Irregular Turbo Codes in Block-Fading Channels

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***Abstract*— We study irregular binary turbo codes over non-ergodic block-fading channels. We first propose an extension of channel multiplexers initially designed for regular turbo codes. We then show that, using these multiplexers, irregular turbo codes that exhibit a small decoding threshold over the ergodic Gaussian-noise channel perform very close to the outage probability on block-fading channels, from both density evolution and finite-length perspectives.**

I. Introduction

The block-fading channel is a simplified channel model that characterizes delay-constrained communication over slowly-varying fading channels [1], [2], [3]. The received signal at block c is given by

$$\boldsymbol{y}_c = \alpha_c \boldsymbol{x}_c + \boldsymbol{w}_c \qquad c = 1, \ldots, n_c \tag{1}$$

where $\boldsymbol{x}_c, \boldsymbol{y}, \boldsymbol{w}_c \in \mathbb{R}^L$ are the input, output and noise vectors at block $c = 1, \ldots, n_c$, and L is the block length. The noise components have zero mean and variance N_0, and α_c is the Rayleigh fading coefficient of block c, assumed to be perfectly known to the receiver. Particular instances of the block-fading channel are orthogonal-frequency multiplexing modulation (OFDM) and frequency-hopping systems, such as mobile data communications in EDGE/3G and WiMax/LTE environments. Despite its simplification, it captures the essential characteristics of delay-constrained wireless communication and yields useful code design criteria. Since this channel is nonergodic, it has zero capacity and the fundamental limit is the outage probability [1], [2]. It has been shown in [4] that the diversity of binary codes of rate R_c over an n_c-block fading channel is given by the Singleton bound

$$\delta = 1 + \lfloor n_c(1 - R_c) \rfloor. \tag{2}$$

The design of binary linear codes for the block-fading channel has been studied in [5], [6], [4], [7], [8], [9]. However, these binary regular codes cannot perform closer than 1 dB from the outage probability. As shown in [8], [9], the effective design procedure for outage-approaching codes follows a two-step process:

1) Design block-wise maximum distance separable (MDS) codes, that achieve the largest possible diversity given by the Singleton bound in the block-erasure channel [10];
2) Reducing the decoding threshold in the AWGN channel.

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In this paper, we design irregular binary turbo codes [11] for block-fading channels. Based on the h-π-diagonal multiplexer [7] we design irregular turbo codes with full diversity. We then find irregular turbo codes with low decoding thresholds over the AWGN. We show that the resulting codes perform within 0.5 dB from the outage probability in both density evolution and finite-length cases, achieving the current best performance reported in the literature.

The organization of the paper is as follows. In Section II, we describe the structure and density evolution of irregular turbo codes. The specific block-fading design and density evolution are described in Section III. Section IV gives the concluding remarks.

II. Basics on Irregular Turbo Codes

In regular parallel turbo codes, the two constituent recursive systematic convolutional (RSC) encoders are identical (*i.e.* same constraint length and generator polynomials) [12]. This is equivalent to merging the two constituent encoders into a single one, and doubling the size of the interleaver. To do so, a 2-fold repeater is added before the interleaver Π, and we obtain a self-concatenated turbo code [13][14] as shown in Fig. 1. In this representation, each information bit is connected to the code trellis via two edges. We hence say that the *degree* of the information bits is $d = 2$ as shown in the propagation tree in Fig. 2, and that the turbo code is *regular*. Using this structure, one can create irregularity by repeating a certain fraction f_i of information bits i times, inducing larger protection for some bits than in the regular case [15]. Like for low-density parity check (LDPC) codes [16], [17], irregularity can enhance the performance of turbo codes for large block lengths [11], [15], [18], [19]. The encoder of an irregular turbo code is similar to that of Fig. 1, with the difference that the information bit stream is fed to a non-uniform repeater that divides the information bits into d classes with $d = 2, ..., d_{\max}$, where $d_{\max}$ is the maximum bit-node degree [11]. The number of bits in a class d is a fraction f_d of the total number of information bits at the turbo encoder input, knowing that bits in class d are repeated d times. Finally, the output of the non-uniform repeater is interleaved and fed to the RSC constituent code. In order to ensure a target rate, puncturing is used, and only a fraction $1 - f_p$ of parity bits are transmitted, where f_p is the fraction of punctured parity bits. Now let K denote the length of the information sequence, N the interleaver size, ρ the rate of the RSC constituent code, and R_c the rate of the turbo code. We have the following

