Asymptotics of the Error Probability in Quasi-Static Binary Symmetric Channels

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***Abstract*—This paper provides an asymptotic expansion of the error probability, as the codeword length n goes to infinity, in quasi-static binary symmetric channels. After the leading term, namely the outage probability, the next two terms are found to be proportional to $\frac{\log n}{n}$ and $\frac{1}{n}$ respectively. Explicit characterizations of the respective coefficients are given. The resulting expansion gives an approximation to the random-coding union bound, accurate even at small codeword lengths.**

I. Introduction

In delay-constrained communication over slowly varying channels, the channel parameters may stay constant over the whole duration of the codeword. The capacity of this nonergodic channel is zero for most channel distributions, since the error probability cannot be made arbitrarily small [1], [2]. The outage probability, i.e, the probability that the intended rate exceeds the instantaneous mutual information of the channel, and the outage capacity, i.e, the largest achievable rate for a fixed outage probability, are the fundamental limits in quasi-static channels [3].

For short codewords, a finer analysis is required to assess the backoff from the outage probability and from the outage capacity at any finite length n. In terms of rates, [4], [5] showed that the achievable rates converge to the outage capacity faster than $\frac{1}{\sqrt{n}}$. In terms of error probability, we considered a weakened version of the random-coding union (RCU) bound [6], [7] based on the Markov's inequality, that suggested a $\frac{1}{n}$ backoff from the outage probability.

In this work, we propose an approximation to the pairwise error probability of the RCU based on the saddlepoint method [8] that accurately captures the properties of nonsingular channels [9], [10]. In particular, we consider the simple quasi-static binary symmetric channel (BSC), whose crossover probability is a random variable that changes from codeword to codeword, to closely approximate the RCU bound by

$$\mathrm{rcu}_n(R) \simeq P_{\mathrm{out}}(R) + \frac{\log n}{n}\phi_{\log}(R) + \phi_0(R)\frac{1}{n}, \tag{1}$$

where $P_{\mathrm{out}}(R)$ is the outage probability, and $\phi_{\log}(R)$ and $\phi_0(R)$ are terms that depend on the rate. We conjecture that the first two terms in (1) also appear in the asymptotic expansion of the optimal error probability.

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II. The Quasi-Static BSC

We consider the transmission of M codewords of length n over a binary symmetric channel (BSC) whose crossover probability q remains constant for the duration of the codeword and changes independently from codeword to codeword according to the probability distribution $p_Q(q)$. We assume that the crossover probability takes values in the interval $q \in (0, \frac{1}{2})$ and that $p_Q(q)$ is continuously differentiable. Under the capacity achieving distribution $p_X(0) = p_X(1) = \frac{1}{2}$ and for any given q, the mutual information of the BSC is given by $I(q) = \log 2 - h(q)$, where $h(q) = -q\log q - (1-q)\log(1-q)$ is the binary entropy function.

For a fixed rate $R = \frac{1}{n}\log M$, random-coding arguments show that the error probability averaged over the code ensemble $\epsilon_n(R)$ satisfies $\lim_{n\to\infty}\epsilon_n(R) = P_{\mathrm{out}}(R)$, where the outage probability $P_{\mathrm{out}}(R)$ is given by

$$P_{\mathrm{out}}(R) = \mathbb{P}[I(q) < R]. \tag{2}$$

Similarly, for a fixed error probability ϵ, the maximum achievable rate $R_n(\epsilon)$ satisfies $\lim_{n\to\infty} R_n(\epsilon) = C_\epsilon$, where the outage capacity C_ϵ is related to $P_{\mathrm{out}}(R)$ through

$$C_\epsilon = \sup\{R : P_{\mathrm{out}}(R) \le \epsilon\}. \tag{3}$$

III. An Error Probability Expansion

In order to derive (1), we study the random-coding union (RCU) bound to the error probability for a fixed rate R and codeword length n, i.e., we upper bound $\epsilon_n(R)$ by [11]

