Large System Analysis of Iterative Multiuser Joint Decoding with an Uncertain Number of Users

Adrià Tauste Campo
Department of Engineering
University of Cambridge
e-mail: adria.tauste@eng.cam.ac.uk

Albert Guillén i Fàbregas
Department of Engineering
University of Cambridge
e-mail: albert.guillen@eng.cam.ac.uk

***Abstract*— We study iterative multiuser joint decoding in large randomly spread code division multiple access systems under the assumption that the number of users accessing the channel is unknown by the receiver. In particular, we focus on the factor graph representation and iterative algorithms based on belief propagation. We study a suboptimal iterative scheme that jointly detects the encoded data and the users' activity. By using the replica method from statistical physics, we analyze the performance of the iterative detector. Using density evolution, we provide a fixed-point equation of the overall iterative system where the probability messages depend on the users' activity. Finally, when the scaling between the log number of users and the block length is below a threshold, we show that in the large-system limit a simple structure on the users' codes yields a multiuser efficiency fixed-point equation that is equivalent to the case of all-active users with a system load scaled by the activity rate.**

I. Introduction

The interplay between multiuser detection (MUD) and channel coding in multiple-access channels has been recently studied from different angles. The capacity region of the Gaussian multiple-access channel is known to be achievable by successive interference cancellation (IC) and single-user decoding [1], [2]. Practical approaches based on code division multiple access (CDMA) and iterative joint decoding have also been studied [3], [4], [5].

The authors in [4] provide a unified framework to analyze the performance of iterative multiuser joint decoding with CDMA in the limit for large block length and system dimensions. Their approach is based on a factor graph representation of the a posteriori probabilities (APP's) of the information symbols using belief propagation [6]. This characterization allows the derivation of iterative algorithms that approximate optimal maximum a posteriori (MAP) decoding. The asymptotic performance of belief propagation can be analyzed by using *density evolution* techniques [7]. Based on results from linear MUD for uncoded systems [8], [4] characterized the performance of large multiuser systems using suboptimal iterative IC and decoding with linear filtering.

Recalling the main result for the large-system analysis of optimal MUD for uncoded systems ([9, Prop. 1]), the authors in [5] provide a modified version by replica method that characterizes the large-system performance of the non-linear data iterative joint decoder. In contrast with previous work on uncoded CDMA [10], the system performance is determined by a *dynamic* fixed-point equation on the multiuser efficiency.

In this paper, we study large-system analysis of iterative multiuser joint decoding under the framework of [11], in which the number of users accessing the channel (among other parameters) is time-varying and unknown, and must be estimated together with the transmitted data. Unlike previous works [12], we adopt here a novel approach that combines a blockwise encoding of activity and a bitwise encoding of data and decode active as well as non-active users based on a prior information on the activity. We analyze the system's performance using density evolution and show that below a threshold between the log number of users and the code block length the corresponding large-system fixed-point equation [10] scales with the average number of active users.

This paper is organized as follows. Section II introduces the system model and the main notations used throughout. Section III describes the iterative MUD factor graph and the belief propagation decoding algorithm. Section IV presents the main results on density evolution showing the system's dynamic fixed-point equations. Finally, section VI draws some concluding remarks.

II. System Model

We consider a synchronous Gaussian CDMA system where K is the maximum number of users entitled to access the system, N is the length of the spreading sequences and L is the length of the users' codewords. The corresponding received signal matrix is given by

$$\boldsymbol{Y} = \boldsymbol{S}\boldsymbol{A}\boldsymbol{X} + \boldsymbol{Z} \tag{1}$$

where $\boldsymbol{Y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L) \in \mathbb{R}^{N\times L}$ is the received signal, $\boldsymbol{S} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_K) \in \mathbb{R}^{N\times K}$ is the matrix of the normalized spreading sequences, $\boldsymbol{A} = \mathrm{diag}(a_1, \ldots, a_K) \in \mathbb{R}^{K\times K}$ is the diagonal matrix of the users' signal amplitudes, $\boldsymbol{Z} \in \mathbb{R}^{N\times L}$ is an additive white Gaussian noise matrix with i.i.d. entries $\sim \mathcal{N}(0,1)$, and $\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K)^{\mathsf{T}} \in \mathbb{R}^{K\times L}$ is the matrix containing the users' coded blocks, with $\boldsymbol{x}_k = (x_{k,1}, \ldots, x_{k,L})^{\mathsf{T}}$ being the codeword of the k-th user.