$$\sum_{d=2}^{d_{\max}} f_d = 1, \quad \sum_{d=2}^{d_{\max}} d \cdot f_d = \overline{d}, \tag{3}$$

$$N = \sum_{d=2}^{d_{\max}} d \cdot (f_d K) = K \cdot \overline{d}, \tag{4}$$

$$R_c = \frac{K}{K + \frac{N}{\rho} - N} = \frac{1}{1 + \left(\frac{1}{\rho} - 1\right)\overline{d}}, \tag{5}$$

$$\rho = \frac{1}{1 + (1 - f_p)\left(\frac{1}{\rho_0} - 1\right)}, \tag{6}$$

where $\rho_0 = k/n$ is the initial rate of the constituent RSC code before puncturing, and $\overline{d}$ is the average degree of information bits. Similar to LDPC codes, the degree distribution from an edge perspective is defined by

$$\lambda_d = \frac{d \cdot f_d}{\overline{d}}, \quad d = 2 \ldots d_{\max}. \tag{7}$$

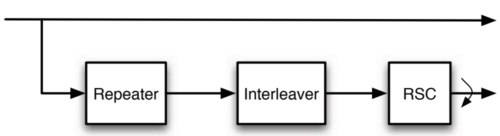

Fig. 1. Systematic self-concatenated turbo encoder. Information bits are sent directly over the channel, and parity bits are generated by first repeating information bits, interleaving, and then recursive systematic convolutional (RSC) encoding.

A. Density Evolution in AWGN

We consider rate-R_c irregular turbo codes built from a rate-ρ RSC constituent code and degree profile $\{f_d\}_{d=2,\ldots,d_{\max}}$. Due to the symmetry of the channel, we assume that the all-zero codeword is modulated into $\boldsymbol{x} = +1, +1, ..., +1$ and transmitted over an AWGN channel with noise variance N_0. At the channel output, each received sample can be written as $y = x + w = 1 + w$, so the log-likelihood ratio (LLR) is given by the well-known expression:

$$\mathcal{M}_0 = \log \frac{p(y|x=+1)}{p(y|x=-1)} = \frac{2}{N_0} y = \frac{2}{N_0}(1+w). \tag{8}$$

We have $\mathcal{M}_0 \sim \mathcal{N}(\frac{2}{N_0}, \frac{4}{N_0})$, the associated probability density function will be denoted by $p_0(x)$.

The local neighborhood tree for an information bit belonging to an acyclic asymptotically large irregular turbo code is shown in Fig. 2. The index i refers to the decoding iteration number. A bitnode of degree d has $d-1$ incoming extrinsic probabilities ξ_i and one outgoing *a priori* probability $\pi_{d,i}$ which also plays the role of a partial *a posteriori* probability (APP). The total APP may be obtained by combining $\pi_{d,i}$ with an extra extrinsic probability. The message associated to ξ_i is $\mathcal{M}_i = \frac{\log(\xi_i(\text{bit}=0))}{\log(\xi_i(\text{bit}=1))}$ and its probability density function is $p_{\mathcal{M}_i}(x)$. Given d and i, the probability density function of log-ratio messages associated to $\pi_{d,i}$ will be denoted by $p_{d,i}(x)$. Following [19] we have that

$$p_{d,i}(x) = \mathcal{F}^{-1}\left[\mathcal{F}\left[p_0(x)\right]\mathcal{F}^{d-1}\left[p_{\mathcal{M}_i}(x)\right]\right] \tag{9}$$

where $\mathcal{F}$ denotes the Fourier transform operator. Based on partial a posteriori probabilities, the average bit error probability at iteration i is defined as

$$P_b(i) = \sum_{d=2}^{d_{\max}} f_d P_b(d,i) \tag{10}$$

where $P_b(d,i)$ is the bit error probability of class d given by the area under the tail of $p_{d,i}(x)$.