$$\mathrm{rcu}_n(R) = \mathbb{E}\big[\min\big\{1, (M-1)\mathrm{pep}_Q^n(X^n, Y^n)\big\}\big], \tag{4}$$

where $\mathrm{pep}_q^n(x^n, y^n)$ is the pairwise error probability, i.e., the probability that an independently randomly generated codeword has a likelihood decoding metric that exceeds the decoding metric of the true codeword, given by

$$\mathrm{pep}_q^n(x^n, y^n) = \mathbb{P}\big[i_q^n(\overline{X}^n; y^n) \ge i_q^n(x^n; y^n)\big], \tag{5}$$

where $i_q^n(x^n; y^n)$ is the information density defined as

$$i_q^n(x^n; y^n) = \sum_{i=1}^n \log\frac{W_q(y_i|x_i)}{\mathbb{E}\big[W_q(y_i|\overline{X}_i)\big]}. \tag{6}$$

In (6), we used the fact that the channel input symbols are identically and independently distributed (i.i.d.). For the BSC, the transition probability distribution is $W_q(y|x) = 1 - q$ for $x = y$ and $W_q(y|x) = q$ for $x \neq y$. Hence,

$$i_q^n(x^n; y^n) = n \log(2 - 2q) + d \log \frac{q}{1-q}, \tag{7}$$

where $d = w_H(x^n \oplus y^n)$ is the Hamming distance between x^n and y^n. Since the information density depends on the channel input sequence and the channel output sequence only through their Hamming distance d, we may indistinguishably write $i_q^n(d)$ and $\mathrm{pep}_q^n(d)$.

A. Saddlepoint Approximation

Let $\ell = w_H(\overline{x}^n \oplus y^n)$ be the Hamming distance between $\overline{x}^n$ and y^n. From (7), we observe that $i_q^n(\ell)$ lies in a lattice of span $\gamma = \log \frac{1-q}{q}$. The tail probability (5) can be written in terms of the inverse Laplace transformation [12] as

$$\mathrm{pep}_q^n(d) = \sum_{\ell=0}^{d} \frac{\gamma}{2\pi j} \int_{\hat{\tau} - j\frac{\pi}{\gamma}}^{\hat{\tau} + j\frac{\pi}{\gamma}} e^{\kappa(\tau) - \tau i_q^n(\ell)} \, \mathrm{d}\tau, \tag{8}$$

where $\hat{\tau}$ is in the region of convergence, and $\kappa(\tau)$ is the cumulant generating function of $i_q^n(\ell)$ for a fixed channel output y^n and crossover probability q, i.e.,

$$\kappa(\tau) = \log \mathbb{E}_{\overline{X}^n}\left[e^{\tau i_q^n(\overline{X}^n; y^n)}\right]. \tag{9}$$

Under the capacity achieving distribution, it is given as

$$\kappa(\tau) = n\tau \log(2 - 2q) + n \log\left(\frac{1}{2} + \frac{1}{2}\left(\frac{q}{1-q}\right)^{\tau}\right). \tag{10}$$

We note that $\kappa(\tau)$ does not depend on y^n. We now approximate the complex integration in (8) by extending the limits of the integration and by expanding the cumulant generating function $\kappa(\tau)$ around $\hat{\tau}$, i.e.,

$$\int_{\hat{\tau} - j\frac{\pi}{\gamma}}^{\hat{\tau} + j\frac{\pi}{\gamma}} e^{\kappa(\tau) - \tau i_q^n(\ell)} \, \mathrm{d}\tau \simeq \int_{\hat{\tau} - j\infty}^{\hat{\tau} + j\infty} e^{\kappa(\hat{\tau}) + \kappa'(\hat{\tau})(\tau - \hat{\tau}) + \frac{1}{2}\kappa''(\hat{\tau})(\tau - \hat{\tau})^2 - \tau i_q^n(\ell)} \, \mathrm{d}\tau, \tag{11}$$

