Error Probability of Bit-Interleaved Coded Modulation using the Gaussian Approximation

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Abstract — **This paper presents a very accurate and simple to compute approximation to the performance of bit-interleaved coded modulation (BICM) systems. The proposed method is based on approximating the binary-input continuous-output equivalent BICM channel by a binary-input AWGN (BI-AWGN) channel with scaled SNR. The scaling factor can be easily computed numerically and depends on the actual channel SNR and on the modulation signal set and binary labeling. The key is that very good approximation results when the Bhattacharyya parameter is used to estimate the variance of the underlying Gaussian channel. Under such approximation, we can use all bounding techniques known for binary codes in Gaussian channels. In particular, we use the union and the tangential-sphere bounds and we apply such results to both convolutional and turbo-like codes. The proposed method represents a simple yet powerful tool for estimating the error probability of finite-length turbo-like codes with BICM.**

I. Introduction and outline of the work

Bit-interleaved coded modulation (BICM) was introduced in [1] and further generalized and elaborated in [2] as a means of coding for spectrally efficient modulations. In essence, it states that nearly optimal performance can be achieved by concatenating a powerful binary code with a non-binary modulator, by the simple addition of a bit-interleaver between these two components. An additional advantage offered by BICM is its inherent flexibility, as a single mother code may be used for several modulations, with no additional adaptations. This is an appealing feature for future communication systems where a large set of spectral efficiencies is needed.

The original works on BICM [1, 2], consider a decoder that for every symbol produces soft statistics for the bits of its binary label, and feeds these values to a ML decoder of the mother code, as if they were outputs of a *virtual* binary-input continuous-output channel. We shall refer to this decoder as BICM-ML decoder and the virtual channel as the equivalent BICM channel.

Several works have also considered iterative decoding of convolutionally encoded BICM with optimized mappings, showing remarkable performance increase with respect to Gray mapping (see e.g. [3, 4]). Simple iterative decoding analysis of such system has been provided in [3] based on an approximation of density evolution techniques [5] for infinite blocklength. However, no such satisfactory results have been observed when using capacity approaching codes such as turbo-like or LDPC codes, where iterative decoding analysis is very complicated (see [6] for recent results on the subject). Therefore it is common practice to couple turbo-like or LDPC codes and BICM with Gray mapping, since it offers the best performance-complexity tradeoff.

Error probability bounds of finite-length BICM under BICM-ML decoding have been derived in [2]. The metric model assumed by the BICM-ML decoder is nearly optimal for Gray mapping and assumes no demapping iterations. A simple union bound based on a bitwise Bhattacharyya factor was found to be quite loose. Several refined techniques, also derived in [2], provided more accurate results, but are much more complex to compute. In this paper, we provide a very simple method that allows for the computation of very accurate approximations on the error probability of BICM with BICM-ML decoding, which is mainly based on a Gaussian approximation (GA) of the binary-input BICM equivalent channel. We verify the validity of the approximation and we apply the results to compute tight union and tangential-sphere bounds for both convolutional and turbo-like codes. We also illustrate how the proposed approximation can be used to compute BICM-ML thresholds. We show that the proposed method is a simple and powerful tool, since it yields very accurate error probability estimates with little computational effort.

II. System model

We consider a classical additive white Gaussian noise (AWGN) channel, for which the received signal at time k, $y_k \in \mathbb{C}$ is given by,

$$y_k = \sqrt{\rho} x_k + z_k \ , \ k = 1, \dots, L \tag{1}$$

where $x_k \in \mathbb{C}$ is the transmitted signal at time k with L the codeword length, $\rho = \frac{E_s}{N_0}$ is the the signal-to-noise ratio (SNR) and $z_k \in \mathbb{C}$ is the complex noise sample at time k i.i.d. $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$. We denote by $\mathcal{X} \in \mathbb{C}$ the complex signal constellation (i.e. PSK, QAM). Without loss of generality we study unit energy constellations, i.e., $\mathbb{E}[|x|^2] = 1$.

The codewords $\mathbf{x} = (x_1, \dots, x_L) \in \mathcal{X}^L$ are BICM codewords obtained by bit-interleaving the codewords $\mathbf{c}$ of the binary code $\mathcal{C} \in \mathbb{F}_2^N$ of rate $r = K/N$, and mapping with

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the mapping rule μ, that maps binary labels of length $M = \log_2 |\mathcal{X}|$ over to the points of $\mathcal{X}$ [1, 2]. The resulting BICM codeword length is L modulation symbols, with $L = N/M$, and the spectral efficiency of such system is $R = rM$ bit/s/Hz.

