Error Exponent Sensitivity of Sequential Probability Ratio Testing

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***Abstract*—We study mismatched sequential hypothesis testing. We analyze the type-I and and type-II error exponents when the actual distributions generating the observation are different from those used in the test. We derive the worst-case error exponents when the actual distributions generating the data are within a relative entropy ball of the test distributions and show the error exponent sensitivity of the test for small relative entropy balls.**

I. Introduction and Prelimenaries

Let X be a random variable distributed according to either P_0 or P_1, where P_0, P_1 are probability measures on $\mathbb{R}$. Consider the binary hypothesis testing problem [1] where an observation $\boldsymbol{x} = (x_1, \dots, x_n)$ is generated from one of the two possible probability measures P_0^n and P_1^n where we assume that $P_0^n(\boldsymbol{x}) = \prod_{i=1}^n P_0(x_i)$ and $P_1^n(\boldsymbol{x}) = \prod_{i=1}^n P_1(x_i)$. We also assume that both P_0 and P_1 are absolutely continuous with respect to each other which ensures that $D(P_0\|P_1) < \infty$, $D(P_1\|P_0) < \infty$ where $D(P\|Q) = \int \log \frac{dP}{dQ} dP$ is the relative entropy between P and Q.

A sequential hypothesis test is a pair $\Phi = (\phi : \mathcal{X}^\tau \to \{0,1\}, \tau)$ where τ is a random variable denoting the stopping time taking values on $\mathbb{Z}_+$; for every $n \geq 0$ the event $\{\tau \leq n\} \in \mathscr{F}_n$ where $\mathscr{F}_n$ is the sigma algebra induced by random variables $X_1, \dots, X_n$, i.e., $\sigma(X_1, \dots, X_n)$. Moreover, ϕ is a $\mathscr{F}_\tau$ measurable decision rule, i.e., the decision rule determined by causally observing the sequence X_i. In other words, at each time instant, the test decides in favor of one of the hypotheses or takes a new sample.

The two possible pairwise error probabilities measure the performance of the test and are defined as

$$\epsilon_0(\Phi) = \mathbb{P}_0\big[\phi(X^\tau) \neq 0\big] \ , \ \epsilon_1(\Phi) = \mathbb{P}_1\big[\phi(X^\tau) \neq 1\big]. \quad (1)$$

There are two possible definitions of achievable error exponents. According to [2] the optimal error exponent is

$$E_1(E_0) \triangleq \sup\Big\{E_1 \in \mathbb{R}^+ : \exists \Phi,\ \exists n \in \mathbb{Z}_+ \ \text{ s.t. } \mathbb{E}_{P_0}[\tau] \leq n, \mathbb{E}_{P_1}[\tau] \leq n,\ \epsilon_0(\Phi) \leq 2^{-nE_0} \ \text{ and } \ \epsilon_1(\Phi) \leq 2^{-nE_1}\Big\}. \quad (2)$$

Alternatively, the expected stopping time can be different under each hypothesis by design in order to increase the reliability under one of the hypotheses. Accordingly, [3] defined the error exponent tradeoff as

$$E_1(E_0) \triangleq \sup\Big\{E_1 \in \mathbb{R}^+ \colon \exists \Phi, \exists n_0, n_1 \in \mathbb{Z}_+, \text{s.t. } \mathbb{E}_{P_0}[\tau] \leq n_0, \mathbb{E}_{P_1}[\tau] \leq n_1,\ \epsilon_0(\Phi) \leq 2^{-n_0 E_0},\ \epsilon_1(\Phi) \leq 2^{-n_1 E_1}\Big\}. \quad (3)$$

In this work we will consider the performance of mismatched sequential ratio test under both definitions.

The sequential probability ratio test was proposed by Wald in [4]. For $\gamma_0, \gamma_1 \in \mathbb{R}^+$,

$$\tau = \inf\{n \geq 1 \colon S_n \geq \gamma_0 \text{ or } S_n \leq -\gamma_1\}, \quad (4)$$

$$\phi = \begin{cases} 0 & \text{if } S_\tau \geq \gamma_0 \\ 1 & \text{if } S_\tau \leq -\gamma_1, \end{cases} \quad (5)$$

with S_n being the accumulated log-likelihood ratio of $\boldsymbol{x}$,

$$S_n = \sum_{i=1}^n \log \frac{dP_0(x_i)}{dP_1(x_i)}. \quad (6)$$

It is shown in [3], [4] that the above test attains the optimal error exponent tradeoff, i.e., as thresholds γ_0, γ_1 approach infinity, the test achieves the best error exponent trade-off in (2) and (3). It is known that the error probabilities of sequential probability ratio test as a function of γ_0 and γ_1 are [5]

$$\epsilon_0 = c_0 \cdot e^{-\gamma_1} \quad , \quad \epsilon_1 = c_1 \cdot e^{-\gamma_0}, \quad (7)$$

as $\gamma_0, \gamma_1 \to \infty$ where c_0, c_1 are positive constants. Moreover, It can also be shown that,

$$\mathbb{E}_{P_0}[\tau] = \frac{\gamma_0}{D(P_0\|P_1)}(1 + o(1)), \quad (8)$$

$$\mathbb{E}_{P_1}[\tau] = \frac{\gamma_1}{D(P_1\|P_0)}(1 + o(1)). \quad (9)$$

Therefore, choosing the thresholds γ_0, γ_1 as

$$\gamma_0 = n\big(D(P_0\|P_1) + o(1)\big),\ \gamma_1 = n\big(D(P_1\|P_0) + o(1)\big), \quad (10)$$