where $\kappa'(\tau)$ and $\kappa''(\tau)$ are the fist and second derivatives of $\kappa(\tau)$, respectively. With the change of variable $\tau = \hat{\tau} + j\tau_i$, equation (11) can be expressed as

$$\int_{\hat{\tau} - j\frac{\pi}{\gamma}}^{\hat{\tau} + j\frac{\pi}{\gamma}} e^{\kappa(\tau) - \tau i_q^n(\ell)} \, \mathrm{d}\tau \simeq e^{\kappa(\hat{\tau}) - \hat{\tau} i_q^n(\ell)} \int_{-\infty}^{\infty} e^{-j\tau_i (i_q^n(\ell) - \kappa'(\hat{\tau})) - \frac{1}{2}\kappa''(\hat{\tau})\tau_i^2} \, j\mathrm{d}\tau_i. \tag{12}$$

Solving the integral in τ_i and putting the resulting expression of (12) back into (8), we obtain the following saddlepoint approximation to the pairwise error probability

$$\mathrm{pep}_q^n(d) \simeq \frac{e^{\kappa(1) - i_q^n(d)}}{\sqrt{2\pi\kappa''(1)}} \sum_{\ell=0}^{d} \gamma \, e^{-(i_q^n(\ell) - i_q^n(d)) - \frac{(i_q^n(\ell) - \kappa'(1))^2}{2\kappa''(1)}}, \tag{13}$$

where we have chosen $\hat{\tau} = 1$ for all values of d. Neglecting the $(i_q^n(\ell) - \kappa'(1))^2$ terms in (13) would lead to [9, Eq. (131)], an approximation to the RCU that is not refined enough for quasi-static channels since the error probability converges to the outage probability and not to zero.

Combining the following equalities

$$\kappa(1) = 0 \tag{14}$$

$$\kappa'(1) = n \log(2 - 2q) - nq\gamma \tag{15}$$

$$\kappa''(1) = nq(1-q)\gamma^2 \tag{16}$$

$$i_q^n(\ell) - i_q^n(d) = \gamma(d - \ell) \tag{17}$$

$$i_q^n(\ell) - \kappa'(1) = \gamma(nq - \ell) \tag{18}$$

with (14)–(16), (13) can be written as

$$\mathrm{pep}_q^n(d) \simeq \frac{k_q^n(d)}{\sqrt{n}} e^{-i_q^n(d)}, \tag{19}$$

where

$$k_q^n(d) = \frac{1}{\sqrt{2\pi V(q)}} \sum_{\ell=0}^{d} \gamma e^{-\gamma(d-\ell) - \frac{\gamma^2 (nq - \ell)^2}{2nV(q)}}, \tag{20}$$

and $V(q) = q(1-q)\gamma^2$ is the variance of the information density thanks to the symmetry of the channel.

B. Asymptotic Expansion

Inserting the saddlepoint approximation (19) into the RCU (4), we obtain

$$\mathrm{rcu}_n(R) \simeq \mathbb{E}\left[\min\left\{1, \frac{M-1}{\sqrt{n}} k_Q^n(D) e^{-i_Q^n(D)}\right\}\right], \tag{21}$$

where d, the Hamming distance between x^n and y^n, has a binomial distribution of n trials with probability of success q. Hence, the expectation in (21) is w.r.t. the joint probability distribution $p_{DQ}(d, q) = p_Q(q) p_{D|Q}(d|q)$ where

$$p_{D|Q}(d|q) = \binom{n}{d} (1-q)^{(n-d)} q^d. \tag{22}$$

Let U be a random variable uniformly distributed in the unit interval. Then, for any random variable A, $\mathbb{E}[\min\{1, A\}] = \mathbb{P}[A \geq U]$. Using this identity on (21), taking logarithms inside the probability and multiplying every term by $\frac{1}{n}$, the RCU bound can be expressed as the following tail probability

$$\mathrm{rcu}_n(R) \simeq \mathbb{P}\left[R - \frac{1}{n} i_Q^n(D) + \frac{1}{n}\left(\log k_Q^n(D) - \frac{1}{2}\log n - \log U\right) \geq 0\right], \tag{23}$$

where we also used that $\frac{1}{n}\log(M-1) \simeq R$.