Due to the presence of the bit-interleaver, ML decoding of BICM is only possible by exhaustive search. Several suboptimal strategies have been used, all derived from the belief-propagation iterative algorithm [7], for which, the bit metrics of a given bit b being in the m-th label position of a given symbol are given by,

$$p(y|b,m) \propto \sum_{z \in \mathcal{X}_b^m} p(y|z)P(z) \tag{2}$$

where $\mathcal{X}_b^m$ is the set of all signal constellation symbols with bit b in the label position m, $p(y|z) = \frac{1}{\pi}\exp(-|y-\sqrt{\rho}z|^2)$ is the channel transition probability density function (pdf), and $P(z)$ denotes the *a priori* probability of the symbol z. When iterative demapping is performed, $P(z)$ are given by the decoder of $\mathcal{C}$. In this paper, we restrict our attention to the case of equally likely symbols where no demapping iterations are performed, i.e. $P(z) = \frac{1}{|\mathcal{X}|}$. The suboptimal BICM-ML decoder is known to perform near optimal for signal constellations with Gray mapping [2, 3]. We will refer to the channel between a given binary codeword $\mathbf{c} \in \mathcal{C}$ and its corresponding bit-metrics, as the equivalent binary-input BICM channel. In particular, for a coded binary symbol mapped to the m-th label position of the k-th modulation symbol, the *bit-wise* posterior log-probability likelihood ratio (LLR) is given by

$$\mathcal{L}_{k,m} = \log \frac{\sum_{x \in \mathcal{X}_0^m} \exp\left(-|y_k - \sqrt{\rho}x|^2\right)}{\sum_{x \in \mathcal{X}_1^m} \exp\left(-|y_k - \sqrt{\rho}x|^2\right)}. \tag{3}$$

We also report the capacity with BICM under suboptimal BICM-ML decoding, which is given by [2],

$$C = M - \frac{1}{2^M}\sum_{m=1}^{M}\sum_{b=0}^{1}\sum_{x \in \mathcal{X}_b^m} \mathbb{E}_n\left[\frac{\sum_{z \in \mathcal{X}} e^{-|\sqrt{\rho}(x-z)+n|^2}}{\sum_{z \in \mathcal{X}_b^m} e^{-|\sqrt{\rho}(x-z)+n|^2}}\right], \tag{4}$$

which is known to be maximized for Gray or quasi-Gray binary labeling rules. For the sake of future reference, we show in Figure 1 the capacity with 16-QAM inputs and the BICM capacity with 16-QAM and Gray mapping.

III. The Bhattacharyya Union Bound

The Bhattacharyya Union Bound for BICM was first proposed in [2] as a simple approach for upper bounding the error probability of BICM under BICM-ML, i.e., ML decoding of $\mathcal{C}$ using the bit-metrics (2) with no demapper iterations. For the frame error probability it is given by,

$$P_e \leq \sum_d A_d B(\rho)^d \tag{5}$$

where $A_d = |\mathcal{S}_d|$ is the weight enumeration function (WEF) of $\mathcal{C}$ and accounts for the number of pairwise error events of $\mathcal{C}$ at Hamming distance d, with $\mathcal{S}_d = \{\mathbf{c} \in \mathcal{C} : w_H(\mathbf{c}) = d\}$ denoting the set of codewords with Hamming weight d, and

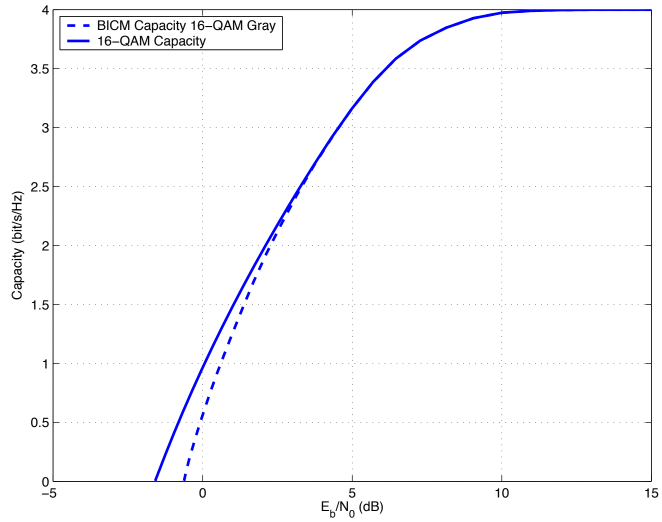

Fig. 1: Capacity for 16-QAM inputs and BICM capacity with Gray mapping.

