An Achievable Error Exponent for the Multiple Access Channel with Correlated Sources

Arezou Rezazadeh*, Josep Font-Segura*, Alfonso Martinez*, Albert Guillén i Fàbregas*†‡
*Universitat Pompeu Fabra, †ICREA and ‡University of Cambridge
arezou.rezazadeh@upf.edu, {josep.font,alfonso.martinez,guillen}@ieee.org

***Abstract*—This paper derives an achievable random-coding error exponent for joint source-channel coding over a multiple access channel with correlated sources. The codebooks are generated by drawing codewords from a multi-letter distribution that depends on the composition of the source message.**

I. Introduction

For point-to-point communication, joint source-channel coding is known to yield a larger exponent than separate source-channel coding [1], [2], even though the latter is optimal in terms of transmissibility. In contrast, for the multiple-access channel with correlated sources, joint source-channel coding leads to a larger transmissibility region [3], [4]. In addition, tuning the random-coding ensemble leads to improved exponents in the point-to-point channel [5], [6] and in the multiple-access channel [7], [8]. Inspired by these facts, we are motivated to consider joint source-channel coding where codewords are generated with a conditional probability distribution of the codeword symbol that depends both on the instantaneous source symbol and on the type of the source sequence. In particular, this paper studies a novel ensemble where codebooks are drawn from a multi-letter distribution that is the product of independent conditional distributions that depend on the corresponding single-letter value of the source message. Generalizations to constant-composition families, and other are possible, although they are not discussed here.

A. Problem set-up, notation and definitions

In the multiple-access channel, two or more terminals send information to a common receiver. Here, we consider simultaneous transmission over the channel of two correlated discrete memoryless sources. The sources are characterized by a probability distribution $P_{U_1U_2}$ on the alphabet $\mathcal{U}_1\times\mathcal{U}_2$, where $\mathcal{U}_1$ and $\mathcal{U}_2$ are the respective alphabets of the two sources. The source sequences $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ have length n. We use a bold letter $\boldsymbol{u}$ to denote a sequence and underline to represent a pair of quantities for users 1 and 2, such as $\underline{u}=(u_1,u_2)$, $\underline{\boldsymbol{u}}=(\boldsymbol{u}_1,\boldsymbol{u}_2)$ or $\underline{\mathcal{U}}=\mathcal{U}_1\times\mathcal{U}_2$.

For user $\nu=1,2$, the source message $\boldsymbol{u}_\nu$ is mapped onto codeword $\boldsymbol{x}_\nu(\boldsymbol{u}_\nu)$, which also has length n and is drawn from the codebook $\mathcal{C}^\nu=\{\boldsymbol{x}_\nu(\boldsymbol{u}_\nu);\boldsymbol{u}_\nu\in\mathcal{U}_\nu^n\}$. Both terminals send the codewords over a discrete memoryless multiple access channel with transition probability $W(y|x_1,x_2)$, input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, and output alphabet $\mathcal{Y}$.

This work has been funded in part by the European Research Council under ERC grant agreement 259663 and by the Spanish Ministry of Economy and Competitiveness under grants RYC-2011-08150, FJCI-2014-22747, and TEC2016-78434-C3-1-R.

Given the received sequence $\boldsymbol{y}$, the decoder estimates the transmitted pair messages $\underline{\boldsymbol{u}}$ based on the maximum a posteriori criterion:

$$\underline{\boldsymbol{u}}'=\arg\max_{\underline{\boldsymbol{u}}\in\underline{\mathcal{U}}^n} P^n_{\underline{U}}(\underline{\boldsymbol{u}})W^n\big(\boldsymbol{y}|\boldsymbol{x}_1(\boldsymbol{u}_1),\boldsymbol{x}_2(\boldsymbol{u}_2)\big). \tag{1}$$

An error occurs if the decoded messages $\underline{\boldsymbol{u}}'$ differ from the transmitted $\underline{\boldsymbol{u}}$. Using the convention that scalar random variables are denoted by capital letter, e.g., X, and capital bold letters denote random vectors, the error probability for a given pair of codebooks is given by

$$\epsilon^n(\mathcal{C}^1,\mathcal{C}^2)\triangleq\mathbb{P}\left[(\boldsymbol{U}_1',\boldsymbol{U}_2')\neq(\boldsymbol{U}_1,\boldsymbol{U}_2)\right]. \tag{2}$$

The pair of sources (U_1,U_2) is transmissible over the channel if there exists a sequence of codebooks $(\mathcal{C}_n^1,\mathcal{C}_n^2)$ such that $\lim_{n\to\infty}\epsilon^n(\mathcal{C}_n^1,\mathcal{C}_n^2)=0$. An exponent $E(P_{\underline{U}},W)$ is achievable if there exists a sequence of codebooks such that

$$\liminf_{n\to\infty}-\frac{1}{n}\log\left(\epsilon^n(\mathcal{C}_n^1,\mathcal{C}_n^2)\right)\geq E(P_{\underline{U}},W). \tag{3}$$

