# Rate-Diversity-Delay Tradeoff for ARQ Systems over MIMO Block-Fading Channels

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***Abstract*— We study the effect of power adaptation on the outage diversity of Incremental-redundancy automatic-repeat-request (INR-ARQ) transmission over the multi-input multi-output (MIMO) block-fading channel. In particular, we derive the optimal outage diversity achieved by power adaptation in INR-ARQ systems, and show that power adaptation provides significant gains in outage diversity. We also prove that the optimal outage diversity is achievable with random coding schemes.**

## I. Introduction

Random fluctuations in signal level, otherwise known as fading, induced by scatterers surrounding the transmitter and receiver make reliable wireless communications particularly challenging [1]. Various transmission and coding schemes have been proposed to mitigate fading, depending on the application requirements and the fading characteristics. For applications without stringent delay constraints or for fast fading channels, the channel is considered *ergodic* and long-interleaved fixed-rate codes at rates not exceeding the channel capacity can be employed [2, 3]. On the other hand, for applications where sufficiently long interleaved codewords are prohibited, the channel is *non-ergodic* since each codeword experiences a finite number of channel realizations. A useful channel model for such communication scenarios is the *block-fading* channel [2, 4], where each codeword is transmitted over a finite number of flat, independently faded blocks. Frequency hopping schemes (e.g. Global System for Mobile Communications (GSM) and the Enhanced Data GSM Environment (EDGE)) can be conveniently modelled as block-fading channels.

In the block-fading channel, the instantaneous communication rate supported by the channel is random, and directly dependent on the channel realizations. For a wide variety of potential fading statistics, it follows that for any positive transmission rate there is a non-zero *outage probability* that the rate is not supported by the channel. The outage probability is a lower bound to the word error probability of codes with sufficiently long block length [5]. Consequently, transmission over non-ergodic channels suffers a rate-reliability tradeoff, where a high transmission rate must accept a large error probability and vice versa. In this case, adaptive transmission techniques are essential in providing reliable and efficient high rate communications.

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The INR-ARQ transmission technique is useful for adaptive transmission over non-ergodic channels (see [6] and the references therein). Additionally MIMO transmission has been employed as an efficient technique to improve the throughput and reliability of communication systems. MIMO transmission has revolutionized modern wireless communications, and is now the key technology used in many of today's standards, e.g. WiFi (IEEE 802.11) and WiMax (IEEE 802.16) [7, 8]. In this paper, we study the performance of MIMO INR-ARQ systems over the block-fading channel. An important performance measure of the ARQ system is the rate-diversity-delay tradeoff in block-fading channels, where diversity is (the negative of) the slope of the outage probability versus signal-to-noise ratio (SNR) curve in the log-log scale. The tradeoff has been studied in [9] for systems with Gaussian inputs with and without power control. However, Gaussian input constellations are infeasible in practice, and in [10], Chuang *et al.* studied the tradeoff of more feasible transmission schemes, which use multi-dimensional rotated discrete input constellations with uniform transmit power. In contrast to [10], we study the optimal rate-diversity-delay tradeoff of INR-ARQ transmission over the block-fading channel using two-dimensional discrete input constellations with adaptive transmit power. We demonstrate that power adaptation offers a significantly large outage diversity and outage performance gain. We also prove that the optimal outage diversity is achievable by employing random codes across the transmit antennas.

The remainder of the paper is organized as follows. Section II describes the system model. In Section III we provide relevant background material on the mutual information and outage probability of the system. Then the main contribution of the paper is given in Section IV, while concluding remarks are then given in Section V.