$B(\rho)$ is the Bhattacharyya factor which is given by,

$$B(\rho) = \mathbb{E}_{y,m,b}\left[\sqrt{\frac{p(y|\bar{b},m)}{p(y|b,m)}}\right]$$

$$= \frac{1}{M2^M}\sum_{m=1}^{M}\sum_{b=0}^{1}\sum_{x \in \mathcal{X}_b^m} \mathbb{E}_n\left[\sqrt{\frac{\sum_{z \in \mathcal{X}_{\bar{b}}^m} e^{-|\sqrt{\rho}(x-z)+n|^2}}{\sum_{z \in \mathcal{X}_b^m} e^{-|\sqrt{\rho}(x-z)+n|^2}}}\right] \tag{6}$$

where $p(y|b,m)$ are given in (2), the expectation is over the joint distribution of the received signal y, the labeling position m and the bit b as value of the m-th label position and $\bar{b}$ denotes the binary complement. Notice that $B(\rho)$ depends actually on the signal constellation and the binary mapping. Also note that this expectation can be easily evaluated using the Gauss-Hermite quadrature rules which are tabulated in [8], since $n \sim \mathcal{N}_{\mathbb{C}}(0,1)$. The results in [2] show that (5) can be very loose, and therefore, not very useful as analytical tool to describe the error probability of BICM. Reference [2] elaborates more refined bounds on the performance of the BICM-ML decoder, which are, however, much harder to compute.

IV. The Gaussian Approximation

In this section we describe how the Gaussian approximation can be used to describe the binary-input equivalent channel of BICM. Notice that the Gaussian approximation of the binary-input BICM channel is commonly employed in convergence analysis of iterative decoding based in density evolution techniques [3, 5] for infinite blocklength. Consider for a moment that the binary code $\mathcal{C}$ mapped over a BPSK signal constellation (i.e., $\mathcal{X} = \{-1, +1\}$) is transmitted across a binary-input AWGN channel with SNR γ. Then, the standard union and Bhattacharyya bounds can be written as,

$$P_e \leq \sum_d A_d Q\left(\sqrt{2d\gamma}\right) \leq \sum_d A_d B_2(\gamma)^d \tag{7}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-\frac{t^2}{2}}dt$$

is the Gaussian tail function, and the Bhattacharyya factor for binary inputs $B_2(\gamma)$ is given by [9],

$$B_2(\gamma) = \mathbb{E}_y\left[\sqrt{\frac{p(y|-1)}{p(y|+1)}}\right] = e^{-\gamma}. \tag{8}$$

It is also well-known that for such binary-input AWGN channel (BI-AWGN), the log-likelihood ratio (LLR) defined as

$$\mathcal{L} = \log\frac{p(y|-1)}{p(y|+1)} \tag{9}$$

is $\sim \mathcal{N}(4\gamma, 8\gamma)$.

Then, by comparing (6) with (8), we can approximate the equivalent binary input BICM channel as a BI-AWGN with SNR γ. Therefore, we can write,

$$B(\rho) = e^{-\gamma} \tag{10}$$

from where we obtain that the signal-to-noise ratio of the equivalent binary-input BICM channel is given by,

$$\gamma = \rho\alpha, \tag{11}$$

where

$$\alpha = -\frac{1}{\rho}\log B(\rho) \tag{12}$$

is the scaling factor with respect to the nominal SNR of the channel ρ.

It is not difficult to show that for $\rho \to \infty$, by simply taking a single dominant term in the numerator and denominator of the term under $\sqrt{\cdot}$ in (6) we have,

$$\alpha \approx -\frac{1}{\rho}\log\left(\frac{1}{M2^M}\sum_{m=1}^{M}\sum_{b=0}^{1}\sum_{x\in\mathcal{X}_b^m}\exp\left(-\frac{1}{4}\rho d^2(x,x')\right)\right) \tag{13}$$

where for each $x \in \mathcal{X}_b^m$, the symbol x' is the point at minimum squared Euclidean distance $d^2(x,x')$ from x in the complement subset $\mathcal{X}_{\bar{b}}^m$. By letting $\rho \to \infty$ and using Varadhan integral lemma, we keep only the dominant term in the above sum and we get

