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Outage Probability of the Free-Space Optical Channel with Doubly Stochastic Scintillation

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Abstract

We study the asymptotic outage probability of multiple-input multiple-output free-space optical communication with pulse-position modulation. In particular, we consider doubly stochastic scintillation models, lognormal-Rice and I-K distributions. First we consider the case when channel state information is available at the receiver only. Then we consider the case when it is also available at the transmitter.

1. Introduction

Free-space optical (FSO) communication offers an attractive alternative to radio frequency (RF) for transmitting data at very high rates. The main drawback of the FSO channel is the detrimental effect the atmosphere has on the propagating laser beam. One such effect is *scintillation*, caused by atmospheric turbulence, and refers to random fluctuations in the irradiance of the received optical laser beam (analogous to fading in RF systems) [1]. Compared to typical signalling rates, scintillation is a slow time-varying process with coherence time on the order of tens of milliseconds. Long deep fades can result in the loss of millions of consecutively transmitted data bits, resulting in system outage. The use of multiple lasers and multiple apertures, creating the well-known multiple-input multiple-output (MIMO) channel, has shown to significantly reduce the effects of scintillation (see [2] and references therein). Of direct relevance to this paper is our previous work in [3], which analysed the asymptotic behaviour of the outage probability of the MIMO FSO channel under the assumptions of pulse-position modulation (PPM) and non-ideal photodetection, where the combined shot and thermal noise processes are considered as additive white Gaussian noise (AWGN). This analysis considered three scintillation distributions: lognormal, exponential and gamma-gamma distributions. The first two model weak and strong turbulence conditions respectively [1]. The gamma-gamma distribution attempts to model scintillation over all turbulence conditions [4], for which a number of other distributions have also been proposed, including the lognormal-Rice [5] and I-K [6] distributions. These universal models are all based on the heuristic argument that scintillation is a *doubly stochastic* random process modelling small and large scale turbulence effects. All three have shown good agreement with experimental results [4–6].

In this paper we extend the analysis in [3] to doubly stochastic scintillation models, lognormal-Rice and I-K distributions. We derive signal-to-noise ratio (SNR) exponents when perfect channel state information (CSI) is known only at the receiver (CSIR case). Then we examine the case when perfect CSI is also known at the transmitter (CSIT) and power is optimally allocated subject to short- and long-term power constraints.

The paper is organised as follows. In Section 2, we define the channel model and assumptions. In Section 3 we review the lognormal-Rice and I-K scintillation models. Then in Sections 4 we present the main results of our asymptotic outage probability analysis. Concluding remarks are then given in Section 5.

2. System Model

We consider an $M \times N$ MIMO FSO system with M transmit lasers and an N aperture receiver. Information data is first encoded by a binary code of rate R_c. The encoded stream is modulated according to a Q-ary PPM scheme, resulting in rate $R = R_c \log_2 Q$ (bits/channel use). Repetition transmission is employed such that the same PPM signal is transmitted in perfect synchronism by each of the M lasers through an atmospheric turbulent channel and collected by N receive apertures. We assume the distance between the individual lasers and apertures is sufficient so that spatial correlation is negligible. At each aperture, the received optical signal is converted to an electrical signal via photodetection. Non-ideal photodetection is assumed such that the combined shot noise and thermal noise processes can be modelled as zero mean, signal independent AWGN (an assumption commonly used in the literature, see e.g. [2, 7]). We model the channel as a non-ergodic *block-fading channel* [8, 9] such that the received signal at aperture n, $n = 1, \dots, N$ is

$$\boldsymbol{y}_b^n[\ell] = \left(\sum_{m=1}^{M} \tilde{h}_b^{m,n} \right) \sqrt{\tilde{p}_b}\, \boldsymbol{x}_b[\ell] + \tilde{\boldsymbol{z}}_b^n[\ell], \qquad (1)$$