We assume that users are active with probability $\alpha \triangleq \Pr\{\text{user } k \text{ active}\}$, $1 \le k \le K$ and employ binary phase-shift keying (BPSK) modulation with uniform probability. We also suppose for ease of analysis equal power users such that $a_k = \sqrt{\gamma}$, where γ is the average received signal-to-noise ratio (SNR). We define the maximum system load as $\beta \triangleq \frac{K}{N}$.

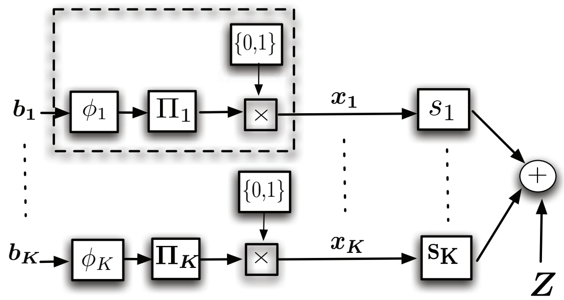

Fig. 1. Block diagram of the transmission.

A. Encoding of Data and Activity

Let $\boldsymbol{B} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_K)^\mathsf{T} \in \mathbb{R}^{K \times B}$ be the information matrix of all users, where B is the length of an information message and $\boldsymbol{b}_k = (b_{k,1}, \ldots, b_{k,B})^\mathsf{T}$ is the k-th user information vector. Whenever user k is active, then $\boldsymbol{b}_k \in \mathbb{F}_2^B$ (Galois field $\mathbf{GF}(2^B)$), otherwise we set $b_{k,i} = 2, \ \forall i = 1, \ldots, B$ for sake of analysis[1]. We assume that active-user vectors appear with probability α and their information vectors are encoded independently. The inactive message appears with probability $1 - \alpha$. The nature of the information messages varies from active to inactive users. For instance, it is easy to see that in the case of inactive users, the information symbols are not independent: if the first one is represented by 2, all the rest are 2 as well. In order to incorporate the activity of users into the decoding process, we add one symbol at the beginning of each coded block, which can be either one or zero, depending on whether the user is active or not, respectively. Hence, if user k is active, we can define an encoding function ϕ_k over each user's message set $\mathcal{M}_k \subseteq \mathbb{F}_2^B$, $\phi_k : \mathcal{M}_k \rightarrow \{-1, +1\}^{L-1}$ such that

$$\phi_k(m_k) = (x_{k,2}, \ldots, x_{k,L}) \tag{2}$$

where $m_k \in \mathcal{M}_k$ is an input message. The code $\mathcal{C}_k$ of an active user is then defined as:

$$\mathcal{C}_k = \{\boldsymbol{x} \in \{-1, +1\}^L : \boldsymbol{x} = (+1, \phi_k(m_k)), \ \ m_k \in \mathcal{M}_k\} \tag{3}$$

For inactive users, $\mathcal{C}_k$ is only modulated as follows:

$$\mathcal{C}_k = \{\boldsymbol{x} = (0, \ldots, 0) \in \mathbb{R}^L\}. \tag{4}$$

Note that this is equivalent to considering a code $\tilde{\mathcal{C}}_k$ that incorporates the all-zero codeword (no signal is transmitted) representing the non-activity. While the presentation given in this paper is general, we will focus our examples on trellis codes. We differentiate two cases: if a user is active, its BPSK stream is interleaved across time. If the user is not active, the all-zero codeword is transmitted. Remark that the resulting vectors accessing the channel are independent but their components might be correlated, due to the temporal correlation induced by the inactive users transmitting the all-zero codeword. The interleaved signals are then spread and transmitted over the channel. The overall transmission scheme is depicted in Fig. 1, where Π_i denotes each user's interleaver.

If the codes $\mathcal{C}_k$ are convolutional codes, the above considerations result in a trellis that combines the activity and encoding functions. The overall trellis can be decoded with the forward-backward algorithm [13].