$$\lim_{\rho\to\infty}\alpha = \frac{d_{\min}^2}{4} = \left(\frac{d_{\min}}{2}\right)^2 \tag{14}$$

where $d_{\min}^2$ is the minimum distance of the constellation. Notice that this asymptotic value does not depend on the binary labeling rule. Also note that the term $\frac{d_{\min}}{2}$ represents the distance from one constellation point to the decision threshold corresponding to its nearest neighbor. Figure 2 shows the SNR scaling α for BICM with 16-QAM with Gray and Set-Partitioning binary labelings as a function of ρ. As predicted by the analysis above, both mappings approach the asymptotic value $\alpha = \frac{d_{\min}^2}{4} = 0.1$. Notice however that Gray mapping shows a smaller scaling thus implying that the equivalent BICM channel is less noisy. This conclusion was already observed directly from the upper bounds with BICM-ML decoding in [2].

We suggest to replace the BICM equivalent channel by a BI-AWGN with scaled SNR $\gamma = \alpha\rho$. Therefore, any suitable bounding technique for binary codes over the BI-AWGN channel can be applied verbatim on the binary code underlying the BICM scheme. The resulting error probability bound will only depend on the SNR of the actual channel ρ, on the scaling factor α,which incorporates the effects of the signal constellation

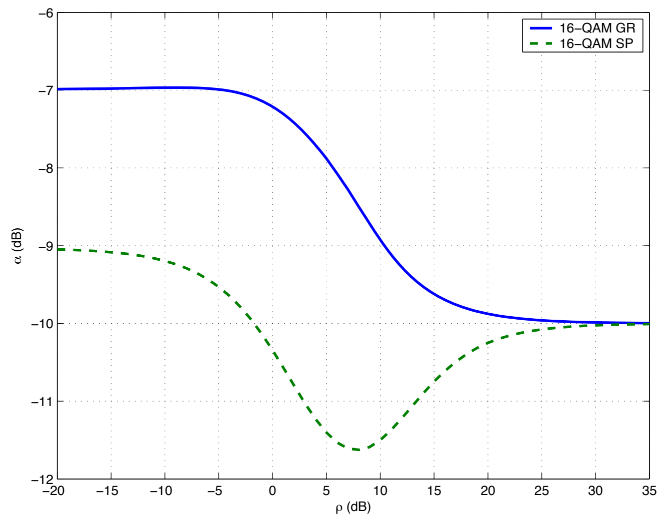

Fig. 2: Signal-to-noise ratio scaling of the equivalent BI-AWGN channel for BICM with 16-QAM and Gray and Set-Partitioning mapping.

$\mathcal{X}$ and the binary labeling μ, and the weight distribution of the underlying binary code.

In order to verify the validity of the proposed Gaussian approximation for the computation of error probabilities, in the following we report several numerical examples. We plot in Figures 3, 4 and 5 the simulated pdfs of the log-likelihood ratio (3) given that a 0 was transmitted[1], denoted by LLR_0, for BICM with 16-QAM and Gray mapping at $\rho = 10$dB and $\rho = 20$dB. We also plot the corresponding Gaussian approximation of the equivalent BI-AWGN channel, i.e., a Gaussian distribution $\mathcal{N}(4\gamma, 8\gamma)$. In the case of $\rho = 10$ dB, $\gamma = 1.07$dB while when $\rho = 20$ dB, $\gamma = 10.16$dB. We observe that in both cases, the error probability behavior, i.e., $\Pr(\text{LLR}_0 < 0)$ is approximately the same for the BICM as for the corresponding Gaussian case, since tails of both distributions are almost identical.

V. Approximations on BICM-ML Error Probability

From the results in the previous section, we here recall some BICM-ML decoding error probability upper bounds for BICM based on the Gaussian approximation of the binary-input BICM equivalent channel. Consider the equivalent binary-input BICM AWGN channel with signal-to-noise ratio γ described in the previous section. Then, the union bound on the frame error probability is given by,

$$P_e \lessapprox \sum_d A_d Q\left(\sqrt{2d\rho\alpha}\right). \tag{15}$$

Notice that in order to compute the bit error probability, we should replace A_d by

$$B_d = \sum_i \frac{i}{K}A_{i,d},$$

where $A_{i,d}$ is the input-output WEF (IOWEF) of $\mathcal{C}$ [9].

[1]Notice that the LLR given that a 1 was transmitted is completely symmetric, since the binary-input BICM channel is output symmetric (BIOS) according to the assumptions made in [2].