We note that as $n \to \infty$, the term $R - \frac{1}{n} i_q^n(d)$ does not tend to zero, and measures the probability that the intended rate R exceeds the normalized information density of the channel for fixed length n. An expansion to (23) was partially addressed in [6], [7] for a weakened version of the RCU bound based on the Markov's inequality and the inverse Laplace method, and a similar tail probability was also addressed in [5], [13] for the meta-converse and $\kappa\beta$ bounds. Here, we propose a third method to expand the RCU bound as follows.

First, we write (23) as

$$\mathrm{rcu}_n(R) \simeq \mathbb{P}\left[A_n + \frac{1}{\sqrt{n}}B_n + \frac{1}{n}C_n \geq 0\right] \tag{24}$$

where the random sequences A_n, B_n and C_n are given by

$$A_n = R - I(Q) \tag{25}$$

$$B_n = \sqrt{n}I(Q) - \frac{1}{\sqrt{n}}i_Q^n(D) \tag{26}$$

$$C_n = \log k_Q^n(D) - \frac{1}{2}\log n - \log U. \tag{27}$$

By writing the tail probability in (24) in terms of the indicator function as $\mathbb{E}[\mathbb{1}\{A_n + \nu B_n + \nu^2 C_n \geq 0\}]$, finding the Taylor series at ∞ in inverse powers of $\sqrt{n}$, and using that the derivative of the indicator function is a Dirac delta function, we obtain (28) at the bottom of the page, where $\delta(x)$ is the Dirac delta function, and $\delta'(x)$ is its derivative. In the following, we study the four terms of the right hand side of (28).

The first term leads to the outage probability, i.e.,

$$\mathbb{P}[A_n \geq 0] = \mathbb{P}\big[R - I(Q) \geq 0\big] \tag{29}$$

$$= P_{\mathrm{out}}(R). \tag{30}$$

The second term, the convergence to the outage probability in $\frac{1}{\sqrt{n}}$, is strictly zero, since

$$\begin{aligned}
&\mathbb{E}[\delta(A_n)B_n] \\
&\quad= \mathbb{E}\left[\delta\big(R - I(Q)\big)\left(\sqrt{n}I(Q) - \frac{1}{\sqrt{n}}i_Q^n(D)\right)\right] &(31)\\
&\quad= \mathbb{E}\left[\delta\big(R - I(Q)\big)\left(\sqrt{n}I(Q) - \sqrt{n}I(Q)\right)\right] &(32)\\
&\quad= 0, &(33)
\end{aligned}$$

where we used that the expectation of $i_Q^n(D)$ w.r.t. the conditional probability distribution $P_{D|Q}$ gives $nI(Q)$.

Using the following identity on the derivative of the Dirac delta function [14, p. 185]

$$\int \delta'\big(f(x)\big)g(x)\,\mathrm{d}x = -\sum_{x_0\,:\,f(x_0)=0} \frac{1}{|f'(x_0)|}\frac{\partial}{\partial x}\frac{g(x)}{f'(x)}\bigg|_{x=x_0}, \tag{34}$$

the next term is given by

$$\begin{aligned}
&\mathbb{E}[\delta'(A_n)B_n^2] \\
&\quad= \mathbb{E}\left[\delta'\big(R - I(Q)\big)\left(\sqrt{n}I(Q) - \frac{1}{\sqrt{n}}i_Q^n(D)\right)^2\right] &(35)\\
&\quad= \mathbb{E}\left[\delta'\big(R - I(Q)\big)V(Q)\right] &(36)\\
&\quad= \mathbb{E}\left[\delta'\left(\frac{R - I(Q)}{\sqrt{V(Q)}}\right)\right] &(37)\\
&\quad= -p'_{\frac{R-I(Q)}{\sqrt{V(Q)}}}(0), &(38)
\end{aligned}$$