We apply random coding to show the existence of such sequences of codebooks. Equivalently, we find a sequence of ensembles whose error probability averaged over the ensemble $\bar{\epsilon}^n$ tends to zero, in order to determine an achievable exponent. For user $\nu=1,2$, we assign to source probability distribution P_{U_ν} a conditional probability distribution $Q_{\nu,P_{U_\nu}}(x|u)$. We represent the set of these distributions by $\{Q_{\nu,P_{U_\nu}}:P_{U_\nu}\in\mathcal{P}_{U_\nu}\}$. For every message $\boldsymbol{u}_\nu^n\in\mathcal{U}_\nu^n$, we randomly generate a codeword $\boldsymbol{x}_\nu(\boldsymbol{u}_\nu)$ according to the probability distribution $Q^n_{\nu,\pi(\boldsymbol{u}_\nu)}(\boldsymbol{x}_\nu|\boldsymbol{u}_\nu)=\prod_{i=1}^n Q_{\nu,\pi(\boldsymbol{u}_\nu)}(x_{\nu,i}|u_{\nu,i})$, where $Q_{\nu,\pi(\boldsymbol{u}_\nu)}$ is a probability distribution that depends on the type of $\boldsymbol{u}_\nu$, denoted by $\pi(\boldsymbol{u}_\nu)$.

We shall also need the symbol $\xi\in\{\{1\},\{2\},\{1,2\}\}$ and its complement ξ^c among the subsets of $\{1,2\}$. For example, $\xi^c=\{2\}$ for $\xi=\{1\}$ and $\xi^c=\emptyset$ for $\xi=\{1,2\}$. In order to simplify some expressions, we adopt the following notational convention,

$$u_\xi=\begin{cases}\emptyset & \xi=\emptyset\\ u_1 & \xi=\{1\}\\ u_2 & \xi=\{2\}\\ \underline{u} & \xi=\{1,2\}\end{cases}, \tag{4}$$

for the variable u_ν, and similarly for the probability distribution Q_ν and the set $\mathcal{X}_\nu$.

The set of all possible distributions of single letter X is denoted by $\mathcal{P}_\mathcal{X}$ and the set of all empirical distributions on a vector in $\mathcal{X}^n$ (i.e. types) is denoted by $\mathcal{P}^n_\mathcal{X}$. Given $\hat{P}_X \in \mathcal{P}^n_\mathcal{X}$, the type class $\mathcal{T}^n(\hat{P}_X)$ is the set of all sequences in $\mathcal{X}^n$ with type $\hat{P}_X$. If $\boldsymbol{x} \in \mathcal{T}^n(\hat{P}_X)$, for any probability distribution $Q^n(\boldsymbol{x}) = \prod_{i=1}^n Q(x_i)$, we have the following facts [9]

$$Q^n(\boldsymbol{x}) = e^{n \sum_{x\in\mathcal{X}} \hat{P}_X(x)\log Q(x)}, \tag{5}$$

$$\frac{e^{nH(\hat{P}_X)}}{(n+1)^{|\mathcal{X}|}} \le \big|\mathcal{T}^n(\hat{P}_X)\big| \le e^{nH(\hat{P}_X)}, \tag{6}$$

where the cardinality of an arbitrary set $\mathcal{Z}$ is denoted by $|\mathcal{Z}|$.

Considering (5) and (6), we have

$$\mathbb{P}\big[\mathcal{T}^n(\hat{P}_X)\big] = \sum_{\boldsymbol{x}\in\mathcal{T}^n(\hat{P}_X)} Q^n(\boldsymbol{x}) \le e^{-nD(\hat{P}_X||Q)}. \tag{7}$$

Given $\hat{P}_{XY} \in \mathcal{P}^n_{\mathcal{X}\times\mathcal{Y}}$ and $\boldsymbol{y} \in \mathcal{T}^n(\hat{P}_Y)$, the conditional type class $\mathcal{T}^n_{\boldsymbol{y}}(\hat{P}_{XY})$ is defined to be the set of all sequences $\boldsymbol{x} \in \mathcal{X}^n$ such that $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{T}^n(\hat{P}_{XY})$. It can be proved that [9]

$$\big|\mathcal{T}^n_{\boldsymbol{y}}(\hat{P}_{XY})\big| = \frac{\big|\mathcal{T}^n(\hat{P}_{XY})\big|}{\big|\mathcal{T}^n(\hat{P}_Y)\big|}. \tag{8}$$

II. An Achievable Exponent

Proposition 1: For the two-user MAC transition probability W and source probability distributions $P_{\underline{U}}$, an achievable exponent $E_1(P_{\underline{U}}, W)$ is given by (9) and (10) at the bottom of the page, where $\lambda(\underline{U},\underline{X},Y) = \log\big(P_{\underline{U}}(\underline{U})W(Y|\underline{X})\big)$, and $[x]^+ = \max\{0,x\}$.