[1]Note that this is simply a convention to represent the inactivity that is hidden in the encoding function. It therefore does not affect the results.

III. Iterative joint decoding under belief propagation

Our goal is to compute the a posteriori p.m.f. of the information symbols:

$$\Pr(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_K | \boldsymbol{Y}, \boldsymbol{S}, \boldsymbol{A}) \tag{5}$$

However, the computation of (5) by brute force is infeasible even for a small maximum number of users due to the large dimensions. In order to obtain a low-complexity detector, we resort the canonical factor graph representation of a multiuser coded system [4] and consider the application of the well-known sum-product algorithm [6]. Although in the model presented above symbols are correlated during the inactive stream, we study symbol-by-symbol belief propagation (BP) decoding as a suboptimal mismatched strategy to approximate iteratively the marginal probabilities of (5). This method would asymptotically replicate large-system optimal detection [9] in the case of a system with collocated users, where coded sequences could be interleaved across the user dimension. However, given our particular scheme, we conjecture that the performance loss is negligible in the large-system limit since the correlation only appears through non-active users, using symbol 0, which does not belong to the data alphabet.

The application of BP to our model results in message passing between the individually optimum multiuser detector (IO-MUD) and the users' soft-input soft-output (SISO) decoders. In this case, our suboptimal detector assumes that each coded symbol can take values in the ternary constellation $\mathcal{X} \triangleq \{-1, 0, +1\}$. We thus use a three-dimensional probability vector to describe the exchanged messages between the two blocks. Hence, the outgoing messages from the IO-MUD at iteration $\ell \in \mathbb{N}$, for user $k = 1, \ldots, K$ and time $l = 1, \ldots, L$ are denoted as $\boldsymbol{q}_{k,l}^{(\ell)} = \left(q_{k,l}^{(\ell)}(-1), q_{k,l}^{(\ell)}(0), q_{k,l}^{(\ell)}(1)\right)$, where $\boldsymbol{q}_{k,l}^{(\ell)}$ stands for the extrinsic probability of the symbol $x_{k,l}$ given the channel observation. The outgoing messages from the SISO decoder are denoted as $\boldsymbol{p}_{k,l}^{(\ell)} = \left(p_{k,l}^{(\ell)}(-1), p_{k,l}^{(\ell)}(0), p_{k,l}^{(\ell)}(1)\right)$, and are the extrinsic probabilities of the coded symbols [7]. When the users' codes are convolutional codes, the messages $\boldsymbol{p}_{k,l}^{(\ell)}$ are obtained by applying the forward-backward algorithm to the combined trellis. Note that the above algorithm is suboptimal since it ignores the correlation introduced by the inactive users, by exchanging different messages over a ternary constellation for every time instance $l = 1, \ldots, L$.

According to [4], [6], the sum-product rules that relate the probabilities $\boldsymbol{q}_{k,l}^{(\ell)}$ and $\boldsymbol{p}_{k,l}^{(\ell)}$ are stated as follows for $x \in \mathcal{X}$:

$$q_{k,l}^{(\ell)}(x) \propto \sum_{\boldsymbol{x}\in\mathcal{X}^K x_{k,l}=x} e^{-\frac{1}{2}||\boldsymbol{y}_l - \sum_{j=1}^K \boldsymbol{s}_j a_j x_{j,l}||^2} \prod_{j\neq k} p_{j,l}^{(\ell-1)}(x_{j,l}), \tag{6}$$

$$p_{k,l}^{(\ell+1)}(x) \propto \sum_{\{\boldsymbol{x}\in\mathcal{C}_k, x_{k,l}=x\}} \prod_{j\neq l} q_{k,j}^{(\ell)}(x) \tag{7}$$

We assume that the message $\boldsymbol{p}_{k,l}^{(0)} = (\frac{\alpha}{2}, 1-\alpha, \frac{\alpha}{2})$ for $k = 1,\ldots,K$ and time $l = 1,\ldots,L$. Finally note that $\boldsymbol{q}_{k,l}^{(\ell)}$ and $\boldsymbol{p}_{k,l}^{(\ell)}$ can be viewed as random variables, which depend on both the channel and code parameters.