where $V(Q)$ is the variance of the information density. The term in (38), the derivative of the probability density function of a normalized rate, was also identified in [13, Eq. (53)], and it can be computed as

$$p'_{\frac{R-I(Q)}{\sqrt{V(Q)}}}(0) = \left(\frac{1}{g_2'(q_0)}\right)^2 p_Q'(q_0) - \frac{g_2''(q_0)}{g_2'(q_0)^3}p_Q(q_0), \tag{39}$$

where the function $g_2(q)$ is given by

$$g_2(q) = \frac{R - \log 2 + h(q)}{\sqrt{q(1-q)}\log\frac{1-q}{q}}, \tag{40}$$

and $q_0 = h^{-1}(\log 2 - R)$.

Finally, the fourth term of the right hand side of (28) is the contribution of the three terms of C_n, given in (27), i.e.,

$$\begin{aligned}
&\mathbb{E}[\delta(A_n)C_n] \\
&\quad= \mathbb{E}\left[\delta\big(R - I(Q)\big)\left(\log k_Q^n(D) - \frac{1}{2}\log n - \log U\right)\right] &(41)\\
&\quad= p_{R-I(Q)}(0)\left(\mathbb{E}\big[\log k_{q_0}^n(D)\big] - \frac{1}{2}\log n + 1\right), &(42)
\end{aligned}$$

where the expectation in (42) is w.r.t. the conditional distribution $P_{D|Q=q_0}$, and we used $\mathbb{E}[\log U] = -1$ and [14, p. 184]

$$\int \delta\big(f(x)\big)g(x)\,\mathrm{d}x = \sum_{x_0\,:\,f(x_0)=0}\frac{g(x_0)}{|f'(x_0)|}. \tag{43}$$

In (42), $p_{R-I(Q)}$ is the probability density function of the random variable $R - I(Q)$, and it can be computed as

$$p_{R-I(Q)}(0) = \frac{p_Q(q_0)}{g_1'(q_0)} \tag{44}$$

where the function $g_1(q)$ is given by

$$g_1(q) = R - \log 2 + h(q). \tag{45}$$

In the Appendix, we show that the term $\mathbb{E}\big[\log k_{q_0}^n(D)\big]$ in (42) can be further expanded, as $n \to \infty$, as

$$\mathbb{E}\big[\log k_{q_0}^n(D)\big] \simeq \log\frac{\gamma(1-e^{-\gamma})^{-1}}{\sqrt{2\pi e V(q_0)}} \tag{46}$$

Placing the probabilistic results obtained in this section, namely equations (30), (33), (38), (42) and (46), back into the expansion (28) given at the bottom of the page, we obtain that the RCU bound to the error probability expands as

$$\mathrm{rcu}_n(R) \simeq P_{\mathrm{out}}(R) + \frac{\log n}{n}\phi_{\log}(R) + \frac{1}{n}\phi_0(R), \tag{47}$$

where the $\frac{\log n}{n}$ factor $\phi_{\log}(R)$ is

$$\phi_{\log}(R) = -\frac{1}{2}p_{R-I(Q)}(0), \tag{48}$$

$$\mathbb{P}\left[A_n + \frac{1}{\sqrt{n}}B_n + \frac{1}{n}C_n \geq 0\right] \simeq \mathbb{P}[A_n \geq 0] + \frac{1}{\sqrt{n}}\mathbb{E}[\delta(A_n)B_n] + \frac{1}{2n}\mathbb{E}[\delta'(A_n)B_n^2] + \frac{1}{n}\mathbb{E}[\delta(A_n)C_n] \tag{28}$$