At an RSC checknode level as illustrated in Fig. 2, based on a priori input π_{i-1} with pdf $p_{i-1}(x)$, an accurate estimation of $p_{\mathcal{M}_i}(x)$ is made via a forward-backward algorithm [20] applied on a sufficiently large trellis window of size W centered around the information bit. Since we are dealing with random ensembles of irregular turbo codes, we have

$$p_i(x) = \sum_{d=2}^{d_{\max}} \lambda_d \; p_{d,i}(x). \tag{11}$$

Given an irregular turbo ensemble, its decoding *threshold* is the minimal signal-to-noise ratio E_b/N_0 for which $P_b(i)$ vanishes with i. The threshold can be determined via Density Evolution (DE) [19], a procedure where $p_i(x)$ is updated from $p_{i-1}(x)$ by propagating probabilistic densities through the tree graph of Fig. 2.

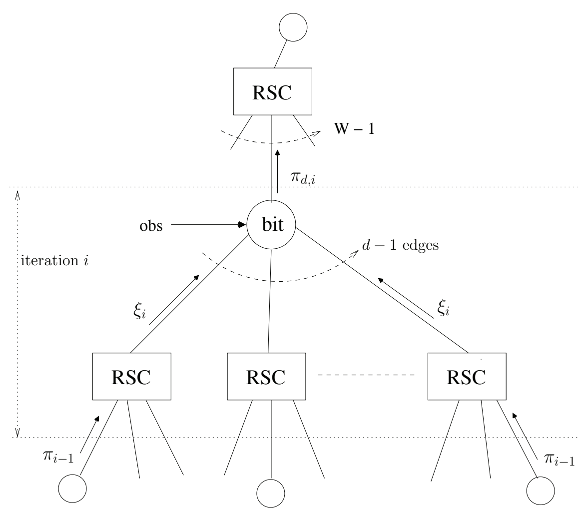

Fig. 2. Propagation tree used in density evolution for an irregular turbo code. The π_i represents *a priori* probability, and the ξ_i the extrinsic probability. Circles represent bitnodes, and rectangles are local neighborhood RSC trellis constraints.

B. Numerical results for AWGN

The DE method gives the limiting convergence behavior of capacity-approaching codes, and it is used to find optimal degree profiles for LDPC codes in [16], [19], [21]. By setting the average degree to be $\overline{d} = 3$ and using the RSC $(13, 15)_8$ constituent code, we obtained powerful half-rate irregular turbo codes with different degree profiles; for example by taking $f_2 = 0.9$, $f_9 = 0.04$, and $f_{15} = 0.06$, the threshold is 0.31 dB. The distributions $f_2 = 0.923$ and $f_{15} = 0.077$ or $f_2 = 0.9$ and $f_{12} = 0.1$, yield a 0.36 dB threshold. Recall that Shannon limit for half-rate coding over the AWGN channel is approximately 0.18 dB. The irregular turbo code defined by $f_2 = 0.9$ and $f_{12} = 0.1$ is used later in section III over the block-fading channel.

III. Irregular Turbo Codes over Block-Fading Channels

In [7], the authors proposed multiplexer design for regular parallel turbo codes that ensure full diversity and optimal coding gain. However, as the self-concatenated structure of the code involves only one constituent code, the generalization of the so-called h-π-diagonal multiplexers initially designed for regular parallel turbo codes is not straightforward. Without loss of generality, we restrict our design to irregular turbo codes over block-fading channels with $n_c = 2$ blocks and rate $R_c = 1/2$. The extension to block-fading channels with more fading blocks follows similar arguments but it is not discussed in this paper. Special care should be taken when designing a turbo code that achieves the Singleton bound without attaining full diversity, i.e., $n_c > \delta \geq 1/R_c$.