IV. Performance Analysis

In order to analyze the performance over the iterations, we employ *density evolution*. Density evolution has been applied to study low-density parity check (LDPC) codes [7] as well as iterative MUD [4], [5]. Density evolution is based on the principle that as the length of the codes is sufficiently large, the p.d.f. of the messages exchanged at each iteration converges to a deterministic one. In our case, we employ statistical physics techniques to characterize the nature of the messages $\boldsymbol{q}_{k,l}^{\ell}$.

A. Large-System Analysis

Large-system analysis is remarkably simpler and accurately mimics the behavior of the system for not-so-large dimensions [4], [5]. In particular, we let $K, N \to \infty$ keeping their ratio, the system load $\beta = K/N$, fixed. Under these conditions, by recalling the decoupling principle [10] for optimum MUD we obtain a single-user equivalent Gaussian characterization of the uncoded CDMA channel. More interestingly, the decoupling principle can be generalized to the case of coded CDMA systems where the users' symbols are not independent from one time to another [14]. In particular, in the large-system limit, for every time index $l = 1,\ldots,L$ we obtain a set of K parallel additive Gaussian channels with colored noise $\mathcal{N}(0,\Sigma_k)$ and time-varying variances given by $(\Sigma_k)_{l,l} = (\gamma_k \eta_l)^{-1}$, $l = 1,\ldots,L$. where $\boldsymbol{\eta} = (\eta_1,\ldots,\eta_L)$ is the multiuser efficiency vector, characterizing the multiuser efficiency at every symbol instant.

We now present our main result that describes the dynamical system behavior by updating the distribution of the IO-MUD messages at each iteration. Note that due to the large-system approach, the extrinsic probabilities provided by the SISO decoders $\boldsymbol{p}_{k,l}^{(\ell)}$ are independent of k, hence resulting in a $3 \times L$ matrix denoted by $\boldsymbol{P}_{\text{ext}}^{(\ell)}$. Generalizing [4], [5], the multiuser efficiency is given in terms of a L-dimensional fixed-point equation $\boldsymbol{\eta}^{(\ell)} = \boldsymbol{\Psi}(\boldsymbol{\eta}^{(\ell-1)}, \beta, \alpha, \gamma)$, that characterizes the density evolution mapping, as $\boldsymbol{\eta}$ evolves through the iterations.

Claim 4.1: Consider an iterative MUD system where the number of users is unknown and parameterized by a Bernoulli variable A_α with success probability α. Then, the multiuser efficiency at iteration ℓ, $\boldsymbol{\eta}^{(\ell)} = \boldsymbol{\Psi}(\boldsymbol{\eta}^{(\ell-1)}, \beta, \alpha, \gamma) \in \mathbb{R}^L$, of a belief-propagation iterative joint multiuser decoder is given by the globally stable solutions of the fixed point equations:

$$\eta_l^{(\ell)} = \left(1 + \beta \mathbb{E}_{A_\alpha, \boldsymbol{P}_{\text{ext}}^{(\ell-1)}, \boldsymbol{z}, \boldsymbol{x}, \gamma}\left[\gamma\left(x_l - \hat{x}_l\right)^2\right]\right)^{-1} \tag{8}$$

for $l = 1,\ldots,L$, where $\hat{x}_l(\boldsymbol{\eta}^{(\ell)}, \boldsymbol{\eta}^{(\ell-1)}, \gamma)$ is the l-th entry of the MMSE (minimum mean square error) symbol estimate $\hat{\boldsymbol{x}} = \mathbb{E}[\boldsymbol{x}|\boldsymbol{y}, \gamma, \boldsymbol{P}_{\text{ext}}^{(\ell-1)}]$ over the single-user equivalent vector Gaussian channel:

$$\boldsymbol{y} = \sqrt{\gamma}\boldsymbol{x} + \boldsymbol{z} \tag{9}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^L$, $\boldsymbol{x} \sim \boldsymbol{P}_{\text{ext}}^{(\ell-1)}$, and $\boldsymbol{z} \sim \mathcal{N}(0,\Sigma)$ with $\Sigma_{l,l} = 1/\eta_l^{(\ell)}$, $l = 1,\ldots,L$.