we find the optimal error exponent tradeoff according to (2),

$$E_0 = D(P_1\|P_0), \quad E_1 = D(P_0\|P_1). \quad (11)$$

Hence, the sequential probability ratio test achieves the optimal error exponents tradeoff of the standard likelihood ratio test [6] simultaneously. Moreover, according to definition (3) the optimal error exponent tradeoff is given by

$$E_0 = \ell D(P_1\|P_0),\ E_1 = \frac{1}{\ell} D(P_0\|P_1), \quad (12)$$

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where $\ell = \frac{n_1}{n_0}$. Equivalently, we have

$$E_0 E_1 = D(P_0\|P_1)D(P_1\|P_0). \tag{13}$$

To achieve (13), thresholds γ_0, γ_1 should be chosen as

$$\gamma_0 = n_0\big(D(P_0\|P_1) + o(1)\big), \gamma_1 = \ell n_0\big(D(P_1\|P_0) + o(1)\big). \tag{14}$$

In this work, we define and analyze the worst case sensitivity (or robustness) of sequential probability ratio test under distribution mismatch. The literature in robust hypothesis testing (see e.g., [7]–[9] and references therein) consists of designing tests that are robust to the inaccuracy of the distributions generating the observation. Instead, we study the error exponent tradeoff of the standard sequential probability ratio test when the two probability measures used by the sequential probability ratio test are not precisely known, and two fixed probability measures $\hat{P}_0$ and $\hat{P}_1$ are used for testing; we assume that $\hat{P}_0, \hat{P}_1, P_0$, and P_1 are all absolutely continuous with respect to each other. In particular, we find the error exponent tradeoff for fixed $\hat{P}_0$ and $\hat{P}_1$ and we study the worst-case tradeoff when the true distributions generating the observation are within a certain relative entropy "distance" of the test distributions. Finally, we study the sensitivity of the test, defined as the deviation of the worst-case error exponent tradeoff for small relative entropy balls.

II. Mismatched Sequential Hypothesis Testing

Let $\hat{P}_0(x)$ and $\hat{P}_1(x)$ be the mismatched measures used in the sequential probability ratio test with thresholds $\hat{\gamma}_0, \hat{\gamma}_1$

$$\hat{\tau} = \inf\{n \geq 1 : \hat{S}_n \geq \hat{\gamma}_0 \text{ or } \hat{S}_n \leq -\hat{\gamma}_1\}, \tag{15}$$

$$\hat{\phi} = \begin{cases} 0 & \text{if } \hat{S}_{\hat{\tau}} \geq \hat{\gamma}_0 \\ 1 & \text{if } \hat{S}_{\hat{\tau}} \leq -\hat{\gamma}_1, \end{cases} \tag{16}$$

where

$$\hat{S}_n = \sum_{i=1}^{n} \log \frac{d\hat{P}_0(x_i)}{d\hat{P}_1(x_i)}. \tag{17}$$

In order to study the sensitivity of the mismatched sequential ratio test, we first study the error exponents tradeoffs analogous to (2) and (3). The next theorem provides the error exponents $\hat{E}_0, \hat{E}_1$ and the average stopping time $\mathbb{E}_{P_0}[\hat{\tau}], \mathbb{E}_{P_1}[\hat{\tau}]$ as the function of thresholds $\hat{\gamma}_0, \hat{\gamma}_1$.

Theorem 1: Let

$$0 < D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0), 0 < D(P_1\|\hat{P}_0) - D(P_1\|\hat{P}_1). \tag{18}$$

Then, as $\hat{\gamma}_0, \hat{\gamma}_1 \to \infty$

$$\hat{\epsilon}_0 = \hat{c}_0 \cdot e^{-\frac{D(P_0\|P_1)}{D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0)}\hat{\gamma}_1}, \tag{19}$$

$$\hat{\epsilon}_1 = \hat{c}_1 \cdot e^{-\frac{D(P_1\|P_0)}{D(P_1\|\hat{P}_0) - D(P_1\|\hat{P}_1)}\hat{\gamma}_0}, \tag{20}$$

where $\hat{c}_0, \hat{c}_1$ are positive constants. Furthermore, the expected stopping times are given by

$$\mathbb{E}_{P_0}[\hat{\tau}] = \frac{\hat{\gamma}_0}{D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0)}(1 + o(1)), \tag{21}$$

$$\mathbb{E}_{P_1}[\hat{\tau}] = \frac{\hat{\gamma}_1}{D(P_1\|\hat{P}_0) - D(P_1\|\hat{P}_1)}(1 + o(1)). \tag{22}$$

The next result states that if the average drift of the likelihood ratio changes sign under mismatch, the probability of error under that hypothesis tends to one.