In an irregular turbo code with average degree $\overline{d}$, a bit is connected to the trellis of the code via $\overline{d}$ edges on average. Following the identity $N = K\overline{d}$, this can be seen as a "parallel" turbo code with β constituent codes, where:

$$\beta = \lceil \overline{d} \rceil \tag{12}$$

In order to achieve high coding gains, the h-π-diagonal multiplexer should be extended to irregular turbo codes. We consider constituent RSC codes with initial coding rate $\rho_0 = 1/2$. To keep the structure of the multiplexer, only half of the parity bits of the first RSC constituent code should be punctured, knowing that the overall rate R_c should remain fixed. Now let ϕ_p be the fraction of parity bits to be punctured from every RSC constituent code starting from the second one. We have that:

$$\phi_p = \frac{\beta f_p - \frac{1}{2}}{\beta - 1} \tag{13}$$

The general h-π-diagonal multiplexer is shown in Fig. 3, where b is the information bit, and s_j is the parity bit of constituent code j. As an example, we consider a half-rate irregular turbo code with $\beta = \overline{d} = 3$. This gives $f_p = 0.66$ and $\phi_p = 0.75$, so 3 parity bits out of 4 are punctured from both RSC 2 and RSC 3. Again, we consider a half-rate irregular turbo code with $\overline{d} = 2.727$. We get $f_p = 0.63$ and $\phi_p = 0.7$. The puncturing pattern is then slightly different from that of the previous example, as in a period of length 20, there is one more bit that is sent over the channel.

RSC 1 (information)	b	1	2	1	2
RSC 1 (parity)	s_1	2	X	2	X
RSC 2 (parity)	$\pi^{-1}(s_2)$	X	1/X	X	1/X
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
RSC β (parity)	$\pi^{-1}(s_\beta)$	X	1/X	X	1/X

Fig. 3. H-π-diagonal multiplexer of a half-rate irregular turbo code built from $\rho_0 = 1/2$ constituent RSC code. The number of rows is $\beta + 1$ where $\beta = \lceil \overline{d} \rceil$. One parity bit out of two is punctured from RSC 1. There is a fraction ϕ_p of punctured parity bits per row (represented by an X) starting from RSC 2.

A. Density evolution on BF channel

In this section we study the word error rate performance of half-rate irregular turbo codes over a two-state block-fading channel via density evolution. As with the AWGN channel, we assume that the all-zero codeword is modulated into $x = +1, +1, ..., +1$ and transmitted over a block-fading channel with n_c states ($n_c = 2$ in our case).

For a given fading instance $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, the irregular turbo code ensemble is observing two types of channel messages, $\mathcal{M}_{0,1} \sim \mathcal{N}(\frac{2\alpha_1}{N_0}, \frac{4\alpha_1^2}{N_0})$ and $\mathcal{M}_{0,2} \sim \mathcal{N}(\frac{2\alpha_2}{N_0}, \frac{4\alpha_2^2}{N_0})$, as in (8). DE is performed in a similar fashion as described in Section II.A after taking into account the multiplexing of bits (i.e. which channel assigned to which bit) as defined in Fig. 3. At a fixed SNR, it is possible to determine via DE whether the average bit error probability $P_b(i)$ vanishes with i or not. When $P_b(i) \nrightarrow 0$ as $i \rightarrow +\infty$, we say that a density evolution outage (DEO) occurs.