## II. System Model

### A. The MIMO block-fading channel

Consider INR-ARQ transmission over the block-fading channel with $N_t$ transmit and $N_r$ receive antennas. Information transmission takes place over $L$ ARQ rounds, where each round is composed of $B$ additive white Gaussian noise (AWGN) blocks of $T$ channel uses. Each block is affected by an independently identically distributed (i.i.d.) flat fading channel gain matrix $\boldsymbol{H}_{\ell,b} \in \mathbb{C}^{N_r \times N_t}$, for $\ell = 1, \dots, L$ and $b = 1, \dots, B$. The baseband equivalent of the channel in the $\ell$-th ARQ round is given by

$$\boldsymbol{Y}_\ell = \sqrt{\frac{P_\ell}{N_t}} \boldsymbol{H}_\ell \boldsymbol{X}_\ell + \boldsymbol{W}_\ell, \tag{1}$$

where $\boldsymbol{X}_\ell \in \mathbb{C}^{BN_t \times T}$ and $\boldsymbol{Y}_\ell, \boldsymbol{W}_\ell \in \mathbb{C}^{BN_r \times T}$ are correspondingly the transmitted signal, the received signal and the AWGN. The overall block diagonal channel gain matrix $\boldsymbol{H}_\ell$ at round $\ell$ is given by

$$\boldsymbol{H}_\ell = \operatorname{diag}(\boldsymbol{H}_{\ell,1}, \dots, \boldsymbol{H}_{\ell,B}).$$

Moreover, the baseband equivalent channel after $\ell$ transmission rounds is

$$\boldsymbol{Y}_{\overline{1,\ell}} = \boldsymbol{H}_{\overline{1,\ell}} \boldsymbol{X}_{\overline{1,\ell}} + \boldsymbol{W}_{\overline{1,\ell}}, \tag{2}$$

with

$$\begin{aligned}
\boldsymbol{Y}_{\overline{1,\ell}} &= \left[\boldsymbol{Y}_1', \dots, \boldsymbol{Y}_\ell'\right]' \\
\boldsymbol{X}_{\overline{1,\ell}} &= \left[\boldsymbol{X}_1', \dots, \boldsymbol{X}_\ell'\right]' \\
\boldsymbol{H}_{\overline{1,\ell}} &= \operatorname{diag}\left(\sqrt{\frac{P_1}{N_t}}\boldsymbol{H}_1, \dots, \sqrt{\frac{P_\ell}{N_t}}\boldsymbol{H}_\ell\right) \\
\boldsymbol{W}_{\overline{1,\ell}} &= \left[\boldsymbol{W}_1', \dots, \boldsymbol{W}_\ell'\right]',
\end{aligned}$$

where $(\cdot)'$ denotes non-conjugate transpose.

We consider transmissions with input constellation $\mathcal{X}$ of size $|\mathcal{X}| = 2^M$ and unit average energy, i.e., entries $x \in \mathcal{X}$ of $\boldsymbol{X}_\ell$ satisfy $\mathbb{E}\left[|x|^2\right] = 1$. Furthermore, we assume that the entries of $\boldsymbol{H}_{\ell,b}$ and $\boldsymbol{W}_{\ell,b}$ are independently drawn from a unit variance Gaussian complex distribution $\mathcal{N}_{\mathbb{C}}(0,1)$ and $\boldsymbol{H}_{\ell,b}$ is known perfectly at the receiver. Therefore, the average SNR at each receive antenna is $P_\ell$.

### B. The ARQ System

Consider a codebook $\mathcal{C}$ of rate $\frac{R_M}{L}$ bits per coded symbol, $R_M \in (0, M)$, that maps a message $m \in \left\{1, \dots, 2^{R_M N_t T B}\right\}$ into a codeword $\boldsymbol{x}(m) \in \mathcal{X}^{N_t T B L}$. Consider the first ARQ round for transmission of message $m$. In this case, the first $N_t T B$ coded symbols of $\boldsymbol{x}(m)$ are formatted into $\boldsymbol{X}_1(m) \in \mathcal{X}^{BN_t \times T}$ and transmitted through the MIMO block-fading channel in (1) within $BT$ channel uses. The realized code rate at the first ARQ round is $R = N_t R_M$ bits per channel use (bpcu). The receiver attempts to decode the noisy received sequence $\boldsymbol{Y}_1$ and, if unsuccessful, feeds back a negative acknowledgement (NACK) to the transmitter; otherwise, the receiver delivers a positive acknowledgement (ACK) back to the transmitter. Upon receiving a NACK, the transmitter starts another round with the next $N_t T B$ coded symbols of $\boldsymbol{x}(m)$. After $\ell$ transmission rounds, the transmit signal is $\boldsymbol{X}_{\overline{1,\ell}}(m)$ and the resulting code rate is $\frac{R}{\ell}$ bpcu. The process continues until a positive acknowledgement (ACK), denoting that the codeword has been correctly decoded, is received or until $L$ transmission rounds have elapsed.