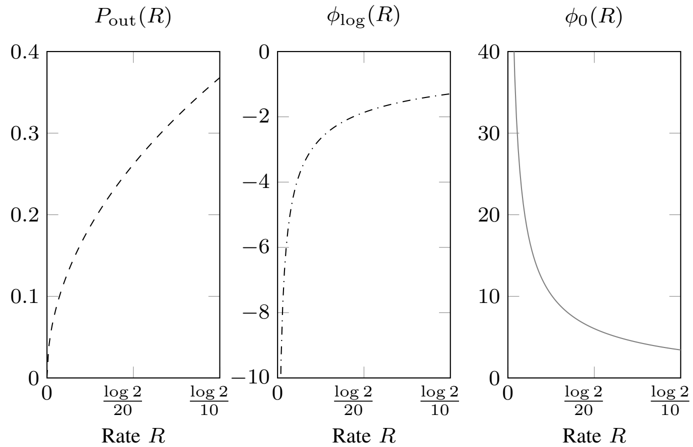

Fig. 1. Expansion terms $P_{\rm out}(R)$, $\phi_{\log}(R)$ and $\phi_0(R)$.

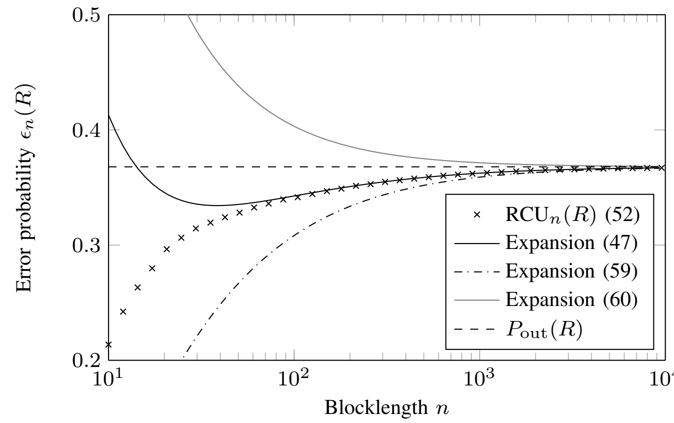

Fig. 2. Error probability versus codeword length n for $R = \frac{1}{10}\log 2$.

and the $\frac{1}{n}$ factor $\phi_0(R)$ is given by

$$\phi_0(R) = p_{R-I(Q)}(0)\left(1+\log\frac{\gamma(1-e^{-\gamma})^{-1}}{\sqrt{2\pi e V(q_0)}}\right) - \frac{1}{2}p'_{\frac{R-I(Q)}{\sqrt{V(Q)}}}(0), \tag{49}$$

where q_0 is the crossover probability satisfying $I(q_0) = R$.

Finally, by expanding all the terms in (47) around the outage capacity $R = C_\epsilon$, we obtain the following expansion

$$\begin{aligned}\mathrm{rcu}_n(R) \simeq P_{\rm out}(C_\epsilon) + \frac{\log n}{n}\phi_{\log}(C_\epsilon) + \frac{1}{n}\phi_0(C_\epsilon) + \\ (R - C_\epsilon)\Big(P'_{\rm out}(C_\epsilon) + \frac{\log n}{n}\phi'_{\log}(C_\epsilon) + \frac{1}{n}\phi'_0(C_\epsilon)\Big),\end{aligned} \tag{50}$$

where $P'_{\rm out}(R)$, $\phi'_{\log}(R)$ and $\phi'_0(R)$ denote the derivatives of $P_{\rm out}(R)$, $\phi_{\log}(R)$ and $\phi_0(R)$ w.r.t. the rate R. Isolating the rate R in the r.h.s. of (50) around $\mathrm{rcu}_n(R) = \epsilon$, and further expanding in inverse powers of the blocklength n, we obtain that the achievable rate $R_n(\epsilon)$ can be expanded as

$$R_n(\epsilon) \simeq C_\epsilon + \frac{1}{2}\frac{\log n}{n} - \frac{1}{n}\frac{\phi_0(C_\epsilon)}{p_{C_\epsilon - I(Q)}(0)}, \tag{51}$$

where we have used that $P'_{\rm out}(C_\epsilon) = p_{C_\epsilon - I(Q)}(0)$. As reported in [5], we recover a zero channel dispersion, i.e., there is no term proportional to $\frac{1}{\sqrt{n}}$. In addition, we obtain two additional terms in the expansion of the rate. The $\frac{\log n}{n}$ factor further coincides with the third-order term of the normal approximation of the BSC [11, Eq. (289)].