We briefly note that by setting $\hat{P}_{\underline{U}\underline{X}Y} = P_{\underline{U}}Q_{1,\hat{P}_{U_1}}Q_{2,\hat{P}_{U_2}}W$ and $\hat{P}_{\underline{U}\underline{X}Y} = \tilde{P}_{\underline{U}\underline{X}Y}$, the exponent in (9) can be shown to recover the achievability region by Cover, El Gamal and Salehi [3].

Proof: We first bound $\bar{\epsilon}^n$, the average error probability over the ensemble, for a given block length n. Applying the random coding union bound [10] for joint source channel coding, we have

$$\bar{\epsilon}^n \le \sum_{\underline{\boldsymbol{u}},\underline{\boldsymbol{x}},\boldsymbol{y}} P^n_{\underline{\boldsymbol{U}}\underline{\boldsymbol{X}}\boldsymbol{Y}}(\underline{\boldsymbol{u}},\underline{\boldsymbol{x}},\boldsymbol{y}) \min\left\{1, \sum_{\underline{\boldsymbol{u}}'\ne\underline{\boldsymbol{u}}} \mathbb{P}\left[\frac{P^n_{\underline{\boldsymbol{U}}}(\underline{\boldsymbol{u}}')W^n(\boldsymbol{y}|\underline{\boldsymbol{X}}')}{P^n_{\underline{\boldsymbol{U}}}(\underline{\boldsymbol{u}})W^n(\boldsymbol{y}|\underline{\boldsymbol{x}})} \ge 1\right]\right\}, \tag{11}$$

where $\underline{\boldsymbol{x}}'$ has the same distribution as $\underline{\boldsymbol{x}}$ but is independent of $\boldsymbol{y}$. We group the error events corresponding to the summation over $(\boldsymbol{u}_1',\boldsymbol{u}_2') \ne (\boldsymbol{u}_1,\boldsymbol{u}_2)$ into three types of error events, namely $(\boldsymbol{u}_1',\boldsymbol{u}_2) \ne (\boldsymbol{u}_1,\boldsymbol{u}_2)$, $(\boldsymbol{u}_1,\boldsymbol{u}_2') \ne (\boldsymbol{u}_1,\boldsymbol{u}_2)$ and $(\boldsymbol{u}_1',\boldsymbol{u}_2') \ne (\boldsymbol{u}_1,\boldsymbol{u}_2)$. We respectively denote these types of error by $\xi \in \{\{1\},\{2\},\{1,2\}\}$. Using that $\min\{1,a+b\} \le \min\{1,a\}+\min\{1,b\}$, we further bound $\bar{\epsilon}^n$ as

$$\bar{\epsilon}^n \le \sum_\xi \bar{\epsilon}^n_\xi, \tag{12}$$

where

$$\bar{\epsilon}^n_\xi \le \sum_{\underline{\boldsymbol{u}},\underline{\boldsymbol{x}},\boldsymbol{y}} P^n_{\underline{\boldsymbol{U}}\underline{\boldsymbol{X}}\boldsymbol{Y}}(\underline{\boldsymbol{u}},\underline{\boldsymbol{x}},\boldsymbol{y}) \min\left\{1, \sum_{\substack{\boldsymbol{u}'_\xi\ne\boldsymbol{u}_\xi \\ \boldsymbol{x}'_\xi:\frac{P^n_{\underline{\boldsymbol{U}}}(\boldsymbol{u}'_\xi,\boldsymbol{u}_{\xi^c})W^n(\boldsymbol{y}|\boldsymbol{x}'_\xi,\boldsymbol{x}_{\xi^c})}{P^n_{\underline{\boldsymbol{U}}}(\underline{\boldsymbol{u}})W^n(\boldsymbol{y}|\boldsymbol{x}_1,\boldsymbol{x}_2)}\ge 1}} Q^n_{\xi,\pi(\boldsymbol{u}'_\xi)}(\boldsymbol{x}'_\xi|\boldsymbol{u}'_\xi)\right\}. \tag{13}$$

