Bit-Interleaved Coded Modulation in the Wideband Regime

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***Abstract*— This paper studies the wideband regime of bit-interleaved coded modulation (BICM) in Gaussian channels. Simple formulas for the minimum energy per bit and the wideband slope, both for coded modulation and for bit-interleaved coded modulation, are given. The wideband slope can be decomposed into the product of two terms, respectively due to the fading characteristics and the modulation and binary labeling rule. BICM is found to be suboptimal in the sense that its minimum energy per bit can be larger than the corresponding value for coded modulation schemes. The minimum energy per bit using standard Gray mapping on M-PAM, or M^2-QAM is given by a simple formula, and shown to approach -0.34 dB as M increases.**

I. Introduction and Motivation

Bit-interleaved coded modulation (BICM) was originally proposed in [1] and further elaborated in [2] as a practical way of constructing efficient coded modulation (CM) schemes over non-binary signal constellations. Reference [2] derived and computed the channel capacity of BICM under a low-complexity sub-optimal *non-iterative* decoder, and compared it to the CM channel capacity, assuming equiprobable signalling over the constellation. When plotted as a function of the signal-to-noise ratio (SNR), BICM capacity was observed to be near optimal when Gray mapping was used (see Fig. 1(a)). Plots as a function of the energy per bit for reliable communication (see Fig. 1(b)) reveal the suboptimality of BICM with the non-iterative decoder of [2] for low rates, or equivalently, in the power-limited regime. In this paper, we take a closer look at this regime and study the minimum energy per bit required for reliable communication as well as the wideband slope.

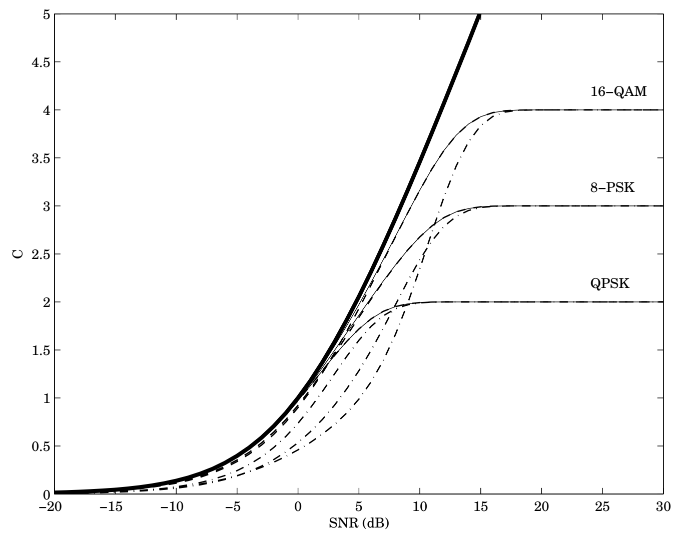
II. Model and Assumptions

We consider an additive Gaussian noise channel with fading, such that discrete-time baseband model is given by

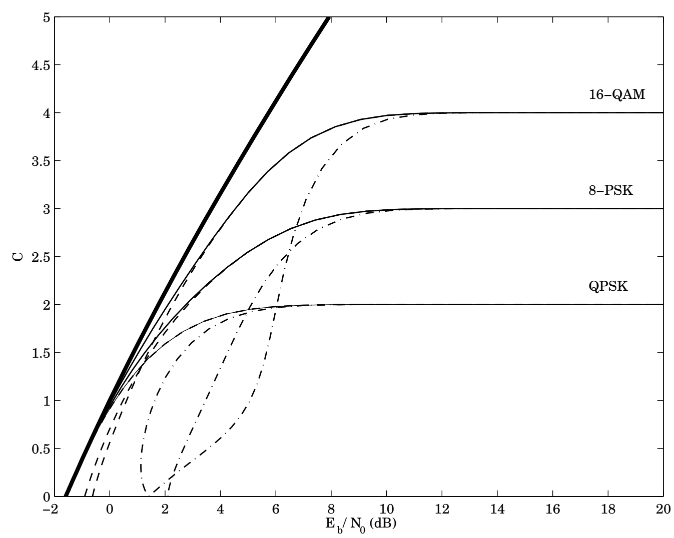
$$Z = H\sqrt{\mathrm{SNR}}\,X + W \tag{1}$$

where Z, X, W are the random variables denoting the received, transmitted and noise signals, and H denotes the fading random variable. We denote by $z_\ell, x_\ell, w_\ell \in \mathbf{C}$ the corresponding realizations at time $\ell = 1, \ldots, L$, where L is the duration of a codeword measured in channel uses. The fading

[1]The work by A. Guillén i Fàbregas work has been partly supported by the Australian Research Council under ARC grant DP0558861.