IV. Numerical Example

In this section, we consider a uniformly distributed crossover probability q within $q \in (0, \frac{1}{2})$.

The RCU for the BSC (4)–(7) can be computed analytically [11, Eq. (162)]. Averaging it over q, we obtain

$$\begin{aligned}\mathrm{rcu}_n(R) = \sum_{t=0}^{n}\binom{n}{t}2B_{\frac{1}{2}}(1+t, 1+n-t) \\ \min\left\{1, (M-1)\sum_{k=0}^{t}\binom{n}{k}2^{-n}\right\},\end{aligned} \tag{52}$$

where $B_z(a,b)$ is the incomplete Beta function, given by

$$B_z(a,b) = \int_0^z u^{a-1}(1-u)^{b-1}\,\mathrm{d}u. \tag{53}$$

Under uniformly distributed crossover probability, (44) and (39) are respectively given by

$$p_{R-I(Q)}(0) = \frac{2}{\log\frac{1-q_0}{q_0}} \tag{54}$$

$$p'_{\frac{R-I(Q)}{\sqrt{V(Q)}}}(0) = 2\left(1 - 2q_0 - \frac{1}{\log\frac{1-q_0}{q_0}}\right). \tag{55}$$

Hence, the terms of the expansion (47) can be computed as

$$P_{\rm out}(R) = 1 - 2q_0 \tag{56}$$

$$\phi_{\log}(R) = \frac{1}{\log\frac{q_0}{1-q_0}} \tag{57}$$

$$\phi_0(R) = 2q_0 + \frac{2 - 2\log(1-2q_0) - \log 2\pi}{\log\frac{1-q_0}{q_0}}. \tag{58}$$

For sake of comparison, we also include an expansion of the RCU neglecting the third term, i.e.,

$$\mathrm{rcu}_n(R) \simeq P_{\rm out}(R) + \frac{\log n}{n}\phi_{\log}(R), \tag{59}$$

as well as a weakened version of the RCU bound based on the Markov's inequality, i.e., setting $\phi_{\log}(R) = 0$ and $k_q^n(d) = 1$ (see [7] for the details)

$$\mathrm{rcu}_n(R) \simeq P_{\rm out}(R) + \frac{1}{n}\Big(p_{R-I(Q)}(0) - \frac{1}{2}p'_{\frac{R-I(Q)}{\sqrt{V(Q)}}}(0)\Big). \tag{60}$$

The behavior of the three terms of the expansion, namely $P_{\rm out}(R)$, $\phi_{\log}(R)$ and $\phi_0(R)$, is illustrated in Fig. 1 as a function of the rate R, whereas we compare expansion (47) with the exact RCU (52) and the partial expansions (59) and (60), at a rate $R = \frac{1}{10}\log 2$, in Fig. 2. It can be appreciated that expansion (47) is an accurate approximation to the RCU, even at small codeword lengths, even though it fails to capture the ripples of the RCU (see [11, Fig. 2]), as both $\phi_{\log}(R)$ and $\phi_0(R)$ are asymptotic terms.