Next, we group the outer and inner summations in (13) based on the empirical distributions of $(\underline{\boldsymbol{u}},\underline{\boldsymbol{x}},\boldsymbol{y})$ and $(\boldsymbol{u}'_\xi,\boldsymbol{x}'_\xi)$, respectively, and then sum over all possible empirical distributions, respectively denoted by $\hat{P}_{\underline{U}\underline{X}Y}$ and $\tilde{P}_{\underline{U}\underline{X}Y}$. We note that the summation over $\hat{P}_{\underline{U}\underline{X}Y}$ runs over the set of all possible empirical distributions, $\mathcal{P}^n_{\underline{U}\times\underline{X}\times Y}$, while the summation over $\tilde{P}_{\underline{U}\underline{X}Y}$ is restricted to the set $\mathcal{L}^n_\xi$, defined as

$$\mathcal{L}^n_\xi(\hat{P}_{\underline{U}\underline{X}Y}) \triangleq \big\{\tilde{P}_{\underline{U}\underline{X}Y} \in \mathcal{P}^n_{\underline{\mathcal{U}}\times\underline{\mathcal{X}}\times\mathcal{Y}} : \tilde{P}_{U_{\xi^c}X_{\xi^c}Y} = \hat{P}_{U_{\xi^c}X_{\xi^c}Y}, \; \mathbb{E}_{\tilde{P}}[\lambda(\underline{U},\underline{X},Y)] \ge \mathbb{E}_{\hat{P}}[\lambda(\underline{U},\underline{X},Y)]\big\}. \tag{14}$$

As a result, we can write the summations in equation (13) respectively as

$$\sum_{\underline{\boldsymbol{u}},\underline{\boldsymbol{x}},\boldsymbol{y}} P^n_{\underline{\boldsymbol{U}}\underline{\boldsymbol{X}}\boldsymbol{Y}}(\underline{\boldsymbol{u}},\underline{\boldsymbol{x}},\boldsymbol{y}) = \sum_{\hat{P}_{\underline{U}\underline{X}Y}\in\mathcal{P}^n_{\underline{\mathcal{U}}\times\underline{\mathcal{X}}\times\mathcal{Y}}} \sum_{(\underline{\boldsymbol{u}},\underline{\boldsymbol{x}},\boldsymbol{y})\in\mathcal{T}^n(\hat{P}_{\underline{U}\underline{X}Y})} P^n_{\underline{\boldsymbol{U}}\underline{\boldsymbol{X}}\boldsymbol{Y}}(\underline{\boldsymbol{u}},\underline{\boldsymbol{x}},\boldsymbol{y}), \tag{15}$$

$$E_1(P_{\underline{U}},W) = \max_{\{Q_{\nu P_{U_\nu}}\}_{\nu=1,2}\, P_{u_\nu}\in\mathcal{P}_{\mathcal{U}_\nu}} \min_{\xi\in\{\{1\},\{2\},\{1,2\}\}} \min_{\hat{P}_{\underline{U}\underline{X}Y}\in\mathcal{P}_{\underline{\mathcal{U}}\times\underline{\mathcal{X}}\times\mathcal{Y}}} D(\hat{P}_{\underline{U}\underline{X}Y}||P_{\underline{U}}Q_{1,\hat{P}_{U_1}}Q_{2,\hat{P}_{U_2}}W) + \left[\min_{\tilde{P}_{\underline{U}\underline{X}Y}\in\mathcal{L}_\xi(\hat{P}_{\underline{U}\underline{X}Y})} D(\tilde{P}_{\underline{U}\underline{X}Y}||\tilde{P}_{U_\xi}Q_{\xi,\tilde{P}_{U_\xi}}\hat{P}_{U_{\xi^c}X_{\xi^c}Y}) - H(\tilde{P}_{U_\xi})\right]^+ \tag{9}$$

$$\mathcal{L}_\xi(\hat{P}_{\underline{U}\underline{X}Y}) \triangleq \left\{\tilde{P}_{\underline{U}\underline{X}Y} \in \mathcal{P}_{\underline{\mathcal{U}}\times\underline{\mathcal{X}}\times\mathcal{Y}} \;:\; \tilde{P}_{U_{\xi^c}X_{\xi^c}Y} = \hat{P}_{U_{\xi^c}X_{\xi^c}Y},\; \mathbb{E}_{\tilde{P}}\lambda(\underline{U},\underline{X},Y) \ge \mathbb{E}_{\hat{P}}\lambda(\underline{U},\underline{X},Y)\right\} \tag{10}$$

and

$$\sum_{\substack{\boldsymbol{u}'_\xi \neq \boldsymbol{u}_\xi \\ \boldsymbol{x}'_\xi : \frac{P^n_{\underline{U}}(\boldsymbol{u}'_\xi, \boldsymbol{u}_{\xi^c}) W^n(\boldsymbol{y}|\boldsymbol{x}'_\xi, \boldsymbol{x}_{\xi^c})}{P^n_{\underline{U}}(\underline{\boldsymbol{u}}) W^n(\boldsymbol{y}|\boldsymbol{x}_1, \boldsymbol{x}_2)} \geq 1}} Q^n_{\xi,\pi(\boldsymbol{u}'_\xi)}(\boldsymbol{x}'_\xi|\boldsymbol{u}'_\xi) = \sum_{\tilde{P}_{\underline{U}\underline{X}Y} \in \mathcal{L}^n_\xi(\hat{P}_{\underline{U}\underline{X}Y})} \sum_{(\boldsymbol{u}'_\xi, \boldsymbol{x}'_\xi) \in \mathcal{T}^n_{\boldsymbol{u}_{\xi^c}\boldsymbol{x}_{\xi^c}\boldsymbol{y}}(\tilde{P}_{\underline{U}\underline{X}Y})} Q^n_{\xi,\pi(\boldsymbol{u}'_\xi)}(\boldsymbol{x}'_\xi|\boldsymbol{u}'_\xi). \tag{16}$$