for $b = 1, \dots, B, \ell = 1, \dots, L$, where $\boldsymbol{y}_b^n[\ell], \tilde{\boldsymbol{z}}_b^n[\ell] \in \mathbb{R}^Q$ are the received and noise signals at block b, time instant ℓ and aperture n, $\boldsymbol{x}_b[\ell], \in \mathbb{R}^Q$ is the transmitted signal at block b and time instant ℓ, and $\tilde{h}_b^{m,n}$ denotes the scintillation fading coefficient between laser m and aperture n. Each transmitted symbol is drawn from a PPM alphabet, $\boldsymbol{x}_b[\ell] \in \mathcal{X}^{\text{ppm}} \triangleq \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_Q\}$, where $\boldsymbol{e}_q$ is the canonical basis vector, i.e., it has all zeros except for a one in position q, the time slot where the pulse is transmitted. The noise samples of $\tilde{\boldsymbol{z}}_b^n[\ell]$ are independent realisations of a random variable $Z \sim \mathcal{N}(0, 1)$, and $\tilde{p}_b$ denotes the received electrical power of block b at each aperture in the absence of scintillation. The fading coefficients $\tilde{h}_b^{m,n}$ are independent realisations of a random variable $\tilde{H}$ with probability density function (pdf) $f_{\tilde{H}}(h)$. Equal gain combining (EGC) is assumed, such that the entire system is equivalent to a single-input single-output (SISO) channel, i.e.

$$\boldsymbol{y}_b[\ell] = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \boldsymbol{y}_b^n[\ell] = \sqrt{p_b} h_b \boldsymbol{x}_b[\ell] + \boldsymbol{z}_b[\ell], \qquad (2)$$

where $\boldsymbol{z}_b[\ell] = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \tilde{\boldsymbol{z}}_b^n[\ell] \sim \mathcal{N}(0, 1)$, and h_b, a realisation of the random variable H, is defined as the normalised combined fading coefficient, i.e.

$$h_b = \frac{c}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \tilde{h}_b^{m,n}, \qquad (3)$$

where $c = 1/(\mathbb{E}[\tilde{H}]\sqrt{1 + \sigma_I^2/(MN)})$ is a constant to ensure $\mathbb{E}[H^2] = 1$, and $\sigma_I^2 \triangleq \mathbb{E}[\tilde{H}^2]/(\mathbb{E}[\tilde{H}])^2 - 1$ is defined as the *scintillation index*. Thus, the total instantaneous received power at block b is $p_b = M^2 N \tilde{p}_b / c$, and the total average received SNR is $\mathsf{snr} \triangleq \mathbb{E}[h_b^2 p_b] = \mathbb{E}[p_b]$.

3. Scintillation Distributions

The distribution of the irradiance fluctuations brought on by optical scintillation has been studied extensively in the literature (see [1] and references therein). For the weak turbulence and strong turbulence regimes, they are well modelled by the lognormal and exponential distributions respectively. In between these two extremes, the fluctuations observed in practise deviate much from these models, and a vast number of distributions have been proposed that attempt to model the fluctuations over much wider set of atmospheric turbulence conditions. Three of the most promising distributions include the I-K [6], lognormal-Rice[5], and gamma-gamma [4] distributions. All three are base on doubly stochastic models, where the irradiance fluctuations are comprised of two independent random processes modelling the effects of small- and large-scale eddy cells. In [3], we analysed the outage probability of (1) for the gamma-gamma case, along with the lognormal and exponential cases. In this paper we consider lognormal-Rice and I-K distributed scintillation. Like the gamma-gamma case analysed in [3], they are doubly stochastic, yet as we shall see, both yield much different asymptotic outage behaviours.

The lognormal-Rice distribution arises from the product of two independent random variables, i.e. $\tilde{H} = XY$, where X is lognormal and $\sqrt{Y}$ is Rice distributed. The resulting pdf of $\tilde{H}$ can be written in integral form [5]

$$f_{\tilde{H}}(h) = \frac{(1+r)e^{-r}}{\sqrt{2\pi}\sigma} \int_0^\infty \frac{1}{z^2} I_0\left(2\sqrt{\frac{(1+r)rh}{z}}\right) \exp\left(-\frac{(1+r)h}{z} - \frac{1}{2\sigma^2}\left(\log z + \frac{1}{2}\sigma^2\right)^2\right) dz \qquad (4)$$

where r is referred to as the *coherence parameter* [5]. The lognormal-Rice includes the lognormal, lognormal-exponential and exponential distributions as special cases.