(a) Capacity as a function of SNR.



(b) Capacity as a function of $\frac{E_\mathrm{b}}{N_0}$.

Fig. 1. BICM channel capacity (in bits per channel use) in the AWGN channel. Gray and set partitioning labeling rules correspond to thin dotted and dashed-dotted lines respectively. Capacity with Gaussian inputs shown in thick solid lines, and CM channel capacity with thin solid lines.

realization at time ℓ is denoted by $h_\ell \in \mathbf{C}$, and is assumed to be perfectly known to the receiver.

We assume that the transmitted symbols x_ℓ belong to a zero-mean complex signal constellation $\mathcal{X} \subset \mathbf{C}$, of cardinality $|\mathcal{X}| = M = 2^m$, and average unit energy. We further assume that the noise $W \sim \mathcal{N}_{\mathbf{C}}(0,1)$ is an i.i.d. zero-mean unit-variance circularly symmetric Gaussian random variable. For a given fading realization $H = h_\ell$, the conditional output probability density $p(Z|X, H = h_\ell)$ is

$$p(Z|X, H = h_\ell) = \frac{1}{\pi}\exp\Bigl(\bigl|Z - h_\ell\sqrt{\mathrm{SNR}}\,X\bigr|^2\Bigr). \quad (2)$$

Furthermore, we assume fully-interleaved Nakagami-ν fading, namely, where the fading coefficients are i.i.d. random variables, whose squared modulus $\chi = |H|^2$ has a Gamma distribution with parameter ν,

$$p_\chi(\chi_\ell) = \frac{\nu^\nu \chi_\ell^{\nu-1}}{\Gamma(\nu)} e^{-\nu\chi_\ell}. \quad (3)$$

where $\Gamma(x) \triangleq \int_0^{+\infty} t^{x-1}e^{-t}dt$ is the Gamma function [3] and $\nu > 0$ [2]. The Nakagami fading subsumes a broad class of fading distributions. In particular, we recover the unfaded AWGN, Rayleigh fading and Rice fading (with factor K) channels by letting $\nu \to \infty$, $\nu = 1$ and $\nu = (K+1)^2/(2K+1)$, respectively [4]. Since $\mathrm{E}[|H|^2] = 1$ and signal and noise have average unit energy, SNR is the average SNR at the receiver.

III. Wideband Regime

In the wideband regime, as defined by Verdú in [5], the energy of a single bit is spread over many channel degrees of freedom, resulting in a low signal-to-noise ratio SNR. It is then convenient to study the asymptotic behavior of the spectral efficiency function as $\mathrm{SNR} \to 0$. In general, for an arbitrary input distribution, the mutual information (in bits), which we denote by C, admits an expansion in terms of SNR[3],

$$\mathrm{C}(\mathrm{SNR}) = \frac{1}{\log 2}\left(c_1\mathrm{SNR} + c_2\mathrm{SNR}^2 + \mathrm{O}\bigl(\mathrm{SNR}^3\bigr)\right). \quad (4)$$

where c_1 and c_2 depend on the modulation format, the receiver design, and the fading distribution. At a fixed capacity C, we define the bit-energy-to-noise ratio $\frac{E_\mathrm{b}}{N_0}$ such that

$$\mathrm{SNR} = \mathrm{C}\,\frac{E_\mathrm{b}}{N_0}. \quad (5)$$

Verdú [5] studied the transformation of expansion (4) into a function of $\frac{E_\mathrm{b}}{N_0}$, chiefly in a logarithmic scale, expressing $\frac{E_\mathrm{b}}{N_0}$ in decibels. We use linear scale rather than logarithmic, since it yields simpler formulas in the case of BICM. To any extent, both expressions are equivalent, as conclusions one can draw from either are identical. In linear scale, one obtains

[2]Typically $\nu > 0.5$, although it is not necessary, in the sense that the fading distribution is well defined and reliable communication is possible for any $0 < \nu \le 0.5$.

[3]The factor $\log 2$ dividing corresponds to the nats-bits conversion. We explicitly factor it out, as it will allow for more compact results. The constants c_1 and c_2 correspond to the first and second order terms of the capacity expansion in nats.