Since the conditional distribution $Q^n_{\xi,\pi(\boldsymbol{u}'_\xi)}(\boldsymbol{x}'_\xi|\boldsymbol{u}'_\xi)$ has the same value for all $(\boldsymbol{u}'_\xi, \boldsymbol{x}'_\xi) \in \mathcal{T}^n_{\boldsymbol{u}_{\xi^c}\boldsymbol{x}_{\xi^c}\boldsymbol{y}}(\tilde{P}_{\underline{U}\underline{X}Y})$, we have

$$\sum_{(\boldsymbol{u}'_\xi, \boldsymbol{x}'_\xi) \in \mathcal{T}^n_{\boldsymbol{u}_{\xi^c}\boldsymbol{x}_{\xi^c}\boldsymbol{y}}(\tilde{P}_{\underline{U}\underline{X}Y})} Q^n_{\xi,\pi(\boldsymbol{u}'_\xi)}(\boldsymbol{x}'_\xi|\boldsymbol{u}'_\xi) = |\mathcal{T}^n_{\boldsymbol{u}_{\xi^c}\boldsymbol{x}_{\xi^c}\boldsymbol{y}}(\tilde{P}_{\underline{U}\underline{X}Y})| Q^n_{\xi,\pi(\boldsymbol{u}'_\xi)}(\boldsymbol{x}'_\xi|\boldsymbol{u}'_\xi). \tag{17}$$

Considering (8) and the fact that $\tilde{P}_{U_{\xi^c}X_{\xi^c}Y} = \hat{P}_{U_{\xi^c}X_{\xi^c}Y}$ in $\mathcal{L}^n_\xi(\hat{P}_{\underline{U}\underline{X}Y})$ in (14), we have the following upper bound

$$\left|\mathcal{T}^n_{\boldsymbol{u}_{\xi^c}\boldsymbol{x}_{\xi^c}\boldsymbol{y}}(\tilde{P}_{\underline{U}\underline{X}Y})\right| = \frac{\left|\mathcal{T}^n(\tilde{P}_{\underline{U}\underline{X}Y})\right|}{\left|\mathcal{T}^n(\tilde{P}_{U_{\xi^c}X_{\xi^c}Y})\right|} \tag{18}$$

$$\leq \frac{e^{nH(\tilde{P}_{\underline{U}\underline{X}Y})+o(n)}}{e^{nH(\hat{P}_{U_{\xi^c}X_{\xi^c}Y})}}, \tag{19}$$

where $o(n)$ is a sequence satisfying $\lim_{n\to\infty} \frac{o(n)}{n} = 0$. In addition, using equation (5) for conditional distributions, for all $(\boldsymbol{u}'_\xi, \boldsymbol{x}'_\xi) \in \mathcal{T}^n_{\boldsymbol{u}_{\xi^c}\boldsymbol{x}_{\xi^c}\boldsymbol{y}}(\tilde{P}_{\underline{U}\underline{X}Y})$, we have the following identity on the conditional probability

$$Q^n_{\xi,\pi(\boldsymbol{u}'_\xi)}(\boldsymbol{x}'_\xi|\boldsymbol{u}'_\xi) = e^{n\sum_{\underline{u},\underline{x},y} \tilde{P}_{\underline{U}\underline{X}Y}(\underline{u},\underline{x},y) \log Q_{\xi,\tilde{P}_{U_\xi}}(x_\xi|u_\xi)}. \tag{20}$$

Combining inequality (19) and identity (20) and into (17), we obtain the following inequality

$$\sum_{(\boldsymbol{u}'_\xi, \boldsymbol{x}'_\xi) \in \mathcal{T}^n_{\boldsymbol{u}_{\xi^c}\boldsymbol{x}_{\xi^c}\boldsymbol{y}}(\tilde{P}_{\underline{U}\underline{X}Y})} Q^n_{\xi,\pi(\boldsymbol{u}'_\xi)}(\boldsymbol{x}'_\xi|\boldsymbol{u}'_\xi) \leq e^{-n\left(D(\tilde{P}_{\underline{U}\underline{X}Y}||\tilde{P}_{U_\xi}Q_{\xi,\tilde{P}_{U_\xi}}\hat{P}_{U_{\xi^c}X_{\xi^c}Y}) - H(\tilde{P}_{U_\xi})\right)+o(n)}. \tag{21}$$