Proposition 1: For fixed $\hat{P}_0, \hat{P}_1$, let

$$D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0) < 0 \ , \ D(P_1\|\hat{P}_0) - D(P_1\|\hat{P}_1) < 0. \tag{23}$$

Then, as thresholds $\hat{\gamma}_0, \hat{\gamma}_1$ approach infinity,

$$\hat{\epsilon}_0 \to 1 \quad , \quad \hat{\epsilon}_1 \to 1. \tag{24}$$

Corollary 1: Under the conditions of Theorem 1, the achievable error exponent tradeoff analagous to (2) is given by

$$\hat{E}_0 = D(P_0\|P_1)\frac{D(P_1\|\hat{P}_0) - D(P_1\|\hat{P}_1)}{D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0)}, \tag{25}$$

$$\hat{E}_1 = D(P_1\|P_0)\frac{D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0)}{D(P_1\|\hat{P}_0) - D(P_1\|\hat{P}_1)}, \tag{26}$$

where to achieve these exponents thresholds $\hat{\gamma}_0, \hat{\gamma}_1$ should be chosen as

$$\hat{\gamma}_0 = n\big(D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0) + o(1)\big), \tag{27}$$

$$\hat{\gamma}_1 = n\big(D(P_1\|\hat{P}_0) - D(P_1\|\hat{P}_1) + o(1)\big). \tag{28}$$

Moreover, the achievable error exponents analogous to (3) satisfy

$$\hat{E}_0 = \ell D(P_0\|P_1), \quad \hat{E}_1 = \frac{1}{\ell}D(P_1\|P_0), \tag{29}$$

where $\ell = \frac{D(P_1\|\hat{P}_0) - D(P_1\|\hat{P}_1)}{D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0)}\frac{n_1}{n_0}$. Equivalently, we have that

$$\hat{E}_0\hat{E}_1 = D(P_0\|P_1)D(P_1\|P_0). \tag{30}$$

To achieve (30), thresholds $\hat{\gamma}_0, \hat{\gamma}_1$ should be chosen as

$$\hat{\gamma}_0 = n_0\big(D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0) + o(1)\big), \tag{31}$$

$$\hat{\gamma}_1 = \ell n_0\big(D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0) + o(1)\big). \tag{32}$$

By comparing (13) and (30) we conclude that mismatched sequential probability ratio test has the same performance as the case with no mismatch, i.e., there exist thresholds $\hat{\gamma}_0, \hat{\gamma}_1$ such that the expected stopping time condition is met, and the error exponents satisfy (30). However, choosing $\hat{\gamma}_0, \hat{\gamma}_1$ to achieve (30) requires the knowledge of true probability measures P_0, P_1 by (31), (32), which might not be possible in a realistic scenario. Having this in mind, we consider the performance of the mismatched probability ratio test when the thresholds are selected from (27), (28), (31) and (32) but replacing P_0, P_1 by their mismatched counterparts $\hat{P}_0, \hat{P}_1$. In this scenario, the mismatch in probability measures will induce a mismatch both in expected stopping time and error exponents. Consider the case where (10) is used with mismatched measures $\hat{P}_0, \hat{P}_1$,

$$\hat{\gamma}_0 = n\big(D(\hat{P}_0\|\hat{P}_1) + o(1)\big) \ , \ \hat{\gamma}_1 = n\big(D(\hat{P}_1\|\hat{P}_0) + o(1)\big). \tag{33}$$

Using (33) and Theorem 1 we obtain

$$\mathbb{E}_{P_0}[\hat{\tau}] = n\frac{D(\hat{P}_0\|\hat{P}_1)}{D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0)}(1+o(1)), \quad (34)$$

$$\mathbb{E}_{P_1}[\hat{\tau}] = n\frac{D(\hat{P}_1\|\hat{P}_0)}{D(P_1\|\hat{P}_0) - D(P_1\|\hat{P}_1)}(1+o(1)). \quad (35)$$

Therefore, the mismatch in the thresholds, induces expected stopping times that may be larger than n. Letting $\eta^{-1} = \max\left\{\frac{D(\hat{P}_0\|\hat{P}_1)}{D(P_0\|\hat{P}_1)-D(P_0\|\hat{P}_0)}, \frac{D(\hat{P}_1\|\hat{P}_0)}{D(P_1\|\hat{P}_0)-D(P_1\|\hat{P}_1)}\right\}$, and according to definition (2) we have the following exponents,

$$\hat{E}_0 = \frac{D(P_0\|P_1)D(\hat{P}_1\|\hat{P}_0)}{D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0)}\eta, \quad (36)$$

$$\hat{E}_1 = \frac{D(P_1\|P_0)D(\hat{P}_0\|\hat{P}_1)}{D(P_1\|\hat{P}_0) - D(P_1\|\hat{P}_1)}\eta. \quad (37)$$

Similarly to (14), for the second definition of exponent, we need to multiply one of the thresholds by ℓ, and the corresponding exponents will be equal to ℓ and $\frac{1}{\ell}$ times the above exponents.

We now analyze the worst-case error exponents, defined as

$$\underline{\hat{E}}_i(r_i) \triangleq \min_{P_i\in\mathcal{B}(\hat{P}_i,r_i)} \hat{E}_i \quad, i=0,1 \quad (38)$$

where $\mathcal{B}(Q,r) = \{P \in \mathcal{P}(\mathcal{X}) : D(Q\|P) \le r\}$ is a relative entropy ball of radius r centered at distribution Q. From (36), we observe error exponents of mismatched sequential probability ratio test are a function of both testing distributions $\hat{P}_0, \hat{P}_1$, as opposed to the fixed sample-size setting where $\hat{E}_0$ is independent of $\hat{P}_1$ [10]. The next theorem shows the behavior of the worst-case exponents when the true distributions are within a small relative entropy ball of radii r_0, r_1 and center $\hat{P}_0, \hat{P}_1$, respectively.