APPENDIX

In this Appendix, we study the following limit

$$\lim_{n\to\infty} \mathbb{E}[\log k_q^n(D)]. \tag{61}$$

The Lebesgue dominated convergence theorem [15] leads to $\lim_{n\to\infty} \mathbb{E}[\log k_q^n(D)] = \mathbb{E}[\log \lim_{n\to\infty} k_q^n(D)]$. Hence, we concentrate in $k_q^n(d)$ as $n \to \infty$. From (20), we first rewrite $k_q^n(d)$ as the following summation

$$k_q^n(d) = \frac{1}{\sqrt{2\pi V}} \sum_{\ell=0}^{d} f(\ell), \tag{62}$$

where the summands $f(\ell)$ are given by

$$f(\ell) = \gamma e^{-\gamma(d-\ell) - \frac{\gamma^2(nq-\ell)^2}{2nV}}. \tag{63}$$

Using the Euler-Maclaurin summation formula [16, Eq. (25.4.7)], the summation (62) can be expressed as a function of its integration and a linear combination of the m-th derivatives of $f(\ell)$, denoted as $f^{(m)}(\ell)$, by

$$\sum_{\ell=0}^{d} f(\ell) = \int_0^d f(\ell)\,\mathrm{d}\ell + \sum_{m=1}^{\infty} \frac{B_m^+}{m!}\left(f^{(m-1)}(d) - f^{(m-1)}(0)\right), \tag{64}$$

where B_m^+ denote the original Bernoulli numbers $B_0^+ = 1$, $B_1^+ = \frac{1}{2}$, $B_2^+ = \frac{1}{6}$, $B_3^+ = 0$, $B_4^+ = -\frac{1}{30}$, etc, defined as $B_m^+ = B_m(1)$ through the generating function [16, Eq. (23.1.1)]

$$\sum_{m=0}^{\infty} \frac{B_m(x)}{m!} t^m = \frac{te^{xt}}{e^t - 1}. \tag{65}$$

The integration term leads to

$$\int_0^d f(\ell)\,\mathrm{d}\ell = \sqrt{2\pi nV} e^{\frac{nV}{2}} e^{-(d-nq)\gamma} \times \left(Q\left(\frac{nq\gamma + nV}{\sqrt{nV}}\right) - Q\left(\frac{nq\gamma + nV - d\gamma}{\sqrt{nV}}\right)\right), \tag{66}$$

where $Q(x)$ is the tail probability of the standard normal distribution, i.e.,

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{u^2}{2}}\,\mathrm{d}u. \tag{67}$$

Making the change of variable $d = \sqrt{n}a + nq$ in (66) and taking the limit, it can be shown that

$$\lim_{n\to\infty} \int_0^d f(\ell)\,\mathrm{d}\ell \Big|_{d=\sqrt{n}a+nq} = e^{-\frac{a^2\gamma^2}{2V}}. \tag{68}$$

Similarly,

$$\lim_{n\to\infty} \left(f^{(m-1)}(d) - f^{(m-1)}(0)\right)\Big|_{d=\sqrt{n}a+nq} = e^{-\frac{a^2\gamma^2}{2V}} \gamma^m, \tag{69}$$

for $m \geq 1$.

Hence,

$$\lim_{n\to\infty} k_q(d)\Big|_{d=\sqrt{n}a+nq} = \frac{e^{-\frac{a^2\gamma^2}{2V}}}{\sqrt{2\pi V}} \sum_{m=0}^{\infty} \frac{B_m^+}{m!}\gamma^m \tag{70}$$

$$= \frac{e^{-\frac{a^2\gamma^2}{2V}}}{\sqrt{2\pi V}} \frac{\gamma}{1 - e^{-\gamma}}, \tag{71}$$

where in (70) we identified the polynomial generating function (65) at the point $t = \gamma$.

Finally, the expectation w.r.t. the random variable A of the logarithm of (71) leads to

$$-\frac{\gamma^2}{2V}\mathbb{E}[A^2] + \log\frac{\gamma(1-e^{-\gamma})^{-1}}{\sqrt{2\pi V}} = \log\frac{\gamma(1-e^{-\gamma})^{-1}}{\sqrt{2\pi e V}}, \tag{72}$$

where we used the fact that $V = q(1-q)\gamma^2$ and

$$\mathbb{E}[A^2] = \mathbb{E}\left[\left(\frac{D - nq}{\sqrt{n}}\right)^2\right] \tag{73}$$

$$= q(1-q), \tag{74}$$

concluding the proof.

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