Now, let us define the following indicator function:

$$\mathbf{1}_{\text{DEO}}(\alpha) = \begin{cases} 0, & P_b(i) \rightarrow 0, \\ 1, & P_b(i) \nrightarrow 0. \end{cases} \tag{14}$$

The probability of a DEO is then given by

$$P_{\text{DEO}} = \int_{\boldsymbol{\alpha} \in \mathbb{R}^2} \mathbf{1}_{\text{DEO}}(\boldsymbol{\alpha}) p(\boldsymbol{\alpha}) d\boldsymbol{\alpha} = \int_{\boldsymbol{\alpha} \in V_o} p(\boldsymbol{\alpha}) d\boldsymbol{\alpha}, \tag{15}$$

where V_o is the outage region for the irregular turbo code ensemble under DE, i.e.,

$$V_o = \left\{ \boldsymbol{\alpha} \in \mathbb{R}_+^{n_c} \mid \mathbf{1}_{\text{DEO}}(\boldsymbol{\alpha}) = 1 \right\}. \tag{16}$$

The $(n_c - 1)$-dimensional surface separating V_o from its complementary in $\mathbb{R}_+^{n_c}$ is called the outage boundary. Thus, DE on a block-fading channel is a method to determine the outage boundary for a given turbo code ensemble at a given SNR. The information-theoretical boundary related to the outage probability is defined by the equality $C(\boldsymbol{\alpha}, E_b/N_0) = R_c$, where C is the channel capacity (or mutual information) under a certain type of input alphabet.

For an infinite-length code ensemble, it is easy to show that the word error probability P_{ew} satisfies [9]

$$P_{\text{DEO}} \leq P_{ew}. \tag{17}$$

Consequently, the outage probability found by DE is a lower bound for the word error probability and can be compared to the information outage probability. Equality in (17) occurs if the block threshold is equal to the bit threshold [22].

B. Numerical results on BF channel

Fig. 5 compares the outage boundary of regular and irregular turbo codes with the 8-state RSC$(13, 15)_8$ constituent code and h-π-diagonal multiplexing at $E_b/N_0 = 8$dB. The irregular turbo code is the best one from Section III-A, with a threshold of 0.31dB on the AWGN channel. The boundaries are computed by picking points orthogonal to the BPSK input outage. Although irregular and regular codes have similar performance for largely unbalanced fading pairs, the irregular turbo code performs better in the neighborhood of the ergodic line. It actually approaches the BPSK input outage border over a large range of fading pairs.

Fig. 6 shows the word error rate performance of the same codes and h-π-diagonal multiplexing under both density evolution and Monte Carlo simulation with $K = 6000$ bits. As we observe, both DE performance and finite-length are very close to the outage probability (within 0.5 dB). Note that, as observed in [4], [7], [8], [9], irregular turbo codes are good for the block-fading channel, in the sense that their performance is insensitive to the block length.

For finite length simulations, the repeater should be designed in a special way, as shown in Fig. 4. Bits are divided into two groups, and only the information bits of the first RSC are transmitted over the channel: circled bits are transmitted over the 1^{st} channel state, and non-circled bits are sent over the 2^{nd} channel state. To guarantee full diversity, the decoder should always find its way through the trellis of the code, thus bits corresponding to the same trellis transition should not be sent over the same channel state [7]. In order to ensure this property, bits of degree greater than 2 are placed in the H positions in the multiplexer of Fig. 4. Repetition is thus done in a way that if the 2nd channel state is unreliable, decoding can be successful through RSC 2 and RSC 3, and if the 1st channel state is unreliable, RSC 1 can decode the received codeword.

Note that although the density evolution convergence criterion is based on bit error probability, it is relevant to assume that the word error probability of irregular turbo codes has an equivalent decoding threshold under density evolution. In fact, it was shown in [22] that the word and bit error probability of certain LDPC codes, among which the class of Irregular Repeat-Accumulate (IRA) codes [23], have identical thresholds. Irregular turbo codes can be seen as IRA codes that are decoded iteratively using a different scheduling, that results from the difference between forward-backward and belief-propagation decoding.

IV. Conclusions

In this paper, we presented irregular turbo codes that are capable of closely approaching the outage probability of the block-fading channel, both in terms of density evolution (infinite length) and finite length. The design method is based on two steps. First, a suitable full-diversity multiplexer was designed. Second, codes were optimized over the AWGN channel through density evolution. This represents the best family of codes over the block-fading channel reported in the literature.