Theorem 2: Let $P_i, \hat{P}_i$ are defined on the probability simplex $\mathcal{P}(\mathcal{X})$ and $r_i \ge 0$, for $i \in \{0,1\}$. Define $\bar{\imath} = 1-i$ to be the complement of index i. Then, the worst-case error exponents can be approximated as

$$\underline{\hat{E}}_i(r_i) = E_i - \min\left\{\sum_{j=0}^{1}\sqrt{r_j\cdot\theta_{i,j}(\hat{P}_0,\hat{P}_1)}, \sqrt{r_{\bar{\imath}}\cdot\theta_{\bar{\imath}}(\hat{P}_0,\hat{P}_1)}\right\} + o\big(\sqrt{r_0}+\sqrt{r_1}\,\big), \quad (39)$$

where

$$\theta_{i,j}(\hat{P}_0,\hat{P}_1) = \begin{cases} 2\mathrm{Var}_{\hat{P}_i}\left(\rho\log\frac{\hat{P}_i(X)}{\hat{P}_{\bar{\imath}}(X)}\right) & i=j \\ 2\mathrm{Var}_{\hat{P}_j}\left(\rho\frac{\hat{P}_i(X)}{\hat{P}_j(X)}\right) & i\ne j \end{cases} \quad (40)$$

$$\theta_{\bar{\imath}}(\hat{P}_0,\hat{P}_1) = 2\mathrm{Var}_{\hat{P}_{\bar{\imath}}}\left(\log\frac{\hat{P}_i(X)}{\hat{P}_{\bar{\imath}}(X)} + \rho\frac{\hat{P}_i}{\hat{P}_{\bar{\imath}}}\right), \quad (41)$$

$$\rho = \frac{D(\hat{P}_1\|\hat{P}_0)}{D(\hat{P}_0\|\hat{P}_1)}. \quad (42)$$

Next assuming $r_0 = r_1 = r$, we obtain the following result.

Corollary 2: For every $r = r_0 = r_1 \ge 0$, and $i \in \{0,1\}, \bar{\imath} = 1-i$,

$$\underline{\hat{E}}_i(r_i) = E_i - \sqrt{r\cdot\theta_{\bar{\imath}}(\hat{P}_0,\hat{P}_1)} + o(\sqrt{r}), \quad (43)$$

where

$$\theta_{\bar{\imath}}(\hat{P}_0,\hat{P}_1) = 2\mathrm{Var}_{\hat{P}_{\bar{\imath}}}\left(\log\frac{\hat{P}_i(X)}{\hat{P}_{\bar{\imath}}(X)} + \rho\frac{\hat{P}_i}{\hat{P}_{\bar{\imath}}}\right). \quad (44)$$

As an example, consider $\hat{P}_0 = \text{Bern}(0.1)$, $\hat{P}_1 = \text{Bern}(0.8)$, and $r = r_0 = r_1$. Figure 1 shows the worst-case error exponent given by solving non-convex optimization problem in (38) with precision of 10^{-3} as well as the approximation $\underline{\tilde{E}}_0$ obtained from (39) by ignoring the $o(\sqrt{r_0}+\sqrt{r_1})$ terms. Observe that there exists some gap between the approximation $\underline{\tilde{E}}_0$ and the actual exponent $\underline{\hat{E}}_0$ in (38). The approximtion consists of a linear approximation of the objective and second order approximation of constraints and computing it is straightforward for arbitrary distributions and radii. Instead, computing the exact optimization problem $\underline{\hat{E}}_0$ (38) is difficult, as it is a nonconvex optimization problem involving a highly nonlinear objective, cf. Eqs. (36)–(38).

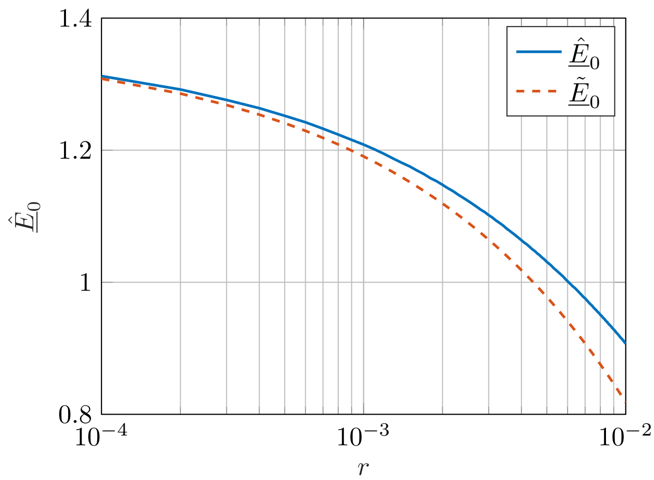

Fig. 1. Worst-case achievable type-I error exponent. The solid line is by solving the optimization problems in (38), and the dashed line is the approximated exponent using Theorem 2.

III. Proof of Theorem 1

From the absolute continuity assumption, let the log-likelihood ratio be bounded by a positive constant c, i.e.,

$$\left|\log\frac{\hat{P}_0(x)}{\hat{P}_1(x)}\right| \le c \quad \forall x. \quad (45)$$

We use the following results.