Proposition 1 *Consider the second order expansion of* C *given in* (4)*. Then* C *admits the the following first-order expansion in terms of* $\frac{E_\mathrm{b}}{N_0}$

$$\mathrm{C}\left(\frac{E_\mathrm{b}}{N_0}\right) = \zeta_0\left(\frac{E_\mathrm{b}}{N_0} - \frac{E_\mathrm{b}}{N_{0\,\mathrm{lim}}}\right) + \mathrm{O}\left(\left(\Delta\frac{E_\mathrm{b}}{N_0}\right)^2\right) \quad (6)$$

where $\Delta\frac{E_\mathrm{b}}{N_0} \triangleq \frac{E_\mathrm{b}}{N_0} - \frac{E_\mathrm{b}}{N_{0\,\mathrm{lim}}}$ *and*

$$\zeta_0 \triangleq -\frac{c_1^3}{c_2\log^2 2}, \qquad \frac{E_\mathrm{b}}{N_{0\,\mathrm{lim}}} \triangleq \frac{\log 2}{c_1}. \quad (7)$$

In the following, we determine the coefficients c_1 and c_2 in the expansion (4), and the values of ζ_0 and $\frac{E_\mathrm{b}}{N_{0\,\mathrm{lim}}}$, for coded modulation and bit-interleaved coded modulation.

For the sake of completeness, we prove the asymptotic expansion of the capacity as a function of $\frac{E_\mathrm{b}}{N_0}$.

Proof: We start with (4) and use Lagrange's inversion formula. The inversion formula transforms a function

$$\mathrm{C} = f_1(\mathrm{SNR}) \quad (8)$$

into its inverse

$$\mathrm{SNR} = f_2(\mathrm{C}). \quad (9)$$

We do an expansion around $\mathrm{SNR} = 0$, which is also $\mathrm{C} = 0$. Applied to our case, the inversion formula becomes

$$\mathrm{SNR} = \left.\frac{\mathrm{SNR}}{f_1(\mathrm{SNR})}\right|_{\mathrm{SNR}\to 0}\mathrm{C} + \frac{1}{2}\frac{d}{d\mathrm{SNR}}\left.\left(\frac{\mathrm{SNR}}{f_1(\mathrm{SNR})}\right)^2\right|_{\mathrm{SNR}\to 0}\mathrm{C}^2 + \mathrm{O}(\mathrm{C}^3). \quad (10)$$

Using the expansion in (4), after some simplifications we get

$$\mathrm{SNR} = \frac{\log 2}{c_1}\mathrm{C} - \frac{c_2\log^2 2}{c_1^3}\mathrm{C}^2 + \mathrm{O}(\mathrm{C}^3). \quad (11)$$

Letting $\mathrm{SNR} = \mathrm{C}\,\frac{E_\mathrm{b}}{N_0}$ and rearranging we obtain

$$\frac{E_\mathrm{b}}{N_0} = \frac{\log 2}{c_1} - \frac{c_2\log^2 2}{c_1^3}\mathrm{C} + \mathrm{O}(\mathrm{C}^2), \quad (12)$$

which leads to

$$\mathrm{C} = -\frac{\log^2 2c_2}{c_1^3}\left(\frac{E_\mathrm{b}}{N_0} - \frac{\log 2}{c_1}\right) + \mathrm{O}\left(\left(\frac{E_\mathrm{b}}{N_0} - \frac{\log 2}{c_1}\right)^2\right),$$

and hence the desired result. ■

The parameter ζ_0 is precisely Verdú's wideband slope in linear scale [5]. We avoid using the word minimum for $\frac{E_\mathrm{b}}{N_{0\,\mathrm{lim}}}$, since there exist communication schemes with a negative slope ζ_0, for which the absolute minimum value of $\frac{E_\mathrm{b}}{N_0}$ is achieved at non-zero rates. Even in these cases, the expansion at low efficiency (low power) is still given by Eq. (6).

For general fading distributions, we have the following result (Theorem 12 of [5]),

Theorem 1 *Let* c_1^{AWGN} *and* c_2^{AWGN} *denote the first two coefficients of the expansion around* $\mathrm{SNR} = 0$ *of the CM capacity in AWGN. Assume the fading distribution satisfies* $\mathrm{E}[|H|^2] =$

$\mathrm{E}[\chi] = 1$ *and its higher order moments are finite. Then, the coefficients* c_1 *and* c_2 *for a general fading distribution are*

$$c_1 = c_1^{\mathit{AWGN}} \quad (13)$$

$$c_2 = \mathrm{E}[\chi^2] c_2^{\mathit{AWGN}}. \quad (14)$$

Note that c_1 is independent of the fading distribution. For Nakagami-ν fading, $\mathrm{E}[\chi^2] = 1 + 1/\nu$ [3]. For unfaded AWGN, when $\nu \to \infty$, we have $\mathrm{E}[\chi^2] = 1$, for Rayleigh fading ($\nu = 1$) we get $\mathrm{E}[\chi^2] = 2$, and the slope ζ_0 halves, and as $\nu \to 0$, the wideband slope $\zeta_0 \to 0$.