The I-K distribution arises from a compound statistical model whereby the conditional irradiance distribution is assumed to be the modified Rice-Nakagami pdf. When averaged over gamma statistics, the unconditional pdf has the form [6],

$$f_{\tilde{H}}(h) = \begin{cases} 2\alpha(1+\rho)\left[\frac{(1+\rho)h}{\rho}\right]^{\frac{\alpha-1}{2}} \mathrm{K}_{\alpha-1}(2\sqrt{\alpha\rho}) \times \mathrm{I}_{\alpha-1}(2\sqrt{\alpha h(1+\rho)}) & h < \frac{\rho}{1+\rho} \\ 2\alpha(1+\rho)\left[\frac{(1+\rho)h}{\rho}\right]^{\frac{\alpha-1}{2}} \mathrm{I}_{\alpha-1}(2\sqrt{\alpha\rho}) \times \mathrm{K}_{\alpha-1}(2\sqrt{\alpha h(1+\rho)}) & h > \frac{\rho}{1+\rho} \end{cases} \qquad (5)$$

where $I_\nu(z)$ and $\mathrm{K}_\nu(z)$ denote the modified Bessel function of the first and second kind [10, p.374] respectively, α is the effective number of scatters, and ρ is also referred to as the coherence parameter. The I-K distribution contains the exponential and K distributions as special cases.

4. Main Results

The channel described by (2) under the quasi-static assumption is not information stable [11] and therefore, the channel capacity in the strict Shannon sense is zero. The codeword error probability of any coding scheme can be lower bounded by the information outage probability [8, 9],

$$P_{\text{out}}(\mathsf{snr}, R) = \Pr(I(\boldsymbol{p}, \boldsymbol{h}) < R), \tag{6}$$

where R is the transmission rate and [12],

$$I(\boldsymbol{p}, \boldsymbol{h}) = \frac{1}{B}\sum_{b=1}^{B} I^{\text{awgn}}(p_b h_b^2), \tag{7}$$

is the instantaneous mutual information for a given power allocation $\boldsymbol{p} = (p_1, \ldots, p_b)$ and vector channel realisation $\boldsymbol{h} \triangleq (h_1, \ldots, h_B)$. We denote $I^{\text{awgn}}(\gamma)$ as the mutual information for the AWGN channel with SNR γ, with PPM [7],

$$\begin{aligned} I^{\text{awgn}}(\gamma) &= \log_2 Q \\ &\quad - \mathbb{E}\left[\log_2\left(1 + \sum_{q=2}^{Q} e^{-\gamma + \sqrt{\gamma}(Z_q - Z_1)}\right)\right], \end{aligned} \tag{8}$$

where $Z_q \sim \mathcal{N}(0, 1)$ for $q = 1, \ldots, Q$.

4.1. Outage Analysis: CSIR Case

For the CSIR case, we employ uniform power allocation, i.e. $p_1 = \ldots = p_B = \mathsf{snr}$. For codewords transmitted over B blocks, obtaining a closed form analytic expression for the outage probability is intractable. Even for $B = 1$, the complicated pdfs (4) and (5) prohibit us from determining the exact distribution of H and therefore the outage probability. Instead, as we shall see, obtaining the asymptotic behaviour of the outage probability is substantially simpler. Towards this end, and following the footsteps of [13, 14], we derive the *SNR exponent*, defined as

$$d_{(\log \mathsf{snr})} \triangleq -\lim_{\mathsf{snr}\to\infty} \frac{\log P_{\text{out}}(\mathsf{snr}, R)}{\log \mathsf{snr}}. \tag{9}$$

Theorem 4.1. *The optimal SNR exponent for a MIMO FSO communications system modelled by* (2) *with lognormal-Rice and I-K scintillation is respectively given by*

$$d^{\text{LN}-\text{R}}_{(\log \mathsf{snr})} = \frac{MN}{2}\left(1 + \lfloor B\,(1 - R_c)\rfloor\right) \tag{10}$$

$$d^{\text{I}-\text{K}}_{(\log \mathsf{snr})} = \frac{MN}{2}\alpha\left(1 + \lfloor B\,(1 - R_c)\rfloor\right) \tag{11}$$

where $R_c = R/\log_2(Q)$ is the rate of the binary code.