Theorem 3 ([5]): Let $S_n = \sum_{i=1}^n Z_i$ be a random walk where Z_i is some non-lattice random variable[1] generated in the i.i.d fashion with $\mathbb{E}[Z_i] > 0$. For $\gamma > 0$, let

$$\tau = \inf\{n \ge 1 : S_n \ge \gamma\}. \quad (46)$$

[1]A random variable Z is said to be lattice if and only if $\sum_{k=-\infty}^{\infty}\Pr[Z = a+kd] = 1$ for some non-negative a, d. Otherwise, it is said to be non-lattice.

Also, let $R_\gamma \triangleq S_\tau - \gamma$. Then R_γ converges in distribution to a random variable R with distribution Q as $\gamma \to \infty$. Moreover, if Z is lattice random variable, then R_γ has a limiting distribution Q_d as $\gamma \to \infty$ through multiples of d.

The next result shows that under conditions (18), the mismatched sequential probability ratio test stops at a finite time.

Lemma 1: Let $\hat{\tau}_0$ be the the smallest time that the mismatched sequential probability ratio test crosses threshold $\hat{\gamma}_0$, i.e.,

$$\hat{\tau}_0 = \inf\{n \geq 1 : \hat{S}_n \geq \hat{\gamma}_0\}. \tag{47}$$

Also, assume that conditions (18) hold. Then,

$$\mathbb{P}_0[\hat{\tau}_0 \geq n] \leq e^{d\hat{\gamma}_0} e^{-(n-1)E(0)}, \tag{48}$$

where $E(0), d > 0$. Also, as $\hat{\gamma}_0 \to \infty$, $\hat{\tau}_0 \to \infty$ almost surely.

We now proceed with the proof of the Theorem. We show the result for the type-II error probability; a similar proof holds for the type-I case. The type-II probability of error of mismatched sequential probability ratio test is

$$\hat{\epsilon}_1 = \mathbb{E}_{P_1}\big[\mathbb{1}\{\hat{S}_{\hat{\tau}} \geq \hat{\gamma}_0\}\big] \tag{49}$$

$$= \mathbb{E}_{P_0}\big[e^{-S_{\hat{\tau}}}\mathbb{1}\{\hat{S}_{\hat{\tau}} \geq \hat{\gamma}_0\}\big], \tag{50}$$

where $S_{\hat{\tau}}$ is the log-likelihood ratio under no mismatch in (6) evaluated at the time where the mismatched test stops. Recall the definition of $\hat{\tau}_0$ in (47) and $\hat{R}_{\hat{\gamma}_0} = \hat{S}_{\hat{\tau}_0} - \hat{\gamma}_0$. Observe that, if $\hat{S}_{\hat{\tau}} \geq \hat{\gamma}_0$, then we have $\hat{\tau} = \hat{\tau}_0$. Multipling the exponent in (50) by $\frac{\hat{S}_{\hat{\tau}_0}}{\hat{S}_{\hat{\tau}_0}}$ and substituting $\hat{R}_{\hat{\gamma}_0}$ we get

$$\hat{\epsilon}_1 = \mathbb{E}_{P_0}\Big[e^{-S_{\hat{\tau}_0}}\mathbb{1}\{\hat{S}_{\hat{\tau}} \geq \hat{\gamma}_0\}\Big] \tag{51}$$

$$= \mathbb{E}_{P_0}\Big[e^{-\frac{S_{\hat{\tau}_0}}{\hat{S}_{\hat{\tau}_0}}.\hat{S}_{\hat{\tau}_0}}\mathbb{1}\{\hat{S}_{\hat{\tau}} \geq \hat{\gamma}_0\}\Big] \tag{52}$$

$$= \mathbb{E}_{P_0}\Big[e^{-\frac{S_{\hat{\tau}_0}}{\hat{S}_{\hat{\tau}_0}}(\hat{R}_{\hat{\gamma}_0}+\hat{\gamma}_0)}\mathbb{1}\{\hat{S}_{\hat{\tau}} \geq \hat{\gamma}_0\}\Big]. \tag{53}$$

Let $\mu = \frac{S_{\hat{\tau}_0}}{\hat{\tau}_0}$, $\hat{\mu} = \frac{\hat{S}_{\hat{\tau}_0}}{\hat{\tau}_0}$. By Lemma 1, $\hat{\tau}_0 \to \infty$ as $\hat{\gamma}_0 \to \infty$ a.s., and therefore by the WLLN

$$\mu \xrightarrow{p} D(P_0\|P_1), \tag{54}$$

$$\hat{\mu} \xrightarrow{p} D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0). \tag{55}$$

Also, since $\hat{\mu} > 0$ almost surely, by using the continuous mapping theorem [11] we have

$$\frac{\mu}{\hat{\mu}} = \frac{S_{\hat{\tau}_0}}{\hat{S}_{\hat{\tau}_0}} \xrightarrow{p} \frac{D(P_0\|P_1)}{D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0)}. \tag{56}$$