Further upper bounding the right hand side of equation (21) by the maximum over the empirical probability distributions $\tilde{P}_{\underline{U}\underline{X}Y} \in \mathcal{L}^n_\xi(\hat{P}_{\underline{U}\underline{X}Y})$, we have

$$\sum_{(\boldsymbol{u}'_\xi, \boldsymbol{x}'_\xi) \in \mathcal{T}^n_{\boldsymbol{u}_{\xi^c}\boldsymbol{x}_{\xi^c}\boldsymbol{y}}(\tilde{P}_{\underline{U}\underline{X}Y})} Q^n_{\xi,\pi(\boldsymbol{u}'_\xi)}(\boldsymbol{x}'_\xi|\boldsymbol{u}'_\xi) \leq \max_{\tilde{P}_{\underline{U}\underline{X}Y} \in \mathcal{L}^n_\xi(\hat{P}_{\underline{U}\underline{X}Y})} e^{-n\left(D(\tilde{P}_{\underline{U}\underline{X}Y}||\tilde{P}_{U_\xi}Q_{\xi,\tilde{P}_{U_\xi}}\hat{P}_{U_{\xi^c}X_{\xi^c}Y}) - H(\tilde{P}_{U_\xi})\right)+o(n)}. \tag{22}$$

Moreover, in view of (7), the second summation of the right hand side of (15) can be expressed as

$$\sum_{(\underline{\boldsymbol{u}},\underline{\boldsymbol{x}},\boldsymbol{y}) \in \mathcal{T}^n(\hat{P}_{\underline{U}\underline{X}Y})} P^n_{\underline{\boldsymbol{U}}\underline{\boldsymbol{X}}\boldsymbol{Y}}(\underline{\boldsymbol{u}},\underline{\boldsymbol{x}},\boldsymbol{y}) \leq e^{-n\left(D(\hat{P}_{\underline{U}\underline{X}Y}||P_{\underline{U}}Q_{1,\hat{P}_{U_1}}Q_{2,\hat{P}_{U_2}}W\right)}, \tag{23}$$

where $\hat{P}_{U_\nu}$ denotes the marginal distribution of $\hat{P}_{\underline{U}}$, for $\nu = 1, 2$. Similarly to (22), we may upper bound the right hand side of (23) by the maximum over the empirical distributions $\hat{P}_{\underline{U}\underline{X}Y} \in \mathcal{P}^n_{\underline{\mathcal{U}}\times\underline{\mathcal{X}}\times\mathcal{Y}}$, i.e.,

$$\sum_{(\underline{\boldsymbol{u}},\underline{\boldsymbol{x}},\boldsymbol{y}) \in \mathcal{T}^n(\hat{P}_{\underline{U}\underline{X}Y})} P^n_{\underline{\boldsymbol{U}}\underline{\boldsymbol{X}}\boldsymbol{Y}}(\underline{\boldsymbol{u}},\underline{\boldsymbol{x}},\boldsymbol{y}) \leq \max_{\hat{P}_{\underline{U}\underline{X}Y} \in \mathcal{P}^n_{\underline{\mathcal{U}}\times\underline{\mathcal{X}}\times\mathcal{Y}}} e^{-n\left(D(\hat{P}_{\underline{U}\underline{X}Y}||P_{\underline{U}}Q_{1,\hat{P}_{U_1}}Q_{2,\hat{P}_{U_2}}W\right)}. \tag{24}$$

Putting back the results obtained in equations (24) and (22) into the respective inner and outer summations (15) and (16), we obtain that the average error probability (13) can be bounded as equations (25) and (26), given at the bottom of the page. In (26), we used that the cardinality of the sets $\mathcal{L}^n_\xi(\hat{P}_{\underline{U}\underline{X}Y})$ and $\mathcal{P}^n_{\underline{\mathcal{U}}\times\underline{\mathcal{X}}\times\mathcal{Y}}$ behave polynomially with the codeword length n, and satisfy

$$\left|\mathcal{L}^n_\xi(\hat{P}_{\underline{U}\underline{X}Y})\right| \leq \left|\mathcal{P}^n_{\underline{\mathcal{U}}\times\underline{\mathcal{X}}\times\mathcal{Y}}\right| \leq e^{o(n)}.$$

Using the identity $\min\{1, e^a\} = e^{[a]^+}$, we may write equation (26) as

$$\bar{\epsilon}^n_\xi \leq e^{-nE^n_\xi + o(n)}, \tag{27}$$

where E^n_ξ is given in (28) at the bottom of the next page.