Proof. See the appendix. □

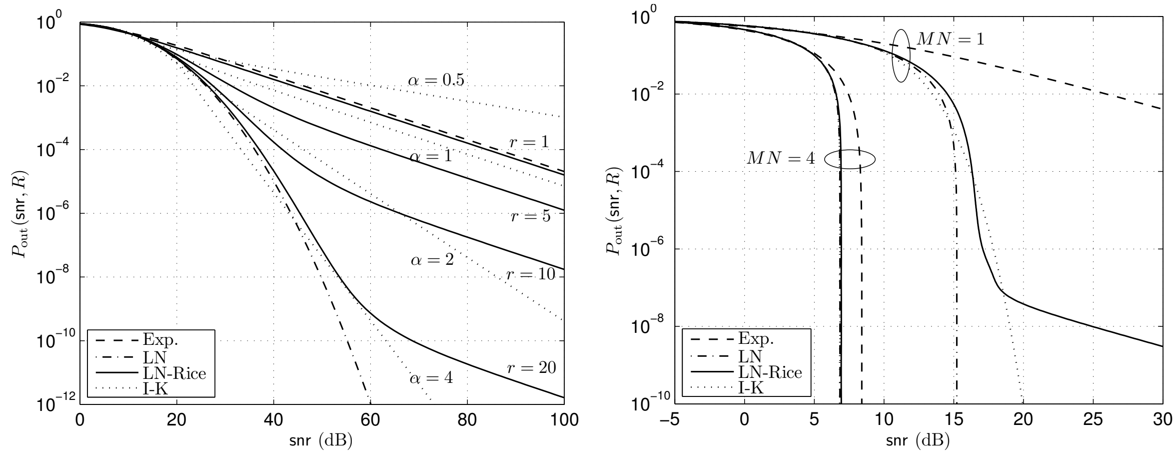
From (10) and (11) we immediately see the benefits of spatial and block diversity on the system. In particular, each exponent is proportional to: the number of lasers times the number of apertures, reflecting the spatial diversity; a channel related parameter that is dependent on the scintillation distribution; and the Singleton bound, which is the optimal rate-diversity tradeoff for Rayleigh-faded block fading channels [14–16]. Interestingly, we see that both exponents are independent of their respective coherence parameters r and ρ. For the lognormal-Rice case, the exponent is the same as the asymptotic strong turbulence case, corresponding to exponential distributed scintillation (derived in [3]). The lognormal component does not influence the asymptotic slope of the outage probability curve. Only the Rice component, believed to be caused by small eddy cells [5], affects the SNR exponent. The Rice component introduces an error floor with the same slope as the exponential case. The SNR at which the error floor begins to dominate depends on the coherence parameter r (see Fig. 1(a)). For small r, the floor totally dominates performance, and the outage behaviour is much like the exponential case. As r increases, the floor dominates at increasingly high SNR values and at low SNR the outage behaves much like the pure lognormal case (analysed in [3]). The exponent of the I-K case shows much different behaviour to the lognormal-Rice. Here, the exponent is proportional to α, which corresponds to the effective number of scatterers [6]. Thus the more scatterers the steeper the outage probability curve (see Fig. 1(a)).

4.2. Outage Analysis: CSIT Case

For the CSIT case, the transmitter finds the optimal power allocation that minimises the outage probability subject to short- and long-term power constraints, i.e. $\frac{1}{B}\sum_{b=1}^{B} p_b \leq P$ and $\mathbb{E}\left[\frac{1}{B}\sum_{b=1}^{B} p_b\right] \leq P$ respectively. In particular, we use results from [17], as done in [3]. The SNR exponent under a short-term constraint is the same as the CSIR case [17]. For the long-term power constraint, the SNR exponent is given by [17]

$$d^{\text{lt}}_{(\log \mathsf{snr})} = \begin{cases} \frac{d^{\text{st}}_{(\log \mathsf{snr})}}{1 - d^{\text{st}}_{(\log \mathsf{snr})}} & d^{\text{st}}_{(\log \mathsf{snr})} < 1 \\ \infty & d^{\text{st}}_{(\log \mathsf{snr})} > 1 \end{cases}, \tag{12}$$

where $d^{\text{st}}_{(\log \mathsf{snr})}$ is the short-term SNR exponent. Note that $d^{\text{lt}}_{(\log \mathsf{snr})} = \infty$ implies the outage probability curve is vertical, i.e. for a given SNR, the power control strategy is able to maintain constant instantaneous mutual information equal to R bits per channel use.



