particular, the additional errors come from the event that the receiver detects an error and sends the negative acknowledgement, $\mathsf{ACKr} = 0$, but the transmitter interprets it as $\mathsf{ACKt} = 1$. Thus, (19) becomes

$$P_e = P_L^{\mathrm{NN}} + \sum_{\ell=1}^{L-1}\Big(\Pr\{\mathsf{ACKt}_\ell = 1,\mathcal{D}_\ell\} + \Pr\{\mathsf{ACKt}_\ell = 1,\mathcal{D}_\ell^c\}\Big) \quad (20)$$

$$= P_L^{\mathrm{NN}} + \sum_{\ell=1}^{L-1} p_{\mathrm{fb}}\Pr\{\mathcal{D}_\ell\} + \sum_{\ell=1}^{L-1}(1-p_{\mathrm{fb}})\Pr\{\mathcal{D}_\ell^c\}. \quad (21)$$

If $\mathsf{ACKr} = 1$ signal is mistakenly interpreted as $\mathsf{ACKt} = 0$ by the transmitter, then the error probability does not improve. However, the extra transmitted rounds increase the latency and lower the throughput. Due to space limitation, we omit latency and throughput results in this paper (see [16] for details).

In the limit of large block length, the error events are typically dominated by the outage events [3], [4], for which only the last round outage events contribute to the overall outage events whenever the feedback is perfect. With imperfect feedback, however, the outage events happening for rounds less than L may contribute to the overall outage events. Using (16) and (18), we can show the achievability of the generalised outage probability, defined as

$$P_{\mathrm{gout}}(R_1) \triangleq P_{\mathrm{gout}}^{L}(R_1) + \sum_{\ell=1}^{L-1} p_{\mathrm{fb}} P_{\mathrm{gout}}^{\ell}(R_1). \quad (22)$$

V. Generalised Outage Diversity

The high SNR behaviour of the generalised outage probability of the INR-ARQ coding system under consideration is characterised by the generalised outage diversity.

Definition 1 (Generalised outage diversity at round ℓ): Generalised outage diversity or generalised outage SNR exponent for ARQ round ℓ is defined as

$$d^\ell \triangleq -\frac{\log P_{\mathrm{gout}}^{\ell}(R_1)}{\log \mathsf{snr}}. \quad (23)$$

The rate-diversity-delay trade-off for ARQ scheme with perfect CSIR and perfect feedback is outlined as follows.

Lemma 1 (Perfect CSIR, Perfect Feedback, [3]–[5]): Consider a MIMO Rayleigh block-fading channel (1) with perfect CSIR and perfect feedback ($p_{\mathrm{fb}} = 0$). The optimal rate-diversity-delay trade-off is given by the SNR exponents

$$d_{\mathrm{arq}} = d_{\mathrm{csir}}^{L} = \begin{cases} LBn_{\mathrm{t}}n_{\mathrm{r}} & \text{for Gaussian inputs,} \\ n_{\mathrm{r}}\left(1+\left\lfloor LB\left(n_{\mathrm{t}}-\frac{R_1}{LM}\right)\right\rfloor\right) & \text{for discrete inputs,}\end{cases} \quad (24)$$

where $M \triangleq \log_2 |\mathcal{X}|$. Furthermore, random codes with Gaussian and discrete signal sets achieve those SNR exponents.

In the case of imperfect CSIR and imperfect BSC feedback, we have the following results.

Theorem 1 (Imperfect CSIR, Imperfect Feedback): Consider a MIMO Rayleigh block-fading channel (1), with imperfect CSIR (6) and imperfect feedback (12). Then, the optimal SNR exponents are given by

$$d_{\mathrm{arq}} = \min\left(d_{\mathrm{icsir}}^{L}, d_{\mathrm{fb}} + d_{\mathrm{icsir}}^{1}\right) \tag{25}$$

where for $\ell = 1, \ldots, L$

$$d_{\mathrm{icsir}}^{\ell} = \min(1, d_e) \times d_{\mathrm{csir}}^{\ell}, \tag{26}$$

$$d_{\mathrm{csir}}^{\ell} = \begin{cases} \ell B n_{\mathrm{t}} n_{\mathrm{r}} & \text{for Gaussian inputs,} \\ n_{\mathrm{r}}\left(1 + \left\lfloor \ell B\left(n_{\mathrm{t}} - \frac{R_1}{\ell M}\right)\right\rfloor\right) & \text{for discrete inputs,} \end{cases} \tag{27}$$

are the diversities for imperfect CSIR and perfect CSIR associated with round ℓ, respectively. The achievability of d_{arq} is shown using random coding.