The above result defines implicitly $\boldsymbol{\Psi}$ and describes the performance of an iterative joint decoder under belief propagation, assuming that the symbols at time $l = 1,\ldots,L$ can be correlated. In terms of convergence, if the above density evolution algorithm has a unique fixed point $\boldsymbol{\eta} = (1,\ldots,1)$, the system approaches single-user performance. On the other hand, if at any iteration ℓ, $\boldsymbol{\Psi}$ has a fixed point in $(0,1)$ in any of its L components, i.e. there exists at least an l such that $\eta_l^{(\ell)} = \eta_l^{(\ell-1)} < 1$, then the multiuser detector cannot remove the interference at time l. Otherwise, the multiuser efficiency $\boldsymbol{\eta}$ converges to $\mathbf{1} \in \mathbb{R}^L$ through the iterations.

Equation (8) can also be interpreted as an analogous case of [15], where the fading distribution takes a specific form characterized by A_α. Notice, however, that the activity factor comes into play in the encoding mapping and is independent of the nature of channel. Therefore, (8) can be extended to fading channels with soft channel estimation feedback, allowing for a further generalization. Furthermore, (8) could potentially encompass correlated and non-ergodic fading, generalizing the study of [15] for i.i.d. fading to more general distributions.

B. Concentration

The concentration theorem in [4] for coded CDMA refers to the existence of a limiting distribution of the output messages $\boldsymbol{p}_{k,l}^{(\ell)}$ for $L \to \infty$ under some conditions on the user codes $\mathcal{C}_k$. From Section II-A, the representation of the inactivity can be characterized as a repetition code of rate $1/L$. We compute now its large-system performance based on the error probability per user, P_e, according to the encoding/decoding scheme described above. The exact expression of $P_{(e|A_\alpha=1)}^{(\ell)}$ and $P_{(e|A_\alpha=0)}^{(\ell)}$, the probabilities of making an incorrect activity decision at iteration ℓ, is in general difficult to obtain. However, we can compute simple upper bounds at any iteration, as the following result shows;

Proposition 4.2: Under the equivalent Gaussian vector channel (9), we have that

$$P_{(e|A_\alpha=1)}^{(\ell)} \leq 2\sqrt{\frac{1-\alpha}{\alpha}} e^{-\frac{L\eta^{(\ell)}\gamma}{8}},$$
$$P_{(e|A_\alpha=0)}^{(\ell)} \leq \sqrt{\frac{\alpha}{1-\alpha}} e^{-\frac{L\eta^{(\ell)}\gamma}{8}} \tag{10}$$

where $\eta^{(\ell)} \leq \min_{1\leq l\leq L} \eta_l^{(\ell)}$.

Assuming that all users transmit their codewords independently, the system's total error probabilities in each case, $P_{(E|A_\alpha=1)}^{(\ell)}$ and $P_{(E|A_\alpha=0)}^{(\ell)}$, satisfy $P_{(E|A_\alpha=1)}^{(\ell)} \leq K P_{(e|A_\alpha=1)}^{(\ell)}$ and $P_{(E|A_\alpha=0)}^{(\ell)} \leq K P_{(e|A_\alpha=0)}^{(\ell)}$ respectively.

Consider now that the blocklength depends on K, $L = L(K)$, and let $L = L(K)$ be such that

$$\lim_{K\to\infty} \frac{\log K}{L(K)} = \rho, \tag{11}$$

where $\rho \geq 0$ expresses the tradeoff between the blocklength and the logarithm of the maximum number of users as K grows large. Remark that $\rho = 0$ implies that the blocklength grows faster than $\log K$. Note that Proposition 4.2 immediately implies that $P^{(\ell)}_{(S|A_\alpha=1)} \triangleq 1 - P^{(\ell)}_{(E|A_\alpha=1)}$ and $P^{(\ell)}_{(S|A_\alpha=0)} = 1 - P^{(\ell)}_{(E|A_\alpha=1)}$ (the conditional probabilities of successful activity detection) can easily be lower-bounded as $P^{(\ell)}_{(S|A_\alpha=1)} \geq 1 - 2\sqrt{\frac{1-\alpha}{\alpha}} e^{-L(\eta^{(\ell)}\gamma/8-\rho)}$ and $P^{(\ell)}_{(S|A_\alpha=0)} \geq 1 - \sqrt{\frac{\alpha}{1-\alpha}} e^{-L(\eta^{(\ell)}\gamma/8-\rho)}$.