This result is true whenever the expansion in Eq. (4) holds, which is the case for coded modulation. We will prove that such an expansion exists also for BICM, and therefore Theorem 1 is also true for BICM. This allows us to focus on the AWGN channel only in the forthcoming sections, while keeping *all results valid for general fading distributions*.

IV. Coded Modulation

Assuming uniformly distributed inputs to the channel described by (1) letting $h_\ell = 1$ for $\ell = 1, \dots, L$, the *coded modulation* AWGN capacity is given by

$$\mathrm{C}_{\mathrm{CM}} = -\,\mathrm{E}_{X,Z}\left[\log_2 \frac{\sum_{x' \in \mathcal{X}} p(x') p(Z|X = x')}{p(Z|X = x)}\right], \quad (15)$$

where $p(Z|X = x)$ is given by Eq. (2).

The coded modulation wideband regime is characterized by

Theorem 2 *The coefficients* c_1 *and* c_2 *in the Taylor expansion around* $\mathit{SNR} = 0$ *of coded modulation schemes over a signal set* $\mathcal{X}$ *with zero-mean and average energy* $\mathrm{E}[|X|^2] = 1$ *are*

$$c_1 = 1, \qquad c_2 = -\frac{1}{2}\left(1 + \left|\mathrm{E}[X^2]\right|^2\right). \quad (16)$$

We postpone a sketch of the proof to the section on BICM, as both cases are very similar.

The formula for c_1 is known, and can be found as Theorem 4 of [5]. For typical proper-complex constellations, which verify $\mathrm{E}[X^2] = (\mathrm{E}[X])^2 = 0$, $c_2 = -\frac{1}{2}$, as found by Prelov and Verdú [6], but Eq. (16) holds more general signal constellations. If we particularize the results of Proposition 2 to signal constellations of practical interest we obtain the following corollaries, whose respective proofs are straightforward.

Corollary 1 *For uniform* M*-PSK,* $c_2 = -1$ *if* $M = 2$ *and* $c_2 = -\frac{1}{2}$ *if* $M > 2$.

This result extends Theorem 11.1 of [5], where the result held for QPSK, a simple example of proper-complex constellation. Theorem 11.2 of [5] is generalized in

Corollary 2 *When* $\mathcal{X}$ *represents a mixture of* N M_n*−PSK constellations for* $n = 1, \dots, N$*,* $c_2 = -\frac{1}{2}$ *if and only if* $M_n > 2$ *for all rings/sub-constellations* $n = 1, \dots, N$.

This applies to APSK modulations, for instance. In [5] the result was stated for mixtures of QPSK constellations only.

V. Bit-Interleaved Coded Modulation

In BICM, the information message is encoded with a binary code $\mathcal{C}$; the choice of the code is independent of $\mathcal{X}$. The mapping between the binary codeword and the symbol set is performed by a binary labeling function $\mu : \{0,1\}^m \to \mathcal{X}$.

The standard BICM decoder [2] performs bitwise demodulation and ML decoding of $\mathcal{C}$. For this reason, we nickname this decoder BICM-ML decoder. Note that this sub-optimal decoder assumes that no iterations are performed at the demapper side. More specifically, BICM-ML decoding computes the following bitwise metrics (in log-likelihood ratio form) for the ℓ-th modulation symbol $\ell = 1, \dots, L$, i-th label position

$$\widetilde{\Lambda}_{\ell,i} = \log \frac{\sum_{x \in \mathcal{X}_0^i} e^{-\left|Z - h_\ell \sqrt{\mathrm{SNR}} x\right|^2}}{\sum_{x \in \mathcal{X}_1^i} e^{-\left|Z - h_\ell \sqrt{\mathrm{SNR}} x\right|^2}} \quad (17)$$

where the $\mathcal{X}_b^i$ are the sets of symbols of $\mathcal{X}$ with bit b in the i-th bit of the binary label.