Proof: The proof is based on the error probability expression in (21). The converse proof follows from the GMI converse for i.i.d. codebooks. For every ARQ round ℓ, $\ell = 1, \ldots, L$, the average error probability can be lower-bounded as [15]

$$\Pr\{\mathcal{E}_\ell\} \geq P_{\mathrm{gout}}^{\ell}(R). \tag{28}$$

Note that the error probability of the threshold decoder (10) can never be smaller than using the maximum metric decoding of (9). This is so because whenever the threshold detector accepts the message, the output is identical to that of maximum metric decoding. Thus, we can lower-bound the error probability of each round using the generalised outage probability of that round. From union error events, we have relationship

$$\Pr\{\mathcal{E}_\ell\} = \Pr\{\mathcal{D}_\ell\} + \Pr\{\mathcal{D}_\ell^c\}. \tag{29}$$

For the converse, we assume perfect error detection; thus, $\Pr\{\mathcal{D}_\ell^c\} = 0$. Following these, we can bound (21) as

$$P_e = P_L^{\mathrm{NN}} + \sum_{\ell=1}^{L-1} p_{\mathrm{fb}} \Pr\{\mathcal{E}_\ell\} \tag{30}$$

$$\geq P_{\mathrm{gout}}^{L}(R) + \sum_{\ell=1}^{L-1} p_{\mathrm{fb}} P_{\mathrm{gout}}^{\ell}(R) \tag{31}$$

$$\doteq \mathsf{snr}^{-d_{\mathrm{icsir}}^{L}} + \mathsf{snr}^{-d_{\mathrm{fb}}} \sum_{\ell=1}^{L-1} \mathsf{snr}^{-d_{\mathrm{icsir}}^{\ell}}. \tag{32}$$

The slowest decaying slope in the sum for the last equation is given by d_{icsir}^{1}. Using the techniques in [15] we obtain that the optimal SNR exponent is upper-bounded by (25).

The achievability for round $\ell = 1, \ldots, L-1$ is proven by using random coding ensembles of i.i.d. inputs, letting $T \to \infty$ and averaging the right-hand side of (16) over all fading and channel estimation error realisations. The achievability for the last round is proven in [15]. ∎

An immediate consequence of Theorem 1 yields the imperfect-CSIR perfect-feedback case.

Corollary 1 (Imperfect CSIR, Perfect Feedback): With perfect feedback, the ARQ system diversity is given by

$$d_{\mathrm{arq}} = d_{\mathrm{icsir}}^{L}. \tag{33}$$

Proof: Set p_{fb} to zero ($d_{\mathrm{fb}} \to \infty$) in Theorem 1. ∎

The performance of ARQ systems is therefore determined by the feedback diversity d_{fb}. The full ARQ diversity is obtained with $d_e \geq 1$ and $d_{\mathrm{fb}} \to \infty$. The first requirement of $d_e \geq 1$ is highlighted in the results of [15]. On the other hand, the second requirement is seen from Corollary 1. Note that the second requirement can be relaxed as long as we have

$$d_{\mathrm{icsir}}^{L} \leq d_{\mathrm{fb}} + d_{\mathrm{icsir}}^{1}. \tag{34}$$

Thus, the minimum d_{fb} to achieve the full ARQ diversity is

$$d_{\mathrm{fb}}^{\star} = d_{\mathrm{icsir}}^{L} - d_{\mathrm{icsir}}^{1}. \tag{35}$$

As L increases, the minimum d_{fb} required to achieve the full diversity is higher. Thus, $d_{\mathrm{fb}}^{\star}$ is an increasing function of L. Therefore, the ARQ diversity gain is achievable with high-diversity feedback links. Perfect feedback is just a special case of high diversity with $d_{\mathrm{fb}} \to \infty$. In practice, however, link reliability depends on the transmission scheme used and on the available resources such as bandwidth and power.

As the feedback diversity gets lower, ARQ may not improve the overall system diversity. In fact, for zero feedback-diversity we have the following result.

Corollary 2 (Fixed crossover probability, p_{fb}): In the case of fixed crossover probability independent of snr, the ARQ system diversity is given by

$$d_{\mathrm{arq}} = d_{\mathrm{icsir}}^{1}. \tag{36}$$

Proof: Set $d_{\mathrm{fb}} = 0$ in Theorem 1. ∎

Corollary 2 suggests that with fixed crossover probability independent of snr, ARQ is useless in terms of diversity improvement when compared to the non-ARQ case.