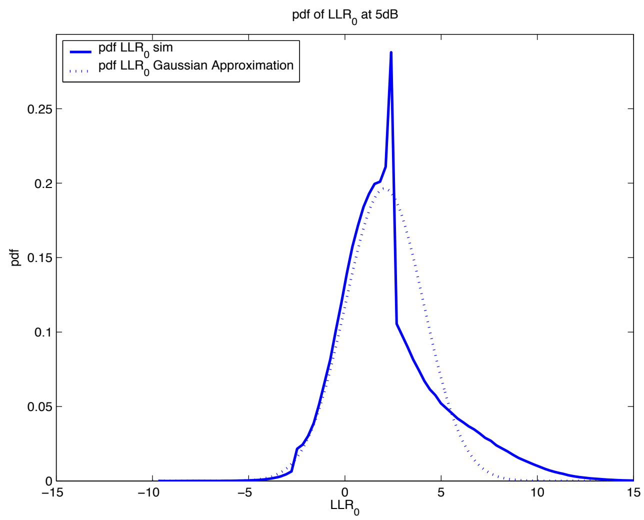

Fig. 3: Simulated pdf of the LLR given that a 0 was transmitted for BICM with 16-QAM and Gray mapping at $\rho = 5$dB.

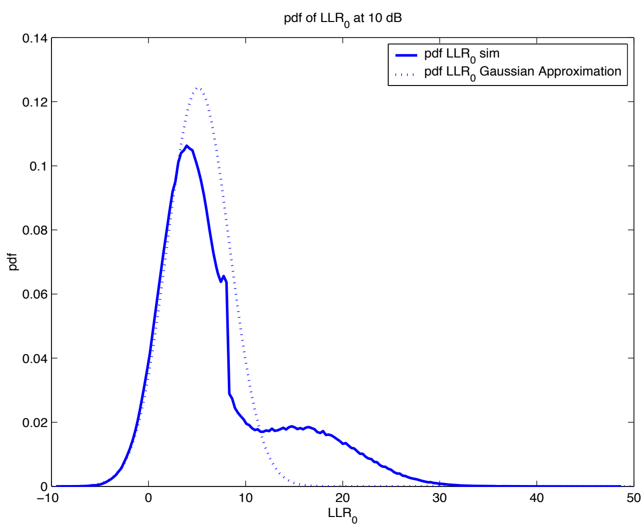

Fig. 4: Simulated pdf of the LLR given that a 0 was transmitted for BICM with 16-QAM and Gray mapping at $\rho = 10$dB.

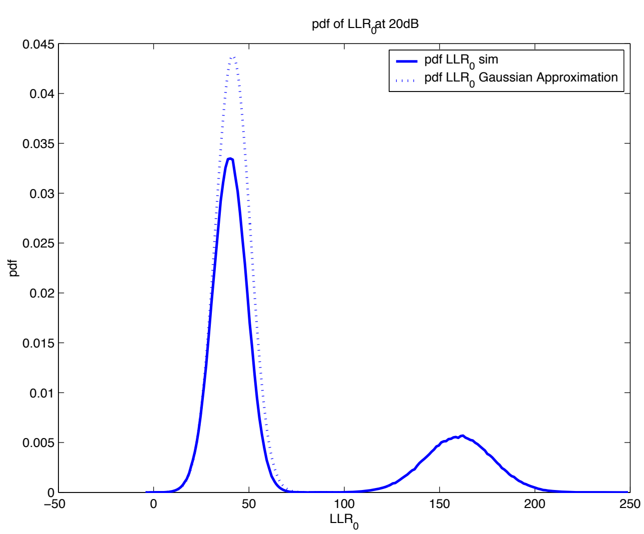

Fig. 5: Simulated pdf of the LLR given that a 0 was transmitted for BICM with 16-QAM and Gray mapping at $\rho = 20$dB.