(a) CSIR

(b) CSIT

Figure 1: Outage probability for lognormal-Rice (solid), exponential (dashed), lognormal (dot-dashed) and I-K scintillation (dotted) with $Q = 2$, $R_c = 1/2$ and $\sigma_I^2 = 1$. The left plot shows the CSIR case with $M = N = B = 1$. The right plot shows the CSIT case for $B = 1$, $MN = 1$ and $MN = 4$.

We define the maximum achievable rate at which this occurs as the *delay-limited capacity* [18]. Since the lognormal-Rice has the same SNR exponent as the exponential case, it follows from [3, Corollary 6.1] that the delay-limited capacity is positive only when $MN > 2(1 + \lfloor B(1 - R_c)\rfloor)^{-1}$. Whereas for I-K scintillation, we require $MN > 2\alpha^{-1}(1 + \lfloor B(1 - R_c)\rfloor)^{-1}$. Otherwise delay-limited capacity is zero, i.e. there exists no threshold SNR at which $P_{\rm out} \to 0$.

For the special case of single block transmission ($B = 1$), in [3] it was shown that

$$P_{\rm out}(\mathsf{snr}, R) = F_H\left(\left(\psi^{-1}(\mathsf{snr})\right)^{-\frac{1}{2}}\right), \qquad (13)$$

where $\psi^{-1}(u)$ is the inverse function of

$$\psi(s) \triangleq \int_{\nu}^{\infty} h^{-2} f_H(h)\, dh, \qquad (14)$$

where $\nu = \sqrt{\frac{\mathsf{snr}_R^{\rm awgn}}{s}}$ and $\mathsf{snr}_R^{\rm awgn}$ is the required SNR to achieve rate R for the non-fading PPM channel (8).

Fig. 1(b) illustrates the outage behaviour of the $B = 1$ CSIT case under the long-term power constraint (13). We see that with $M = N = 1$, the delay-limited capacity is zero for the lognormal-Rice case, as it has an error floor with the same slope as the exponential case. For the I-K case with $\alpha = 2$ and $M = N = 1$, delay-limited capacity is also zero (since we require $MN > 1$ in this case). Increasing MN to 4, we see that the outage curves are vertical in all cases, as predicted by our analysis.

5. Conclusion

In this paper we analysed the asymptotic outage probability behaviour of the FSO MIMO channel under the assumption of PPM in the presence of lognormal-Rice and I-K distributed scintillation, both of which are doubly stochastic. For the case when CSIR is known at the receiver, we showed that the SNR exponent for the lognormal-Rice case is the same as the exponential case (which corresponds to very strong turbulence). For the I-K case, the SNR exponent is proportional to the effective number of scatters α. Both of these results show much different behaviour to the gamma-gamma distribution analysed in [3], which illustrates the importance of correct modelling using these universal scintillation distributions. Choosing the incorrect model can lead to a vastly different outage probability characterisation.

Appendix: Proof of Theorem 4.1

I-K distribution

We begin by defining a normalised (with respect to SNR) fading coefficient, $\zeta_b^{m,n} = -\frac{2\log \tilde{h}_b^{m,n}}{\log \mathsf{snr}}$, which has a pdf

$$f_{\zeta_b^{m,n}}(\zeta) = \frac{\log \mathsf{snr}}{2} e^{-\frac{1}{2}\zeta \log \mathsf{snr}} f_{\tilde{H}}\left(e^{-\frac{1}{2}\zeta \log \mathsf{snr}}\right). \qquad (15)$$