$$\bar{\epsilon}^n_\xi \leq \sum_{\hat{P}_{\underline{U}\underline{X}Y} \in \mathcal{P}^n_{\underline{\mathcal{U}}\times\underline{\mathcal{X}}\times\mathcal{Y}}} \max_{\hat{P}_{\underline{U}\underline{X}Y} \in \mathcal{P}^n_{\underline{\mathcal{U}}\times\underline{\mathcal{X}}\times\mathcal{Y}}} e^{-n\left(D(\hat{P}_{\underline{U}\underline{X}Y}||P_{\underline{U}}Q_{1,\hat{P}_{U_1}}Q_{2,\hat{P}_{U_2}}W)\right)} \min\left\{1, \sum_{\tilde{P}_{\underline{U}\underline{X}Y} \in \mathcal{L}^n_\xi(\hat{P}_{\underline{U}\underline{X}Y})} \max_{\tilde{P}_{\underline{U}\underline{X}Y} \in \mathcal{L}^n_\xi(\hat{P}_{\underline{U}\underline{X}Y})} e^{-n\left(D(\tilde{P}_{\underline{U}\underline{X}Y}||\tilde{P}_{U_\xi}Q_{\xi,\tilde{P}_{U_\xi}}\hat{P}_{U_{\xi^c}X_{\xi^c}Y}) - H(\tilde{P}_{U_\xi})\right)+o(n)}\right\} \tag{25}$$

$$\leq \max_{\hat{P}_{\underline{U}\underline{X}Y} \in \mathcal{P}^n_{\underline{\mathcal{U}}\times\underline{\mathcal{X}}\times\mathcal{Y}}} e^{-n\left(D(\hat{P}_{\underline{U}\underline{X}Y}||P_{\underline{U}}Q_{1,\hat{P}_{U_1}}Q_{2,\hat{P}_{U_2}}W)\right)+o(n)} \min\left\{1, \max_{\tilde{P}_{\underline{U}\underline{X}Y} \in \mathcal{L}^n_\xi(\hat{P}_{\underline{U}\underline{X}Y})} e^{-n\left(D(\tilde{P}_{\underline{U}\underline{X}Y}||\tilde{P}_{U_\xi}Q_{\xi,\tilde{P}_{U_\xi}}\hat{P}_{U_{\xi^c}X_{\xi^c}Y}) - H(\tilde{P}_{U_\xi})\right)+o(n)}\right\} \tag{26}$$

Since the average error probability over the ensemble is bounded by the summation over the error events, we further upper bound the summation by the worst type of error, i.e.,

$$\sum_{\xi} \bar{\epsilon}_{\xi}^{n} \leq e^{-n \min_{\xi} E_{\xi}^{n} + o(n)}. \tag{29}$$

Hence, from (12), we conclude that $\bar{\epsilon}^n$ is upper bounded by the right hand side of (29), i.e.,

$$\bar{\epsilon}^{n} \leq e^{-n \min_{\xi} E_{\xi}^{n} + o(n)}. \tag{30}$$

Using the following properties

$$\liminf_{n\to\infty} (a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \tag{31}$$

$$\liminf_{n\to\infty} \min\{a_n, b_n\} = \min\big\{ \liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n \big\}, \tag{32}$$

we obtain that $\bar{\epsilon}^n$ asymptotically satisfies equation (33) at the bottom of the page. We note that the inequality

$$\liminf_{n\to\infty} \max\{a_n, b_n\} \geq \max\big\{ \liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n \big\}, \tag{34}$$

implies that

$$\liminf_{n\to\infty} [a_n]^{+} \geq \big[\liminf_{n\to\infty} a_n \big]^{+}. \tag{35}$$

We further note that the set of all empirical distributions is dense in the set of all possible probability distributions, and that the functions involved in (33) are uniformly continuous over their arguments. Hence, we may replace the optimization over empirical distributions by an optimization over the set of all possible distributions. Using (35) in (33), we obtain (36) at the bottom of the page, where $\mathcal{L}_{\xi}(\hat{P}_{\underline{U}\underline{X}Y})$ is defined in (10).

Finally, we may optimize the asymptotic error probability over the set of input distributions $\{Q_{\nu P_{U_\nu}}\}$ for $\nu = 1, 2$ and $P_{U_\nu} \in \mathcal{P}_{\mathcal{U}_\nu}$. This concludes the proof, since

$$\liminf_{n\to\infty} -\frac{1}{n} \log(\bar{\epsilon}^n) \geq E_1(P_{\underline{U}}, W). \tag{37}$$

■

III. Conclusion

We have derived an achievable random-coding error exponent for joint source-channel coding over a multiple access channel with correlated sources. We have adopted a novel ensemble where codebooks are generated according to a conditional distribution that depends not only on the instantaneous source symbol, but also on the type of the whole source sequence. The derived exponent may be used to recover the Cover-El Gamal-Salehi achievability region, and generalizations to other random-coding ensembles is left to future work.

References

[1] R. Gallager, *Information Theory and Reliable Communication*. John Wiley & Sons, 1968.

[2] I. Csiszár, "Joint source-channel error exponent," *Probl. Control Inf. Theory*, vol. 9, no. 1, pp. 315–328, 1980.

[3] T. M. Cover, A. El Gamal, and M. Salehi, "Multiple access channels with arbitrarily correlated sources," *IEEE Trans. Inf. Theory*, vol. 26, no. 6, pp. 648–657, Nov. 1980.