Union bound-based techniques are known not to provide good estimates of the error probability of capacity-approaching codes over AWGN channels. On the contrary, improved bounding techniques such as the tangential-sphere bound (TSB) [10, 11] have been shown to provide very accurate results. In our case, the tangential-sphere bound is given by,

$$P_e \lessapprox \int_{-\infty}^{+\infty} \frac{dz_1}{\sqrt{2\pi\sigma^2}} e^{-z_1^2/2\sigma^2} \Bigg\{ 1 - \bar{\Gamma}\left(\frac{N-1}{2}, \frac{r_{z_1}}{2\sigma^2}\right) + \\ + \sum_{d:\,\delta/2<\alpha_\delta} A_d \bar{\Gamma}\left(\frac{N-2}{2}, \frac{r_{z_1}^2 - \beta_\delta(z_1)^2}{2\sigma^2}\right) \cdot \\ \cdot \left[Q\left(\frac{\beta_\delta(z_1)}{\sigma}\right) - Q\left(\frac{r_{z_1}}{\sigma}\right)\right] \Bigg\} \tag{16}$$

where $\delta = 2\sqrt{d}$ is the Euclidean distance corresponding to a pairwise error event at Hamming distance d with unit energy, $R^2 = N$ is the squared sphere radius,

$$\sigma^2 = (2\rho\alpha)^{-1} \tag{17}$$

is the variance Gaussian noise corresponding to the equivalent binary-input BICM channel,

$$\bar{\Gamma}(a, x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt$$

is the normalized incomplete gamma function and

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$$

is the gamma function,

$$r_{z_1} = r\big(1 - \frac{z_1}{R}\big),$$

$$\beta_\delta(z_1) = \frac{r_{z_1}}{\sqrt{1 - \frac{\delta^2}{4R^2}}} \frac{\delta}{2r},$$

$$\alpha_\delta = r\sqrt{1 - \frac{\delta^2}{4R^2}}$$

and r, the cone radius, is the solution of

$$\sum_{d:\,\delta/2<\alpha_\delta} A_d \int_0^{\theta_k} \sin^{N-3}\phi d\phi = \frac{\sqrt{\pi}\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)} \tag{18}$$

with

$$\theta_k = \cos^{-1}\left(\frac{\delta}{2r} \frac{1}{\sqrt{1 - \frac{\delta^2}{4R^2}}}\right). \tag{19}$$

Again, notice that the integral in (16) can be efficiently computed using the Gauss-Hermite quadratures.

In Figure 6 we illustrate the BICM-ML error probability approximations presented above for the 64 states rate 1/2 convolutional code with 16-QAM with Gray mapping, with a frame of $K = 128$ information bits. The overall spectral efficiency is $R = 2$ bit/s/Hz. We have used in all bounds the *truncated* bit-error distance spectrum of the code, i.e., we have considered all error events for which $d \leq 256$. We show the bounds for the bit-error probability and we compare them with the bit error rate simulation. We observe that the

Bhattacharyya union bound is quite loose [2], while the union bound with the Gaussian approximation denoted by UB-GA is much tighter. Moreover, as expected, the TSB with the Gaussian approximation, denoted by TSB-GA, is the tightest and offers a better estimate in the low-SNR regime. This suggests that there is no loss in tightness if we use the Bhattacharyya bound, provided that we use it as a means of estimating the variance of an underlying Gaussian channel, and then use the standard bounding functions to estimate the error probability of the channel.

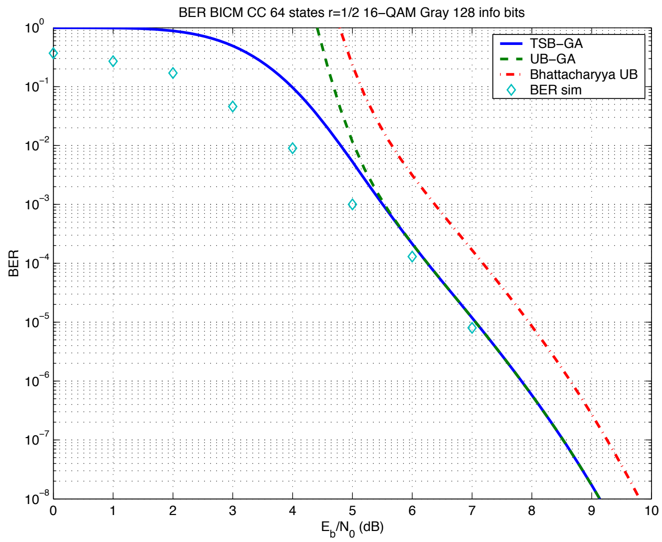

Fig. 6: BER ML BICM Bounds for the 64 states convolutional code of rate 1/2 with 16-QAM and Gray mapping.