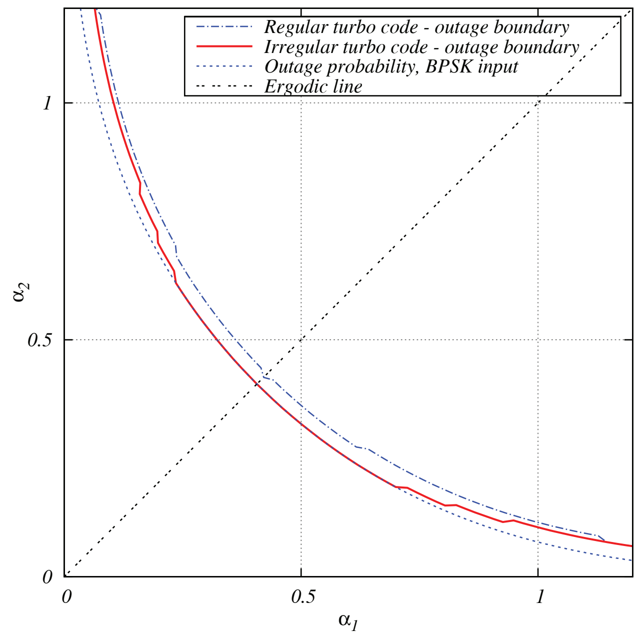

Fig. 5. Outage boundary of regular and irregular turbo codes under h-π-diagonal multiplexing and with the RSC $(13, 15)_8$ constituent code at $E_b/N_0 = 8$dB. Circles filled with crosses correspond to the fading pairs in which irregular turbo codes outperform regular codes. Although the two codes have similar performance with largely unbalanced fading pairs, the irregular code outperforms the regular code in the vicinity of the ergodic line.

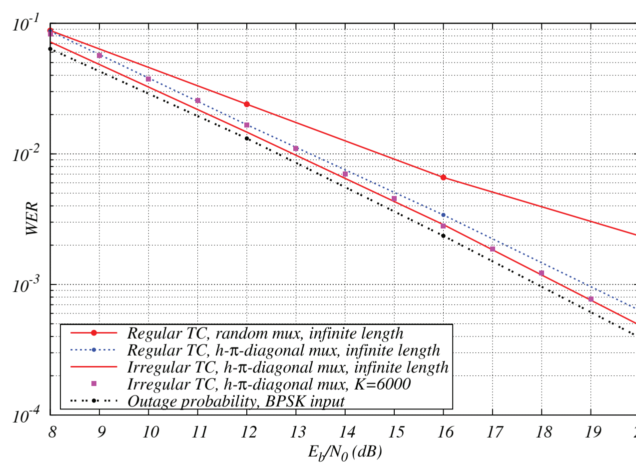

Fig. 6. Word error rate for $R_c = \frac{1}{2}$ turbo codes over the block-fading channel with $n_c = 2$, RSC $(13, 15)_8$ constituent code and BPSK modulation. Performance of codes is invariant with codeword length, and it was estimated using both the density evolution algorithm and Monte Carlo simulations.

	RSC 1						RSC 2						RSC 3					
I	(1)	2	(3)	4	(5)	6	(1)	2	(H)	(H)	(5)	6	(H)	(H)	(3)	4	(H)	(H)
P	p_1	X	p_3	X	p_5	X	X	(p_2)	X	X	X	(p_6)	X	X	X	(p_4)	X	X

Fig. 4. H-π-diagonal multiplexer of a half-rate irregular turbo code with $\overline{d} = 3$ transmitted on a 2-state block-fading channel using a punctured half-rate constituent RSC code. The irregular turbo encoder is built using 3 constituent encoders, where only the information bits (on the line labeled with I) of RSC 1 are transmitted over the channel. The bits p_i correspond to parity bits, the X represents punctured parity bits, and the bits labeled H correspond to bits with degree higher than 2. The circled bits are sent over the the 1^{st} channel state, the other bits are sent over the 2^{nd} state. In order to achieve full diversity, some of the circled information bits should be repeated more than twice and fed to RSC 2 and RSC 3.

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