The instantaneous SNR for block b is given by

$$\gamma_b = \mathsf{snr} h_b^2 = \left(\frac{c}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} \mathsf{snr}^{\frac{1}{2}\left(1-\zeta_b^{m,n}\right)}\right)^2 \qquad (16)$$

for $b = 1, \ldots, B$. Therefore,

$$\lim_{\mathsf{snr}\to\infty} I^{\mathrm{awgn}}(\gamma_b) = \log_2 Q\left(1 - \mathbb{1}\{\boldsymbol{\zeta}_b \succ \mathbf{1}\}\right)$$

where $\boldsymbol{\zeta}_b \triangleq (\zeta_b^{1,1}, \ldots, \zeta_b^{M,N})$, $\mathbb{1}\{\cdot\}$ denotes the indicator function, $\mathbf{1} \triangleq (1, \ldots, 1)$ is a $1 \times MN$ vector of 1's, and the notation $\boldsymbol{a} \succ \boldsymbol{b}$ for vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^k$ means that $a_i > b_i$ for $i = 1, \ldots, k$.

From the definition of outage probability (6),

$$P_{\mathrm{out}}(\mathsf{snr}, R) = \Pr(I_{\boldsymbol{h}}(\mathsf{snr}) < R) = \int_{\mathcal{A}} f(\boldsymbol{\zeta}) d\boldsymbol{\zeta} \tag{17}$$

where $\boldsymbol{\zeta} \triangleq (\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_B)$ is a $1 \times BMN$ vector of normalised fading coefficients, $f(\boldsymbol{\zeta})$ denotes their joint pdf, and

$$\mathcal{A} = \left\{ \boldsymbol{\zeta} \in \mathbb{R}^{BMN} : \sum_{b=1}^{B} \mathbb{1}\{\boldsymbol{\zeta}_b \succ \mathbf{1}\} > B\left(1 - R_c\right) \right\} \tag{18}$$

is the asymptotic outage set. We now compute the SNR exponent as

$$d_{(\log \mathsf{snr})} = \lim_{\mathsf{snr}\to\infty} -\frac{\log \int_{\mathcal{A}} f(\boldsymbol{\zeta})\, d\boldsymbol{\zeta}}{\log \mathsf{snr}}. \tag{19}$$

For the I-K distribution (5), as $\mathsf{snr} \to \infty$, we need only consider the case when $h < \rho/(1+\rho)$. Thus as $\mathsf{snr} \to \infty$, from (5) and (15), and using $I_\nu(z) \approx \left(\frac{z}{2}\right)^\nu / \Gamma(\nu+1)$ for small z [10, 9.6.7], we have

$$f_{\zeta_b^{m,n}}(\zeta) \doteq \exp\left(-\frac{\alpha}{2}\zeta \log \mathsf{snr}\right), \tag{20}$$

and the joint density

$$f(\boldsymbol{\zeta}) \doteq \exp\left(-\frac{\alpha}{2} \log \mathsf{snr} \sum_{b=1}^{B}\sum_{m=1}^{M}\sum_{n=1}^{N} \zeta_b^{m,n}\right). \tag{21}$$

Thus, combining (21) and (19) and using Varadhan's lemma [19] we have

$$d_{(\log \mathsf{snr})}^{\mathrm{I-K}} = \inf_{\mathcal{A}} \left\{ \frac{\alpha}{2} \sum_{b=1}^{B}\sum_{m=1}^{M}\sum_{n=1}^{N} \zeta_b^{m,n} \right\} \tag{22}$$

To solve the above infimum, we chose any $\kappa = 1 + \lfloor B(1 - R_c) \rfloor$ of the $\boldsymbol{\zeta}_b$ such that $\boldsymbol{\zeta}_b = \mathbf{1}$ and the other $B - \kappa$ vectors such that $\zeta_b = \mathbf{0}$ (where $\mathbf{0}$ denotes a $1 \times MN$ vector of all zeros). Hence we obtain (11) as stated in the theorem.