We consider the binary-input continuous output channel that arises between any given codeword $\mathbf{c} \in \mathcal{C}$ and its corresponding LLRs $\Lambda_{\ell,i}$ [2]. For sufficiently long interleavers, the BICM-ML equivalent channel can be considered as a set of m parallel channels, whose index can be characterized by the random variable I, taking values on $\{1, \dots, m\}$ [2]. The input to the channel is denoted by the random variable B that takes values on $\{0, 1\}$. When the inputs to the binary-input channel induced by BICM with BICM-ML decoding are uniform, one can compute a BICM capacity $\mathrm{C}_{\mathrm{BICM}}$ [2], given by

$$\mathrm{C}_{\mathrm{BICM}} = -m\,\mathrm{E}_{B,Z,I}\left[\log_2 \frac{\frac{1}{2}\sum_{x' \in \mathcal{X}} p(Z|X = x')}{\sum_{x'' \in \mathcal{X}_B^I} p(Z|X = x'')}\right], \quad (18)$$

where $p(Z|X = x')$ is given by Eq. (2). Note that now, differently from CM, the random elements in the channel output include the transmitted modulation symbol x, in addition to the noise and fading realizations.

Theorem 3 *Consider a bit-interleaved coded modulation scheme over* $\mathcal{X}$*. It admits an expansion in Taylor series around* $\mathit{SNR} = 0$*, whose two first terms* c_1 *and* c_2 *are respectively*

$$c_1 = m\,\mathrm{E}\left[\left|\overline{X}_B^I\right|^2\right] \quad (19)$$

where

$$\overline{X}_b^i = \mathrm{E}[X | I = i, B = b] = \frac{2}{M} \sum_{x \in \mathcal{X}_b^i} x \quad (20)$$

is the "average" symbol with bit b *in label position* i*, and*

$$c_2 = m c_2^{\mathit{CM}} + m\,\mathrm{E}\left[\left|\overline{X}_B^I\right|^4 + \left(\mathrm{Re}(X X'^*)\right)^2 - 2\left(\mathrm{Re}(X^* \overline{X}_B^I)\right)^2\right], \quad (21)$$

c_2^{CM} *is the coefficient for coded modulation in Eq.* (16)*, and the expectations are over the joint distribution of* I, B, X, X'.

Proof: We do not give all the details of the proof, as it constitutes an involved and tedious exercise in analysis, essentially the computation of a Taylor series. We list however the main steps, so that it can be easily reproduced.

The BICM capacity includes the expectation

$$\mathrm{E}_{B,Z|I=i}\left[\log_2 \frac{\sum_{x'\in\mathcal{X}} \frac{1}{M} p(Z|X=x')}{\sum_{x''\in\mathcal{X}_B^i} \frac{2}{M} p(Z|X=x'')}\right], \qquad (22)$$

where we multiplied numerator and denominator inside the logarithm in Eq. (18) by a factor M. The probability density $p(Z|X=x')$ is given by Eq. (2). A similar form appears in the CM case, with a single term $p(Z|X=x')$ in the denominator.

Normalizing numerator and denominator in Eq. (22) by a constant factor $e^{-|w|^2}$, each density is replaced by

$$\frac{p(Z|X=x')}{e^{-|w|^2}} = e^{-\gamma^2|x-x'|^2 - 2\,\mathrm{Re}(\gamma(x-x')w^*)}, \qquad (23)$$

namely an exponential function e^t, with t small. For the sake of compactness, we have defined $\gamma = \sqrt{\mathrm{SNR}}$.

We use the Taylor series of the exponential around $\gamma = 0$. To catch all necessary terms, we compute an expansion up to the fourth order, i.e., $e^t = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \mathrm{O}(t^5)$. Defining a function $r(t)$, given by $r(t) \triangleq \mathrm{Re}(tw^*)$, and grouping common terms, Eq. (23) becomes

$$\begin{aligned}1-&\gamma^2|X-X'|^2 - 2\gamma r(X-X') + \frac{1}{2}\gamma^4|X-X'|^4\\ &+ \gamma^2 2r^2(X-X') + \gamma^3 2|X-X'|^2 r(X-X')\\ &- \gamma^4 2|X-X'|^2 r^2(X-X') - \gamma^3\frac{4}{3}r^3(X-X')\\ &+ \gamma^4\frac{2}{3}r^4(X-X') + \mathrm{O}(\gamma^5). \end{aligned} \qquad (24)$$