VI. Numerical Results and Discussion

Fig. 1 shows the results for the generalised outage probability for Gaussian inputs. In particular, we choose a reliable channel estimator with $d_e = 1$ and evaluate the different outage curves for different BSC feedbacks. ARQ scheme is useless to improve the diversity if the crossover probability is fixed. It has large SNR slope which is same as those conventional transmission without ARQ ($L = 1$). This implies that any BSC feedback must improve with the forward SNR. We have shown in the curve that as long as d_{fb} satisfies (35), then the full ARQ diversity is achievable. Otherwise, the slope is given by the sum of the feedback diversity and the ARQ diversity for the first round.

Fig. 2 shows clearly the diversity penalty due to imperfect feedback for discrete inputs. With $d_{\mathrm{fb}} = 0$, the system is unable to utilise the benefits of ARQ in improving diversity. As d_{fb} increases the diversity performance improves. High feedback diversity is required to fully utilise the ARQ scheme, especially when L is large. For a given d_{fb}, the full ARQ diversity is typically achieved when R_1 is sufficiently large.

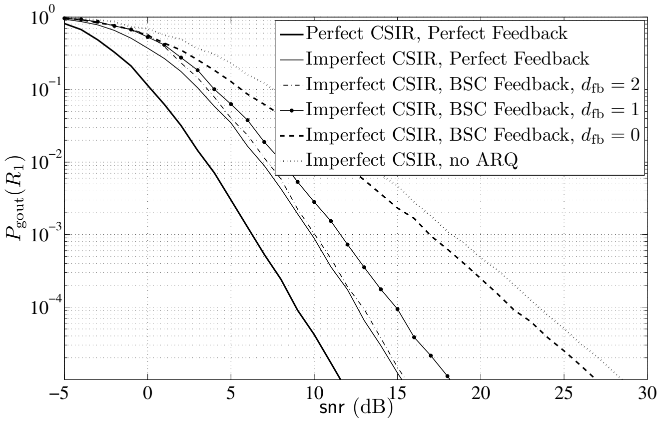

Fig. 1. Generalised outage probability for Gaussian inputs transmitted over a single-input single-output (SISO) Rayleigh block-fading channel, $B = 2$, $L = 2$, $R_1 = 1$ bit/channel use and $p_0 = 0.5$ with various BSC feedback mechanisms.

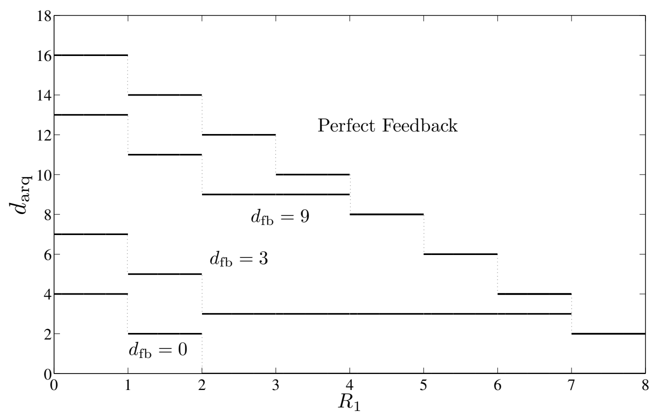

Fig. 2. Generalised outage diversity for BPSK inputs MIMO Rayleigh block-fading channel, $B = 1$, $L = 4$, $n_{\mathrm{t}} = 2$ and $n_{\mathrm{r}} = 2$, with various BSC feedback mechanisms.

The analysis on simple BSC feedback gives insights on other forms of feedback. The diversity performance is closely associated with p_{fb}, particularly due to transmitter's failure to decode the correct negative acknowledgement. Thus, the diversity results are applicable for feedback in the form of binary erasure channel (BEC) with erasure probability p_{fb} and of Z-channel with flip probability p_{fb}. The results also provide some insights on the imperfect feedback due to fading. For example, if the reverse SNR does not improve with the forward SNR, then it has the same phenomenon as when p_{fb} is an SNR-independent constant because the probability of the transmitter's failure to decode the correct negative acknowledgement is fixed. If the reverse SNR improves with the forward SNR, then the ARQ scheme may gain some diversity and the gain is indicated by d_{fb} which is determined by the fading type and the transmission scheme used to transmit the acknowledgement signals.

VII. Conclusions

We have analysed the performance of INR-ARQ over MIMO block-fading channels with imperfect CSIR and BSC feedback. Specifically, we have characterised the diversity penalty caused by imperfect feedback. Our results suggest that the feedback SNR must improve with the forward SNR in order for ARQ to be able to exploit the available diversity; otherwise INR-ARQ is not able to improve the diversity. In particular, we derive the condition for which full ARQ diversity may be exploited. We also learn that in order to achieve the full diversity, the required feedback transmission must provide an additional diversity which is linearly increasing with the maximum number of ARQ rounds.

Acknowledgments

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