Notice that it follows that $P^{(\ell)}_{(S|A_\alpha=1)}$ and $P^{(\ell)}_{(S|A_\alpha=0)}$ converge to 1 as $L \to \infty$, $\forall \rho < \rho_{\mathrm{th}}(\eta^{(\ell)}, \gamma)$, where

$$\rho_{\mathrm{th}}(\eta^{(\ell)}, \gamma) \triangleq \frac{\eta^{(\ell)}\gamma}{8}, \tag{12}$$

whereas for $\rho \geq \rho_{\mathrm{th}}(\eta^{(\ell)}, \gamma)$ this is not true in general. Remark that ρ_{th} is fully determined by the actual SNR of the equivalent large-system Gaussian channel, i.e. by increasing $\gamma\eta^{(\ell)}$, the range of feasible scaling factors can be increased. In particular, by fixing γ, ρ_{th} depends exponentially on the level of interference $\eta^{(\ell)}$ and achieves its maximum value when the interference is removed (i.e., at $\eta^{(\ell)} = 1$).

When $\rho < \rho_{\mathrm{th}}(\eta^{(\ell)}, \gamma)$ the density evolution assumption ($L \to \infty$) allows us to conclude that correct activity detection is achieved with probability $P_s \to 1$. This implies that for every inactive user, $\boldsymbol{p}^{(\ell)}_{k,l} \to (0, 1, 0)$ in the limit of large codeword length. We therefore consider a compound of two types of message probabilities that switch depending on whether soft decoding operates on an active or an inactive coded block. Consequently, the limiting distribution of the messages over an active block exists under the same conditions as in the general case [4], whereas the limiting distribution of the messages over an inactive one exists since it concentrates all the probability in the symbol 0. As a result of the decoupling structure in the $\boldsymbol{p}^{(\ell)}_{k,l}$ distribution for $L \to \infty$, the application of claim 4.1 to our system yields a simplified one-dimensional fixed-point equation. We therefore define $\boldsymbol{p}^{(\ell)}_{\mathrm{ext}} = \boldsymbol{p}^{(\ell)}_{k,l} \in \mathbb{R}^3$ to simplify the notation. In fact, since the activity is perfectly detected after the first iteration for arbitrary SNR and $L \to \infty$, the effect of the correlation among symbols becomes negligible and belief propagation can approximate the optimal detection of the interleaved active-user codewords [16]. Symbols are no longer correlated due to the effect of the interleaver, and the resulting density evolution mapping takes a unique one-dimensional form $\eta^{(\ell)} = \Psi(\eta^{(\ell-1)}, \beta, \alpha, \gamma, \rho)$ given in the following corollary:

Corollary 4.3: The large-system fixed-point equation of a system with unknown number of equal-power users that perfectly estimates their activity at a particular iteration $\ell \geq 1$ due to $\rho < \rho_{\mathrm{th}}(\eta^{(\ell-1)}, \gamma)$ converges with probability 1 to:

$$\eta^{(\ell)} = \left(1 + \beta'\gamma\left(1 - \mathbb{E}_{(\boldsymbol{p}_{\mathrm{ext}},z,x|A_\alpha=1)}\left[\gamma\hat{x}^2\right]\right)\right)^{-1} \tag{13}$$

where $\beta' = \beta\alpha$ and $\hat{x}(\eta^{(\ell-1)}, \eta^{(\ell)}, \gamma) = \mathbb{E}[x|y, \gamma, \boldsymbol{p}^{(\ell-1)}_{\mathrm{ext}}]$ is the MMSE estimate for the single-user scalar Gaussian channel:

$$y = \sqrt{\gamma}x + z \tag{14}$$

where $x, y \in \mathbb{R}$, $x \sim \boldsymbol{p}^{(\ell-1)}_{\mathrm{ext}}$, $\boldsymbol{p}^{(\ell-1)}_{\mathrm{ext}} \in \mathbb{R}^3$ and $z \sim \mathcal{N}(0, \frac{1}{\eta^{(\ell)}})$.