Moreover, by Theorem 3, $\hat{R}_{\hat{\gamma}_0}$ converges in distribution to a random variable $\hat{R}_0$ with limiting distribution $\hat{Q}_0$ under P_0 (through multiples of d in the lattice case). Again by Slutsky's theorem,

$$\frac{S_{\hat{\tau}_0}}{\hat{S}_{\hat{\tau}_0}} \cdot \hat{R}_{\hat{\gamma}_0} \xrightarrow{d} \frac{D(P_0\|P_1)}{D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0)}\hat{R}_0. \tag{57}$$

Thus, letting $\hat{\gamma}_0 \to \infty$ in (53) we get

$$\lim_{\hat{\gamma}_0\to\infty} \hat{\epsilon}_1 = \mathbb{E}_{P_0}\Big[e^{-\frac{D(P_0\|P_1)}{D(P_0\|\hat{P}_1)-D(P_0\|\hat{P}_0)}(\hat{R}_0+\hat{\gamma}_0)}\Big] \tag{58}$$

$$= \hat{c}_1 \cdot e^{-\frac{D(P_0\|P_1)}{D(P_0\|\hat{P}_1)-D(P_0\|\hat{P}_0)}\hat{\gamma}_0}. \tag{59}$$

To prove (21), we show the converges of $\hat{\tau}_0$ in probability as well as its uniform integrability. Therefore, we can conclude its convergence in L^1 norm (and hence in expectation). Finally, from the convergence of $\hat{\tau}_0$, we obtain the convergence of $\hat{\tau}$. First, by the finiteness of $\hat{\tau}_0$ for every $\hat{\gamma}_0$ and definition of $\hat{\tau}_0$, there exist a finite $\hat{\tau}_0$ with probability one such that

$$\hat{s}_{\hat{\tau}_0-1} < \hat{\gamma}_0 \leq \hat{s}_{\hat{\tau}_0} \quad \text{w.p.1.} \tag{60}$$

Also, by the WLLN and Lemma 1 as $\hat{\gamma}_0 \to \infty$, we get

$$\frac{\hat{S}_{\hat{\tau}_0}}{\hat{\tau}_0} \xrightarrow{p} D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0), \tag{61}$$

$$\frac{\hat{S}_{\hat{\tau}_0-1}}{\hat{\tau}_0-1} \xrightarrow{p} D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0). \tag{62}$$

Therefore, by (60), (61), (62) we can conclude that

$$\frac{\hat{\tau}_0}{\hat{\gamma}_0} \xrightarrow{p} \frac{1}{D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0)} \tag{63}$$

as $\hat{\gamma}_0 \to \infty$.

To show the convergence in L^1 we only need to prove the uniform integrability of the sequence of random variables $\frac{\hat{\tau}_0}{\hat{\gamma}_0}$, where $\hat{\tau}_0$ is a random variable that depends on parameter $\hat{\gamma}_0$. Equivalently, we need to show that,

$$\lim_{t\to\infty}\sup_{\hat{\gamma}_0}\mathbb{E}_{P_0}\left[\frac{\hat{\tau}_0}{\hat{\gamma}_0}\mathbb{1}\Big\{\frac{\hat{\tau}_0}{\hat{\gamma}_0} \geq t\Big\}\right] = 0. \tag{64}$$

We can upper bound the given expectation in (64) as

$$\underbrace{\mathbb{E}_{P_0}\left[\frac{\hat{\tau}_0 - \lfloor t\hat{\gamma}_0\rfloor}{\hat{\gamma}_0}\mathbb{1}\Big\{\hat{\tau}_0 \geq \lfloor t\hat{\gamma}_0\rfloor\Big\}\right]}_{A} + \underbrace{t\mathbb{E}_{P_0}\left[\mathbb{1}\Big\{\frac{\hat{\tau}_0}{\hat{\gamma}_0} \geq t\Big\}\right]}_{B}. \tag{65}$$

The second term can be upper bounded by (48) as

$$B = t\mathbb{P}_0[\hat{\tau}_0 \geq t\hat{\gamma}_0] \leq te^{E(0)}e^{-\hat{\gamma}_0(tE(0)-d)}. \tag{66}$$

The first expectation can be also written as the following sum

$$A = \frac{1}{\hat{\gamma}_0}\sum_{m=1}^{\infty}\mathbb{P}_0\big[\hat{\tau}_0 - \lfloor t\hat{\gamma}_0\rfloor \geq m\big], \tag{67}$$

and by (48)

$$A \leq \frac{1}{\hat{\gamma}_0}te^{-\hat{\gamma}_0(tE(0)-d)}\sum_{m=1}^{\infty}e^{-(m-2)E(0)}. \tag{68}$$

Hence A and B are vanishing as $t \to \infty$ for every $\hat{\gamma}_0$ giving the uniform integrability of $\frac{\hat{\tau}_0}{\hat{\gamma}_0}$, and hence convergence in L^1 [12], i.e,

$$\lim_{\hat{\gamma}_0\to\infty}\mathbb{E}_{P_0}\left[\left|\frac{\hat{\tau}_0}{\hat{\gamma}_0} - \frac{1}{D(P_0\|\hat{P}_1) - D(P_0\|\hat{P}_0)}\right|\right] = 0. \tag{69}$$