Figures 7 and 8 illustrate the same BICM-ML bit error probability approximations and simulation for a repeat-accumulate (RA) code [12] of rate 1/4 with $K = 512$ and $K = 1024$ information bits respectively with 16-QAM with Gray mapping. The overall spectral efficiency is $R = 1$ bit/s/Hz. Notice that for such code ensembles, the weight enumerator can be computed explicitly [12, 13]. For the sake of comparison, in Figure 7 we plot also the corresponding (true ML) bounds and simulation for the BPSK case. We observe the same behavior of the binary case, i.e., the TSB-GA yields a good estimate of the waterfall region while the UB-GA and the Bhattacharyya are only valid for estimating the error floor. Similar comments apply to Figure 9, where we show the BER performance for the quasi-repeat and accumulate (QRA) code ensemble[2] with $K = 1024$ and 16-QAM with Gray mapping. The capacity at $R = 1$ bit/s/Hz is also shown. This constitutes a simple and yet accurate finite length analysis for turbo-like codes with BICM, since convergence analysis of iterative decoding can be very complicated task [6].

VI. BICM-ML Thresholds for Turbo-coded BICM

In [14], the author proposed a tight upper bound on the ML decoding signal-to-noise ratio threshold γ_{th} for binary codes.

[2]We denote the quasi-repeat and accumulate (QRA) ensemble as the serially concatenated convolutional code ensemble of rate $r = 1/q$ that has as outer code generators (in octal form) $(\underbrace{1,\dots,1}_{q-1},3)_8$ and inner accumulator.

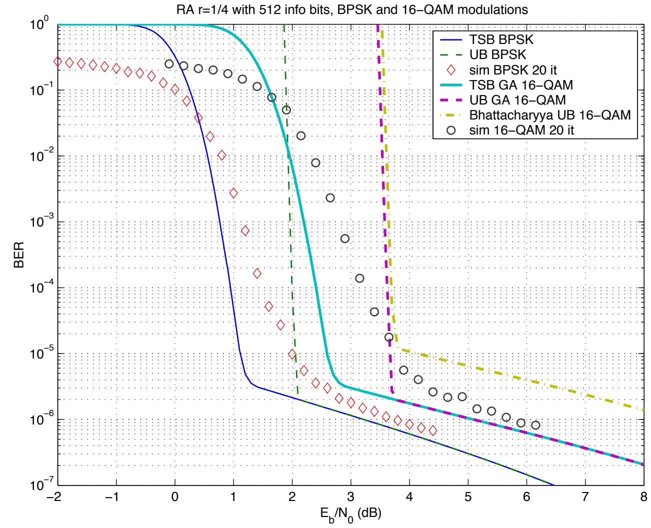

Fig. 7: BER ML BICM Bounds for the a repeat-and-accumulate code of rate 1/4 and $K = 512$ with 16-QAM with Gray mapping.

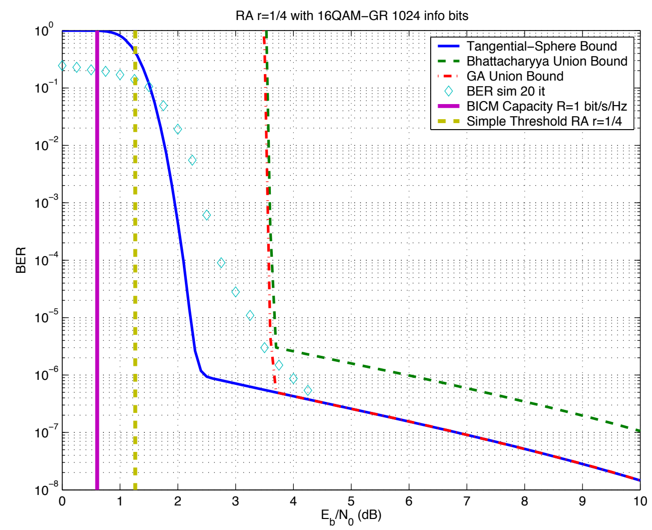

Fig. 8: BER ML BICM Bounds for the a RA code of rate 1/4 and $K = 1024$ with 16-QAM with Gray mapping.