With the substitutions,

$$\begin{aligned}A'_1 &= -2r(X-X')\\ A'_2 &= -|X-X'|^2 + 2r^2(X-X')\\ A'_3 &= 2|X-X'|^2 r(X-X') - \frac{4}{3}r^3(X-X')\\ A'_4 &= \frac{1}{2}|X-X'|^4 - 2|X-X'|^2 r^2(X-X') + \frac{2}{3}r^4(X-X'),\end{aligned}$$

the summations over x' in numerator and denominator inside the logarithm of Eq. (22) respectively become

$$1 + \gamma A_1^{\mathrm{num}} + \gamma^2 A_2^{\mathrm{num}} + \gamma^3 A_3^{\mathrm{num}} + \gamma^4 A_4^{\mathrm{num}} + \mathrm{O}(\gamma^5), \qquad (25)$$

$$1 + \gamma A_1^{\mathrm{den}} + \gamma^2 A_2^{\mathrm{den}} + \gamma^3 A_3^{\mathrm{den}} + \gamma^4 A_4^{\mathrm{den}} + \mathrm{O}(\gamma^5), \qquad (26)$$

where we have defined $A_l^{\mathrm{num}} = \frac{1}{M}\sum_{x'\in\mathcal{X}} A'_l$, for $l = 1,2,3,4$, and similarly $A_l^{\mathrm{den}} = \frac{2}{M}\sum_{x'\in\mathcal{X}_{B=b}^{I=i}} A'_l$.

Taking logarithms of Eqs. (25) and (26) yields expressions of the form $\log(1+t)$, with t small. We use again a Taylor expansion, $\log(1+t) = t - \frac{1}{2}t + \frac{1}{3}t^2 - \frac{1}{4}t^4 + \mathrm{O}(t^5)$. In order to save space, we remove the superscripts "num" and "den" in the following. We thus have

$$\begin{aligned}\log&\Big(1 + \gamma A_1 + \gamma^2 A_2 + \gamma^3 A_3 + \gamma^4 A_4 + \mathrm{O}(\gamma^5)\Big)\\ &= \gamma A_1 + \gamma^2\left(A_2 - \frac{1}{2}A_1^2\right) + \gamma^3\left(A_3 - A_1A_2 + \frac{1}{3}A_1^3\right)\\ &+ \gamma^4\left(A_4 - \frac{1}{2}A_2^2 - A_1A_3 + A_1^2A_2 - \frac{1}{4}A_1^4\right) + \mathrm{O}(\gamma^5).\end{aligned}$$

The remaining steps are the expectation over the noise realization (which we denote by E_W), the bit and label position (for BICM only), and the input symbol (which we denote by E_X). If present, fading may also be included here. If the higher order moments are finite, they are included in the term $\mathrm{O}(\gamma^5)$, and only the terms $\mathrm{E}[\chi] = 1$ and $\mathrm{E}[\chi^2]$ remain.

As for the noise, one can compute

$$\begin{aligned}&\mathrm{E}_W[A_1] = \mathrm{E}_W[A_2] = \mathrm{E}_W[A_3] = \mathrm{E}_W[A_4] = 0\\ &\mathrm{E}_W\left[-\frac{1}{2}A_1^2\right] = -\big|X - \overline{X}_b^i\big|^2\\ &\mathrm{E}_W[-A_1A_2] = \mathrm{E}_W\left[\frac{1}{3}A_1^3\right] = \mathrm{E}_W\Big[-A_1A_3\Big] = 0.\end{aligned}$$

The average symbol $\overline{X}_b^i$ is zero in the coefficients of the numerator and $\overline{X}_b^i = \frac{2}{M}\sum_{x'\in\mathcal{X}_b^i} x'$ in the denominator.

Carrying out the averaging over X' and X'', as well as the expectation over the input X, we have

$$\begin{aligned}\mathrm{E}\left[-\frac{1}{2}A_2^2\right] &= -\,\mathrm{E}_{X,X',X''}\Big[|X|^4 + 3\big(\mathrm{Re}(XX''^*)\big)^2\\ &\quad - 4|X|^2\,\mathrm{Re}(X^*\overline{X}_b^i) + 2|X|^2|\overline{X}_b^i|^2\\ &\quad - 2\big(\mathrm{Re}(X'^*\overline{X}_b^i)\big)^2\Big]\end{aligned} \qquad (27)$$

$$\begin{aligned}\mathrm{E}\Big[A_1^2A_2\Big] &= 4\,\mathrm{E}_{X,X'}\Big[|X|^4 + \big(\mathrm{Re}(XX'^*)\big)^2\\ &\quad + 2\big(\mathrm{Re}(\overline{X}_b^iX^*)\big)^2 - 4|X|^2|\,\mathrm{Re}(X^*\overline{X}_b^i)\\ &\quad + 2|X|^2||\overline{X}_b^i|^2 - 2|\overline{X}_b^i|^4\Big]\end{aligned} \qquad (28)$$