Finally, we prove the convergence of $\hat{\tau}$. By (19), (63) and the union bound, we obtain

$$\mathbb{P}_0\left[\left|\frac{\hat{\tau}}{\hat{\gamma}_0}-\frac{1}{D(P_0\|\hat{P}_1)-D(P_0\|\hat{P}_0)}\right|\geq\epsilon\right] \tag{70}$$

$$\leq\mathbb{P}_0\left[\left|\frac{\hat{\tau}}{\hat{\gamma}_0}-\frac{1}{D(P_0\|\hat{P}_1)-D(P_0\|\hat{P}_0)}\right|\geq\epsilon,\hat{\phi}=0\right] \tag{71}$$

$$+\,\mathbb{P}_0[\hat{\phi}=1] \tag{72}$$

$$=\mathbb{P}_0\left[\left|\frac{\hat{\tau}_0}{\hat{\gamma}_0}-\frac{1}{D(P_0\|\hat{P}_1)-D(P_0\|\hat{P}_0)}\right|\geq\epsilon\right]+\hat{\epsilon}_0, \tag{73}$$

which tends to 0 as $\hat{\gamma}_0\to\infty$, establishing the convergence of $\frac{\hat{\tau}}{\hat{\gamma}_0}$ in probability. Now, using that $\hat{\tau}\leq\hat{\tau}_0$ we have

$$\mathbb{E}_{P_0}\left[\frac{\hat{\tau}}{\hat{\gamma}_0}\mathbb{1}\left\{\frac{\hat{\tau}}{\hat{\gamma}_0}\geq t\right\}\right]\leq\mathbb{E}_{P_0}\left[\frac{\hat{\tau}_0}{\hat{\gamma}_0}\mathbb{1}\left\{\frac{\hat{\tau}_0}{\hat{\gamma}_0}\geq t\right\}\right]. \tag{74}$$

Therefore, uniform integrability of $\hat{\tau}_0$ gives the uniform integrability of $\hat{\tau}$, and hence convergence in L^1 norm and also expectation of $\frac{\hat{\tau}}{\hat{\gamma}_0}$, which concludes the proof.

IV. Proof of Theorem 2

We show the result under hypothesis 0, and similar steps are valid for hypothesis 1. Observe that (36) can be written as

$$\hat{E}_0=D(P_0\|P_1)\cdot\cdot\min\left\{\frac{D(\hat{P}_1\|\hat{P}_0)}{D(\hat{P}_0\|\hat{P}_1)},\frac{D(P_1\|\hat{P}_0)-D(P_1\|\hat{P}_1)}{D(P_0\|\hat{P}_1)-D(P_0\|\hat{P}_0)}\right\}. \tag{75}$$

From (38) and (75), we need to compute two minimizations, the first of which over P_0. To this end, we exchange the order of these minimizations and apply a Taylor series expansion to the first term of (75) around $P_0=\hat{P}_0, P_1=\hat{P}_1$ we obtain

$$\hat{E}_0=D(\hat{P}_1\|\hat{P}_0)+\min\left\{\rho\boldsymbol{d}_0^T\boldsymbol{\theta}_{P_0}+\rho\boldsymbol{d}_1^T\boldsymbol{\theta}_{P_1},\boldsymbol{d}_2^T\boldsymbol{\theta}_{P_1}\right\}+o(\|\boldsymbol{\theta}_{P_0}\|_\infty+\|\boldsymbol{\theta}_{P_1}\|_\infty), \tag{76}$$

where for $i=0,1$,

$$\boldsymbol{\theta}_{P_i}=\big(P_i(x_1)-\hat{P}_i(x_1),\dots,P_i(x_{|\mathcal{X}|})-\hat{P}_i(x_{|\mathcal{X}|})\big)^T, \tag{77}$$

$$\boldsymbol{d}_0=\left(1+\log\frac{\hat{P}_0(x_1)}{\hat{P}_1(x_1)},\dots,1+\log\frac{\hat{P}_0(x_1)}{\hat{P}_1(x_1)}\right)^T, \tag{78}$$

$$\boldsymbol{d}_1=\left(-\frac{\hat{P}_0(x_1)}{\hat{P}_1(x_1)},\dots,-\frac{\hat{P}_0(x_{|\mathcal{X}|})}{\hat{P}_1(x_{|\mathcal{X}|})}\right)^T, \tag{79}$$

$$\boldsymbol{d}_2=\left(1+\log\frac{\hat{P}_1(x_1)}{\hat{P}_0(x_1)},\dots,1+\log\frac{\hat{P}_1(x_1)}{\hat{P}_0(x_1)}\right)^T+\rho\boldsymbol{d}_1, \tag{80}$$

and $\rho=\frac{D(\hat{P}_1\|\hat{P}_0)}{D(\hat{P}_0\|\hat{P}_1)}$. By substituting expansion (76) into (38) we obtain

$$\underline{\hat{E}}_0(r_0)=D(\hat{P}_1\|\hat{P}_0)+\min\Big\{\rho\min_{\substack{P_0\in\mathcal{B}(\hat{P}_0,r_0)\\P_1\in\mathcal{B}(\hat{P}_1,r_1)}}\boldsymbol{d}_0^T\boldsymbol{\theta}_{P_0}+\boldsymbol{d}_1^T\boldsymbol{\theta}_{P_1},\min_{\substack{P_0\in\mathcal{B}(\hat{P}_0,r_0)\\P_1\in\mathcal{B}(\hat{P}_1,r_1)}}\boldsymbol{d}_2^T\boldsymbol{\theta}_{P_1}\Big\}+o(\|\boldsymbol{\theta}_{P_0}\|_\infty+\|\boldsymbol{\theta}_{P_1}\|_\infty). \tag{81}$$