When $\rho \geq \rho_{\mathrm{th}}$, there is less knowledge about the actual limiting distribution of $\boldsymbol{p}^{(\ell)}_{\mathrm{ext}}$ as $L \to \infty$. However, we can provide a lower bound on the system's performance using uncoded-activity detection with $\boldsymbol{p}^{(\ell)}_{\mathrm{ext}} = \left[\alpha p^{(\ell)}_{\mathrm{ext}|\hat{A}_\alpha=1}(1), 1-\alpha, \alpha p^{(\ell)}_{\mathrm{ext}|\hat{A}_\alpha=1}(3)\right]^{\mathsf{T}}$ for $A_\alpha = 1$ and $\boldsymbol{p}^{(\ell)}_{\mathrm{ext}} = \left[\frac{\alpha}{2}, 1-\alpha, \frac{\alpha}{2}\right]^{\mathsf{T}}$ for $A_\alpha = 0$ $\forall \ell \geq 0$, where $\hat{A}_\alpha$ is the hard decision of the decoder on A_α. Since the message probabilities do not exploit the correlation between inactive symbols, the resulting lower bound on the density evolution mapping also takes a unique one-dimensional form.

As ρ_{th} is a function of η at any iteration ℓ, and $\eta \in (0, 1]$, the impact of ρ_{th} into any system with tradeoff ρ can be better analyzed through the threshold value

$$\eta_{\mathrm{th}}(\rho, \gamma) \triangleq \min\left\{\frac{8\rho}{\gamma}, 1\right\}, \tag{15}$$

which establishes the minimum multiuser efficiency above where the system's activity detection satisfies $P_S \to 1$ as $L \to \infty$ and consequently, the fixed-point (13) holds.

Remark that all systems that encounter a fixed-point $\Psi(\eta^{(\ell^\star)}) = \eta^{(\ell^\star)}$ above η_{th} for some iteration $\ell^\star$ (i.e., $\eta^{(\ell^\star)} > \eta_{\mathrm{th}}$), undergo two different phases. The first phase is found for $\rho \geq \rho_{\mathrm{th}}$ during the initial iterations of the decoder, where $\eta^{(\ell-1)} \in (0, \eta_{\mathrm{th}}]$, $\ell < \ell^\star$, hold. As a result, the error due to activity detection cannot be neglected and leads to a performance loss bounded by the uncoded-activity scenario. The second phase corresponds to $\eta^{(\ell-1)}$ being sufficiently large such that $\eta^{(\ell-1)} > \eta_{\mathrm{th}}$, $\ell \leq \ell^\star$. Then, over the following iterations, Corollary 4.3 provides the fixed-point equation of a system with a known number of users and load $\beta' = \beta\alpha$ [5]. Notice that the same arguments on the convergence of Ψ_l, $l = 1, \ldots, L$, in the general case hold here for Ψ.

V. Numerical results

The above results imply that under some conditions the analysis of a coded multiuser system with user-and-data detection can be converted into the analysis of a standard multiuser system where the number of users is fixed and known. The activity can be detected perfectly after a few iterations, and the behavior of the dynamical fixed-point equation has the form of a data detector with a scaled system load.

In Fig. 2 we illustrate Corollary 4.3. We first show the density evolution mapping function corresponding to an overloaded system with $\beta = 4.5$ and a 0.02 resolution grid on the η-axis for the standard MUD case (all users active $\alpha = 1$, thinner solid line) and a case where all users are active with probability $\alpha = 0.5$ and $\rho = 0.0$ ($\eta_{\mathrm{th}} = 0.0$, solid line). We also plot two lower bounds based on density evolution for $\alpha = 0.5$ with non-vanishing tradeoff between the block length and the logarithm of the number of users: one with