$$\begin{aligned}\mathrm{E}\left[-\frac{1}{4}A_1^4\right] &= -3\,\mathrm{E}_{X,X'}\Big[|X|^4 - 3|\overline{X}_b^i|^4 + 4\big(\mathrm{Re}(X^*\overline{X}_b^i)\big)^2\\ &\quad + 2|X|^2|\overline{X}_b^i|^2 - 4|X|^2\,\mathrm{Re}(X^*\overline{X}_b^i)\Big].\end{aligned} \qquad (29)$$

We evaluate these expressions in detail only for the numerator, which corresponds to CM. Then $\overline{X}_b^i = 0$ and we obtain

$$-\,\mathrm{E}\big[|X|^2\big]\gamma^2 + \gamma^4\,\mathrm{E}_{X,X'}\Big[\big(\mathrm{Re}(XX''^*)\big)^2\Big] + \mathrm{O}(\gamma^5).$$

Using the expression

$$\mathrm{E}_{X,X'}\big(\mathrm{Re}(X'X^*)\big)^2 = \frac{1}{2}\Big(\big|\mathrm{E}_X[X^2]\big|^2 + 1\Big). \qquad (30)$$

yields the formula for the coefficients c_1 and c_2 in the CM capacity, Theorem 2.

For BICM both numerator and denominator in Eq. (22) contribute to the final expression. In addition, track must be kept of all coefficients $\overline{X}_b^i$. Collecting all terms and simplifying, the coefficient c_1 in the BICM capacity is

$$\sum_{i=1}^{m} \mathrm{E}\big[|X|^2\big] - \mathrm{E}\big[|X - \overline{X}_b^i|^2\big] = \sum_{i=1}^{m} |\overline{X}_b^i|^2. \qquad (31)$$

Grouping the various contributions and cancelling common terms, the coefficient c_2 is as in Theorem 3. ■

The formula for c_1 can be evaluated for M-PAM or M^2-QAM modulations using Gray mapping.

Theorem 4 *For M-PAM and M^2-QAM, with M is a power of 2 and natural Gray mapping, the minimum $\frac{E_{\mathrm{b}}}{N_{0\,\mathrm{lim}}}$ is*

$$\frac{E_{\mathrm{b}}}{N_{0\,\mathrm{lim}}} = \frac{4(M+1)(M-1)}{3M^2}\log 2. \tag{32}$$

Proof: Gray labelling for $m = \log_2 M$ bits is generated recursively by prefixing a binary 0 to the Gray code for $m-1$ bits, then prefixing a binary 1 to the reflected (i. e. listed in reverse order) Gray code for $m-1$ bits. For M-PAM, this Gray mapping construction makes $\overline{X}_b^i = 0$, for $b = 0, 1$ and $i > 1$. Therefore,

$$c_1 = \frac{1}{2}\left|\overline{X}_1^0\right|^2 + \frac{1}{2}\left|\overline{X}_1^1\right|^2 = \left|\overline{X}_1^0\right|^2 = \left|\overline{X}_1^1\right|^2, \tag{33}$$

the last equations follow from the symmetry between 0 and 1.

Symbols lie on a line with values $\pm\beta\big(1, 3, 5, \ldots, M-1\big)$, with β an energy normalization factor $\beta^2 = 3/((M-1)(M+1))$. The average symbol is $\overline{X}_1^0 = \beta M/2$, and therefore

$$c_1 = \left|\overline{X}_1^0\right|^2 = \frac{3M^2}{4(M-1)(M+1)}. \tag{34}$$

Extension to M-QAM is clear, by taking the Cartesian product along real and imaginary parts. Now, two indices i contribute, each with a similar form to PAM, but as the energy along each axis of half that of PAM, the normalization factor β^2_{QAM} also halves, and overall c_1 does not change. ■

It is somewhat surprising that the loss with respect to coded modulation at low SNR is bounded,

Corollary 3 *As $M \to \infty$, and under the conditions of Theorem 4, the minimum bit-energy to ratio ratio approaches*

$$\frac{E_{\mathrm{b}}}{N_{0\,\mathrm{lim}}} \to \frac{4}{3}\log 2 \simeq -0.3424\,\mathrm{dB}. \tag{35}$$

The loss represents about $1.25\,\mathrm{dB}$ with respect to the classical CM limit, namely $\frac{E_{\mathrm{b}}}{N_{0\,\mathrm{lim}}} = -1.59\,\mathrm{dB}$.