Now, we further approximate the outer minimization constraint in (38), or, equivalently, the minimizations over the relative entropy balls in (81). By approximating $D(\hat{P}_i\|P_i)$ for $i=0,1$ up to second order we get [13]

$$D(\hat{P}_i\|P_i)=\frac{1}{2}\boldsymbol{\theta}_{P_i}^T\boldsymbol{J}(\hat{P}_i)\boldsymbol{\theta}_{P_i}+o(\|\boldsymbol{\theta}_{P_i}\|_\infty^2), \tag{82}$$

where

$$\boldsymbol{J}_i=\operatorname{diag}\left(\frac{1}{\hat{P}_i(x_1)},\dots,\frac{1}{\hat{P}_i(x_{|\mathcal{X}|})}\right) \tag{83}$$

is the Fisher information matrix corresponding to hypothesis i. For $i=0,1$, define the sets

$$\underline{\mathcal{B}}(\hat{P}_i,r_i)=\big\{\boldsymbol{\theta}_i\in\mathbb{R}^{|\mathcal{X}|}:\boldsymbol{\theta}_{P_i}^T\boldsymbol{J}_i\boldsymbol{\theta}_{P_i}\leq 2r_i,\boldsymbol{1}^T\boldsymbol{\theta}_{P_i}=0\big\}. \tag{84}$$

It is easy to show that the error term of $\underline{\hat{E}}_0(r_0)$ caused by approximating the constraints $\mathcal{B}(\hat{P}_i,r_i)$ is $o(\|\boldsymbol{\theta}_{P_i}\|_\infty)$. Hence, we can further approximate (81) as

$$\underline{\hat{E}}_0(r_0)=D(\hat{P}_1\|\hat{P}_0)+\min\Big\{\rho\min_{\substack{P_0\in\underline{\mathcal{B}}(\hat{P}_0,r_0)\\P_1\in\underline{\mathcal{B}}(\hat{P}_1,r_1)}}\boldsymbol{d}_0^T\boldsymbol{\theta}_{P_0}+\boldsymbol{d}_1^T\boldsymbol{\theta}_{P_1},\min_{\substack{P_0\in\underline{\mathcal{B}}(\hat{P}_0,r_0)\\P_1\in\underline{\mathcal{B}}(\hat{P}_1,r_1)}}\boldsymbol{d}_2^T\boldsymbol{\theta}_{P_1}\Big\}+o(\|\boldsymbol{\theta}_{P_0}\|_\infty+\|\boldsymbol{\theta}_{P_1}\|_\infty). \tag{85}$$

The optimization problem in (85) is convex and hence the KKT conditions are also sufficient. The corresponding Lagrangian for the first optimization is given by

$$L_0(\boldsymbol{\theta}_{P_0},\boldsymbol{\theta}_{P_1},\lambda_0,\lambda_1,\nu_0,\nu_1)=\sum_{i=0}^{1}\boldsymbol{d}_i^T\boldsymbol{\theta}_{P_i}+\lambda_i\Big(\frac{1}{2}\boldsymbol{\theta}_{P_i}^T\boldsymbol{J}_i\boldsymbol{\theta}_{P_i}-r_i\Big)+\nu_i(\boldsymbol{1}^T\boldsymbol{\theta}_{P_i}). \tag{86}$$

Differentiating with respect to $\boldsymbol{\theta}_{P_i}$ and setting to zero, gives

$$\boldsymbol{d}_i+\lambda_i\boldsymbol{J}_i\boldsymbol{\theta}_{P_i}+\nu_i\boldsymbol{1}=\boldsymbol{0}. \tag{87}$$

Therefore,

$$\boldsymbol{\theta}_{P_i}=-\frac{1}{\lambda_i}\boldsymbol{J}_i^{-1}\big(\boldsymbol{d}_i+\nu_i\boldsymbol{1}\big). \tag{88}$$

Observe that if $\lambda_i=0$ then from (87) $\boldsymbol{d}_i=-\nu_i\boldsymbol{1}$ which cannot be true since $\hat{P}_0\neq\hat{P}_1$. Therefore, from the complementary slackness condition [14] the inequality constraints (85) should be satisfied with equality. By solving $\frac{1}{2}\boldsymbol{\theta}_{P_i}^T\boldsymbol{J}_i\boldsymbol{\theta}_{P_i}=r_i$ and $\boldsymbol{1}^T\boldsymbol{\theta}_{P_i}=0$ and substituting λ_i,ν_i in (88), we obtain

$$\boldsymbol{\theta}_{P_i}=-\frac{\boldsymbol{\psi}_i}{\sqrt{\boldsymbol{\psi}_i^T\boldsymbol{J}_i\boldsymbol{\psi}_i}}\sqrt{2r_i}, \tag{89}$$

where

$$\boldsymbol{\psi}_i=\boldsymbol{J}_i^{-1}\Big(\boldsymbol{d}_i-\boldsymbol{1}^T\boldsymbol{J}_i^{-1}\boldsymbol{d}_i\boldsymbol{1}\Big). \tag{90}$$

The second optimization is solved similarly. Substituting (89) into (85) yields (39).

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