$\rho = 0.05$ ($\eta_{\text{th}} = 0.1$, dash-dotted line) and another with $\rho = 0.5$ ($\eta_{\text{th}} = 1$, dotted line). The codeword length used here is $L = 4000$, for $E_b/N_0 = 6$dB and the number of realizations are 100. Remark that for $\alpha = 1$, the system converges to a fixed point at very low multiuser efficiency and results in the system not converging to remove the interference. On the other hand, when users are active with probability $\alpha = 0.5$ and $\rho = 0.0$, we have that $\rho < \rho_{\text{th}}(\eta, \gamma)$ for $\eta \in (0, 1]$, the unique solution of $\Psi(\eta) = \eta$ is $\eta = 1$ and the system converges to single-user performance. The same convergence is achieved when the system has $\rho = 0.05$. In this case, the true system performance is lower-bounded by uncoded-activity detection for $\eta \in (0, 0.1]$, and coincides with the case $\rho = 0.0$ for $\eta \in (0.1, 1]$ since perfect activity detection is achieved due to $\rho < \rho_{\text{th}}$. Note that in both cases the activity rate scales the system load after the first iteration, and thus, the curve with $\alpha = 0.5$ significantly improves the performance of the $\alpha = 1$-case. However, when $\alpha = 0.5$ and $\rho = 0.5$, we have that $\rho \geq \rho_{\text{th}}$ $\forall \eta \in (0, 1]$ since $\max_\eta \rho_{\text{th}} = \rho_{\text{th}}(1, \gamma) = 0.49$ and the system only experiments the stage where the lower bound holds. In that case, there is a remaining constant error in the whole range $\eta \in [0, 1]$ and the lower bound encounters a fixed point at approx. $\eta = 0.125$.

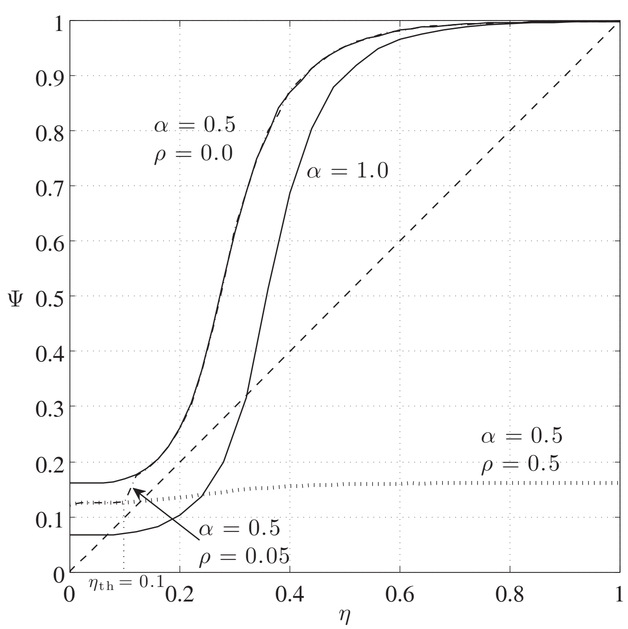

Fig. 2. Mapping function $\Psi(\eta, \beta, \alpha)$ with $\alpha = 1.0$ and $\alpha = 0.5$ for $\rho = 0.0, 0.01, 0.5$, at $E_b/N_0 = 6$dB and $\beta = 4.5$. Solid lines represent the density evolution with the $(5, 7)_8$ convolutional code for a codeword length of $L = 4000$ and 100 realizations for $\alpha = 1.0$ and for $\alpha = 0.5$ with $\rho = 0.0$. For $\alpha = 0.5$ and $\rho = 0.01$ (dash-dotted line), it is shown the lower bound on $\Psi(\eta, \beta, \alpha)$ for $\eta \in (0, 0.1]$ and the exact mapping for $\eta \in (0.1, 1]$. For $\alpha = 0.5$ and $\rho = 0.5$, the lower bound is shown $\forall \eta \in (0, 1]$.

VI. Conclusions

We have studied the large-system performance of iterative multiuser joint decoding under belief propagation using density evolution when the number of active users accessing the channel is unknown at the receiver. Due to inactive users transmitting the all-zero codeword, the channel model is no longer memoryless, since symbols are correlated over time. We employ a low-complexity symbol-by-symbol iterative multiuser detector that ignores this correlation and analyze its large system performance. In particular, using the replica method, we obtain the multidimensional fixed-point equation of the multiuser efficiency for finite block length. We then study the limiting performance for large block length using density evolution techniques. In this case, we first show that in the limit for large block length, the system effectively performs perfect activity detection when the maximum number of users and the blocklength scale appropriately below a threshold. Otherwise, we provide a lower bound on the performance based on uncoded-activity detection. When perfect-activity detection is achieved, the fixed-point equation is equivalent to that of a system where the number of users is fixed and known, but with a scaled system load.

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