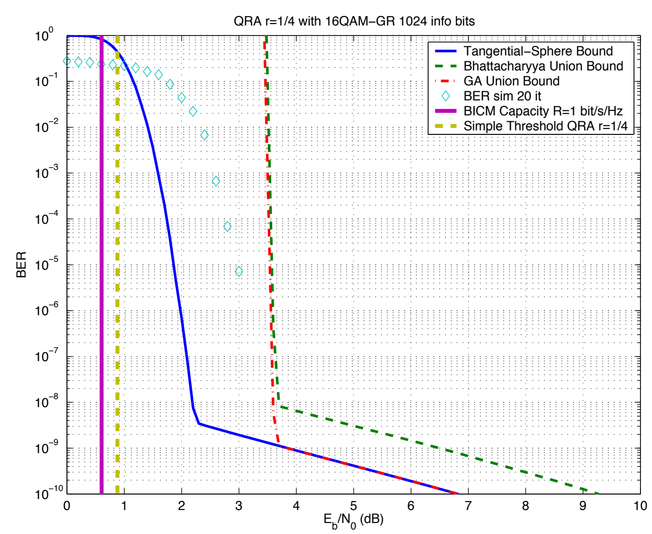

Fig. 9: BER ML BICM Bounds for the a QRA code of rate 1/4 and $K = 1024$ with 16-QAM with Gray mapping.

For signal-to-noise ratios $\gamma > \gamma_{\text{th}}$ the exponent of the simple bound of Divsalar is positive, and therefore, for large block-length $P_e \to 0$ [14]. Following the footsteps of the previous sections, we can easily extend this result to the binary input BICM channel through the Gaussian approximation, which yields that,

$$\left|\frac{E_b}{N_0}\right|_{\text{th}} \leq \frac{1}{\alpha R} \max_{0\leq\omega\leq 1} \left[\frac{(1-e^{-2a(\omega)})(1-\omega)}{2\omega}\right] \quad (20)$$

where $\omega = d/N$ is the normalized output Hamming weight, and

$$a(\omega) = \lim_{N\to\infty} \frac{1}{N} \log A_\omega$$

is the asymptotic growth rate of the normalized weight enumerator A_ω of $\mathcal{C}$ [13]. For the binary RA and QRA code ensembles, table I in [13] reports the upper bounds on the ML $\left|\frac{E_b}{N_0}\right|_{\text{th}}$ thresholds for BPSK modulation. As remarked in [13] the QRA ensemble significantly outperforms the RA ensemble as far as ML thresholds are concerned.

Based on the results obtained in the previous sections, we can easily establish the thresholds for BICM-ML decoding of turbo-coded BICM using the Gaussian approximation. For example, by performing simple computations, when we use BICM with 16-QAM with Gray mapping with a RA code of rate $r = 1/4$, as done in Figures 7 and 8, the BICM-ML with the Gaussian approximation threshold is $\left|\frac{E_b}{N_0}\right|_{\text{th}} = 1.2648$ dB, while the BICM capacity with BICM-ML decoding (4) for 16-QAM with Gray mapping for $R = 1$ bit/s/Hz is at 0.6050 dB (see Figure 1). For the sake of comparison, we show the BICM capacity limit and the simple threshold in Figures 8 and 9. Table VI summarizes the BICM-ML decoding simple bound thresholds (20) using the Gaussian approximation for the RA and QRA ensembles with 16-QAM with Gray mapping with corresponding spectral efficiencies of $R = 1, 2$ bit/s/Hz.

Rate R	Capacity (4)	RA	QRA
1 bit/s/Hz	0.6050 dB	1.2648 dB	0.8820 dB
2 bit/s/Hz	2.2671 dB	6.1512 dB	3.1409 dB

Tab. 1: Upper bounds on the BICM-ML decoding $\left|\frac{E_b}{N_0}\right|_{\text{th}}$ thresholds (20) using the Gaussian approximation for 16-QAM with Gray mapping compared to the BICM capacity limit (4).

Reference [15] provides some ML thresholds based on the union bound. For example, for the RA code ensemble of rate $r = 1/4$ with 16-QAM modulation with Gray mapping, the $\left|\frac{E_b}{N_0}\right|_{\text{th}} = 5.91$ dB. As we can observe, the proposed thresholds are much tighter than those proposed in [15], due to the improved threshold based on the simple bound of [14] and to the accuracy of the Gaussian approximation of the binary-input BICM channel.

VII. Conclusions

We have presented a very accurate and simple to compute approximation to the error probability of BICM under BICM-ML decoding using the Gaussian approximation of the binary-input BICM equivalent channel. We have verified the validity of the approximation to compute error probabilities and we have applied it to compute simple and accurate estimates of the error probability with BICM based on union and tangential-sphere bounds for both convolutional and turbo-like codes. We have also found accurate estimates of the BICM-ML decoding threshold. The key result is that very good approximation is given when the Bhattacharyya bound is used to estimate the variance of the underlying binary-input BICM equivalent channel. The proposed method constitutes a simple and very powerful tool for finite-length ML analysis of capacity approaching codes with BICM.

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