Table I reports the numerical values for the coefficients c_1 and c_2, as well as the minimum bit signal-to-noise ratio $\frac{E_{\mathrm{b}}}{N_{0\,\mathrm{lim}}}$ and wideband slope for various cases, namely QPSK, 8-PSK and 16-QAM modulations and Gray and Set Partitioning mappings. For BPSK, QPSK (2-PAM×2-PAM), and 16-QAM (4-PAM×4-PAM), the values coincide with Eq. (32).

TABLE I

$\frac{E_{\mathrm{b}}}{N_{0\,\mathrm{lim}}}$ AND WIDEBAND SLOPE COEFFICIENTS c_1, c_2 FOR BICM IN AWGN.

	Modulation and Mapping					
	QPSK		8-PSK		16-QAM	
	GR	SP	GR	SP	GR	SP
c_1	1.000	0.500	0.854	0.427	0.800	0.500
$\frac{E_{\mathrm{b}}}{N_{0\,\mathrm{lim}}}$ (dB)	-1.592	1.419	-0.904	2.106	-0.627	1.419
c_2	-0.500	0.250	-0.239	0.005	-0.160	-0.310
ζ_0	4.163	-1.041	5.410	-29.966	6.660	0.839

The numerical results are compared in Figure 2 with the capacity curves, where perfect match is observed. We use labels to identify the specific cases: labels 1 and 2 are QPSK, 3 and 4 are 8-PSK and 5 and 6 are 16-QAM. Also depicted is the linear approximation to the capacity around $\frac{E_{\mathrm{b}}}{N_{0\,\mathrm{lim}}}$, given by Eq. (6). Two cases with Nakagami fading are also included in Figure 2, which also show good match with the estimate. An exception is 8-PSK with set-partitioning, for which the approximation is valid for a very small range of rates, before changing the slope to a positive value.

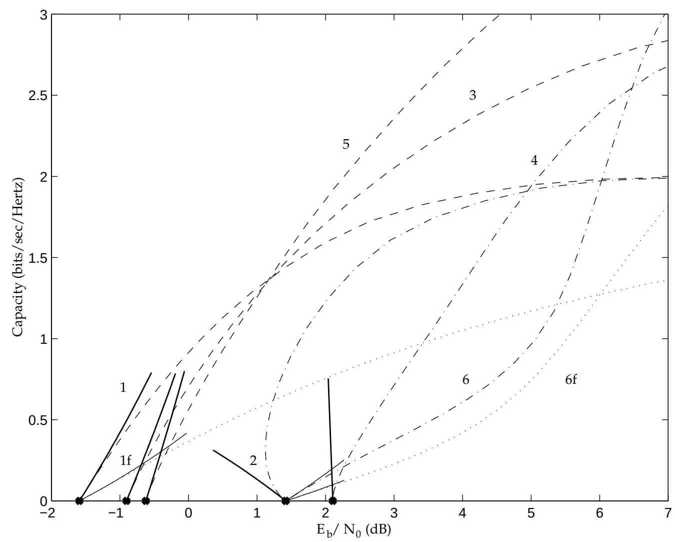


Fig. 2. BICM channel capacity (in bits per channel use). Labels 1 and 2 are QPSK, 3 and 4 are 8-PSK and 5 and 6 are 16-QAM. Gray and set partitioning labeling rules correspond to dashed (and odd labels) and dashed-dotted lines (and even labels) respectively. Dotted lines are cases 1 and 6 with Nakagami-0.3 and Nakagami-1 (Rayleigh) fading (an 'f' is appended to the label index). Solid lines are linear approximation around $\frac{E_{\mathrm{b}}}{N_{0\,\mathrm{lim}}}$.

VI. Conclusions

We have reviewed the analysis of the wideband power-limited regime for CM [5]. We have presented the analysis in such a way that the analysis for BICM appears as a natural extension of the former. Formulas for the wideband slope, for both coded modulation and bit-interleaved coded modulation have been provided. We have also determined the minimum energy per bit required for reliable communication. This energy is larger than or equal to that of coded modulation schemes; for Gray labelling, the loss is bounded by $10\log_{10} 4/3 \simeq 1.25\,\mathrm{dB}$. This fact may be useful for the design of systems operating at low signal-to-noise ratios.

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