Lognormal-Rice distribution

For the lognormal-Rice case, rather than dealing with the integral representation of its distribution (4), we use the fact that a lognormal-Rice random variable can be expressed as the product of two independent random variables, each with a closed form distribution. Thus, we write $\tilde{h}_b^{m,n} = x_b^{m,n} y_b^{m,n}$, where $x_b^{m,n}$ and $y_b^{m,n}$ are lognormal and Rice distributed random variables respectively, i.e.

$$f_{x_b^{m,n}}(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}\left(\log x + \frac{\sigma^2}{2}\right)^2\right) \tag{23}$$

$$f_{y_b^{m,n}}(y) = (r+1)e^{(-r-(r+1)y)} I_0\left(\sqrt{4r(r+1)y}\right) \tag{24}$$

Now define normalised fading coefficients, $\zeta_b^{m,n}$ and $\xi_b^{m,n}$, where $\zeta_b^{m,n} = -\frac{2\log x_b^{m,n}}{\log \mathsf{snr}}$ and $\xi_b^{m,n} = -\frac{2\log y_b^{m,n}}{\log \mathsf{snr}}$. Thus the instantaneous SNR is,

$$\gamma_b = \frac{c}{MN} \sum_{m=1}^{M}\sum_{n=1}^{N} \mathsf{snr}^{1-\frac{1}{2}(\zeta_b^{m,n}+\xi_b^{m,n})}. \tag{25}$$

Hence,

$$\lim_{\mathsf{snr}\to\infty} I^{\mathrm{awgn}}(\gamma_b) = \log_2 Q\left(1 - \mathbb{1}\left\{\frac{1}{2}(\boldsymbol{\zeta}_b + \boldsymbol{\xi}_b) \succ \mathbf{1}\right\}\right). \tag{26}$$

Therefore the outage probability is

$$P_{\mathrm{out}}(\mathsf{snr}, R) = \Pr(I_{\boldsymbol{h}}(\mathsf{snr}) < R) = \int_{\mathcal{A}} f(\boldsymbol{\zeta}, \boldsymbol{\xi})\, d\boldsymbol{\zeta} d\boldsymbol{\xi}, \tag{27}$$

where the asymptotic outage set is

$$\mathcal{A} \triangleq \Bigg\{ \boldsymbol{\zeta}, \boldsymbol{\xi} \in \mathbb{R}^{BMN} : \sum_{b=1}^{B} \mathbb{1}\left\{\frac{1}{2}(\boldsymbol{\zeta}_b + \boldsymbol{\xi}_b) \succ \mathbf{1}\right\} > B(1 - R_c) \Bigg\}. \tag{28}$$

From (23) and (24) we have

$$f_{\zeta_b^{m,n}}(\zeta) \doteq \exp\left(-\frac{1}{8\sigma^2}(\log \mathsf{snr})^2 \zeta^2 + \frac{1}{2}\zeta \log \mathsf{snr}\right) \tag{29}$$

$$f_{\xi_b^{m,n}}(\xi) \doteq \exp\left(-\frac{1}{2}\xi \log \mathsf{snr}\right). \tag{30}$$

Hence we have the joint distribution,

$$f(\boldsymbol{\zeta}, \boldsymbol{\xi}) \doteq \exp\Bigg(-\frac{1}{8\sigma^2}(\log \mathsf{snr})^2 \sum_{b=1}^{B}\sum_{m=1}^{M}\sum_{n=1}^{N} (\zeta_b^{m,n})^2 - \frac{1}{2}\log \mathsf{snr} \sum_{b=1}^{B}\sum_{m=1}^{M}\sum_{n=1}^{N} (\xi_b^{m,n} - \zeta_b^{m,n})\Bigg). \tag{31}$$

Now, using Varadhan's lemma [19] we have

$$d_{(\log \mathsf{snr})}^{\mathrm{LN-R}} = \frac{1}{2} \inf_{\mathcal{A}} \left\{ \frac{1}{4\sigma^2} \log \mathsf{snr} \sum_{b=1}^{B} \sum_{m=1}^{M} \sum_{n=1}^{N} (\zeta_b^{m,n})^2 + \sum_{b=1}^{B} \sum_{m=1}^{M} \sum_{n=1}^{N} (\xi_b^{m,n} - \zeta_b^{m,n}) \right\}, \quad (32)$$

Assuming $\sigma^2 < \infty$, immediately we see that the above infimum is achieved by setting $\boldsymbol{\zeta}_b = \mathbf{0}$ for all $b = 1, \ldots, B$, and the $\boldsymbol{\xi}_b$ vectors as in the I-K case case. Hence we obtain (10) as stated in the theorem.

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