[4] G. Dueck, "A note on the multiple access channel with correlated sources (corresp.)," *IEEE Trans. Inf. Theory*, vol. 27, no. 2, pp. 232–235, Mar 1981.

[5] Y. Zhong, F. Alajaji, and L. L. Campbell, "On the joint source-channel coding error exponent for discrete memoryless systems," *IEEE Trans. Inf. Theory*, vol. 52, no. 4, pp. 1450–1468, April 2006.

[6] A. Tauste Campo, G. Vazquez-Vilar, A. Guillén i Fàbregas, A. Martinez, and T. Koch, "A derivation of the source-channel error exponent using nonidentical product distributions," *IEEE Trans. Inf. Theory*, vol. 60, no. 6, pp. 3209–3217, June 2014.

[7] J. Scarlett, A. Martinez, and A. Guillén i Fàbregas, "Multiuser coding techniques for mismatched decoding," *IEEE Trans. Inf. Theory*, vol. 62, no. 7, pp. 3950–3970, July 2016.

[8] Y. Liu and B. Hughes, "A new universal random coding bound for the multiple-access channel," *IEEE Trans. Inf. Theory*, vol. 42, no. 2, pp. 376–386, March 1996.

[9] I. Csiszár and J. Körner, *Information Theory: Coding Theorems for Discrete Memoryless Systems*. Cambridge University Press, 2011.

[10] Y. Polyanskiy, H. V. Poor, and S. Verdú, "Channel coding rate in the finite blocklength regime," *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2307–2359, May 2010.

$$E_{\xi}^{n} = \min_{\hat{P}_{\underline{U}\underline{X}Y} \in \mathcal{P}^{n}_{\underline{\mathcal{U}} \times \underline{\mathcal{X}} \times \mathcal{Y}}} D(\hat{P}_{\underline{U}\underline{X}Y} || P_{\underline{U}} Q_{1,\hat{P}_{U_1}} Q_{2,\hat{P}_{U_2}} W) + \Big[\min_{\tilde{P}_{\underline{U}\underline{X}Y} \in \mathcal{L}^{n}_{\xi}(\hat{P}_{\underline{U}\underline{X}Y})} D(\tilde{P}_{\underline{U}\underline{X}Y} || \tilde{P}_{U_\xi} Q_{\xi,\tilde{P}_{U_\xi}} \hat{P}_{U_{\xi^c} X_{\xi^c} Y}) - H(\tilde{P}_{U_\xi}) \Big]^{+} \tag{28}$$

$$\liminf_{n\to\infty} -\frac{1}{n} \log(\bar{\epsilon}^n) \geq \min_{\xi \in \{\{1\},\{2\},\{1,2\}\}} \liminf_{n\to\infty} \min_{\hat{P}_{\underline{U}\underline{X}Y} \in \mathcal{P}^{n}_{\underline{\mathcal{U}} \times \underline{\mathcal{X}} \times \mathcal{Y}}} D(\hat{P}_{\underline{U}\underline{X}Y} || P_{\underline{U}} Q_{1,\hat{P}_{U_1}} Q_{2,\hat{P}_{U_2}} W)$$
$$+ \liminf_{n\to\infty} \Big[\min_{\tilde{P}_{\underline{U}\underline{X}Y} \in \mathcal{L}^{n}_{\xi}(\hat{P}_{\underline{U}\underline{X}Y})} D(\tilde{P}_{\underline{U}\underline{X}Y} || \tilde{P}_{U_\xi} Q_{\xi,\tilde{P}_{U_\xi}} \hat{P}_{U_{\xi^c} X_{\xi^c} Y}) - H(\tilde{P}_{U_\xi}) \Big]^{+} \tag{33}$$

$$\liminf_{n\to\infty} -\frac{1}{n} \log(\bar{\epsilon}^n_{\xi}) \geq \min_{\xi \in \{\{1\},\{2\},\{1,2\}\}} \min_{\hat{P}_{\underline{U}\underline{X}Y} \in \mathcal{P}_{\underline{\mathcal{U}} \times \underline{\mathcal{X}} \times \mathcal{Y}}} D(\hat{P}_{\underline{U}\underline{X}Y} || P_{\underline{U}} Q_{1,\hat{P}_{U_1}} Q_{2,\hat{P}_{U_2}} W)$$
$$+ \Big[\min_{\tilde{P}_{\underline{U}\underline{X}Y} \in \mathcal{L}_{\xi}(\hat{P}_{\underline{U}\underline{X}Y})} D(\tilde{P}_{\underline{U}\underline{X}Y} || \tilde{P}_{U_\xi} Q_{\xi,\tilde{P}_{U_\xi}} \hat{P}_{U_{\xi^c} X_{\xi^c} Y}) - H(\tilde{P}_{U_\xi}) \Big]^{+} \tag{36}$$