At round $\ell$, the receiver attempts to decode the message from the accumulated received signal $\boldsymbol{Y}_{\overline{1,\ell}}$. We employ the decoding described in [6], where the decoder outputs $\hat{m}$ if $\boldsymbol{x}(\hat{m})$ is the only codeword such that $\boldsymbol{X}_{\overline{1,\ell}}(\hat{m})$ and $\boldsymbol{Y}_{\overline{1,\ell}}$ are jointly typical [11]; otherwise, an error is detected. An outage is declared if the receiver fails to decode after $L$ transmission rounds, where $L$ represents the delay constraint of the system.

Let $P_e(\ell)$ be the probability that an error is detected at round $\ell$. If $\ell < L$, $P_e(\ell)$ is also the probability of having to transmit at round $\ell + 1$. We consider systems where the average power for each code word is limited to $P$, so the transmit powers $P_\ell, \ell = 1, \dots, L$ must satisfy

$$\sum_{\ell=1}^{L} P_e(\ell - 1) P_\ell \le P, \tag{3}$$

where $P_e(0) = 1$ by definition.

## III. Mutual Information and Outage Probability

The input-output mutual information of the MIMO ARQ block-fading channel (1) is [12]

$$I_\ell = \frac{1}{B} \sum_{b=1}^{B} I\left(\sqrt{\frac{P_\ell}{N_t}} \boldsymbol{H}_{\ell,b}\right), \tag{4}$$

where $I\left(\sqrt{\frac{P_\ell}{N_t}}\boldsymbol{H}_{\ell,b}\right)$ is the input-output mutual information (in bpcu) [11] of a MIMO channel with input constellation $\mathcal{X}$, unit-variance additive white Gaussian noise and channel gain matrix $\sqrt{\frac{P_\ell}{N_t}}\boldsymbol{H}_{\ell,b}$. The average input-output mutual information after $\ell$ transmission rounds is $\frac{1}{\ell} I_{\overline{1,\ell}}$ (bpcu), where

$$I_{\overline{1,\ell}} \triangleq \sum_{j=1}^{\ell} I_\ell \tag{5}$$

is defined as the accumulated mutual information after $\ell$ rounds. Since the realized code rate at round $\ell$ is $\frac{R}{\ell}$, the information outage probability at round $\ell$ is given by

$$p(\ell) = \Pr\left\{\frac{1}{\ell}\sum_{j=1}^{\ell} I_j < \frac{R}{\ell}\right\} = \Pr\left\{I_{\overline{1,\ell}} < R\right\}. \tag{6}$$

Under the assumption of an infinitely long block length $T$ and optimal coding scheme, an error is detected with probability 1 if an information outage occurs [6]; otherwise, the codeword is correctly decoded. Therefore, the achievable error probability $P_e(\ell)$ is given by the outage probability in (6). In other words, $P_e(\ell) = p(\ell)$ when $T \to \infty$. In the sequel, we analyze the corresponding outage diversity of this ARQ system.

## IV. Outage Diversity Analysis

The outage diversity [13] represents the slope of the outage versus SNR curve on a log-log scale. Let $d_\ell(R)$ be the outage diversity of the ARQ system at round $\ell$, then the outage probability is asymptotically (for large SNR) given by

$$p(\ell) \doteq P^{-d_\ell(R)}, \tag{7}$$

where the exponential equality ($\doteq$) denotes that [13]

$$d_\ell(R) = \lim_{P\to\infty} \frac{-\log p(\ell)}{\log P}. \tag{8}$$

We first study the rate-diversity tradeoff of a block-fading MIMO channel. The results are also fundamental in the analysis of the rate-diversity-delay tradeoff of ARQ systems in the subsequent sections.

### A. Rate-Diversity Tradeoff of MIMO Block-Fading Channel

The rate diversity tradeoff of the MIMO block-fading channel is given by $d_1(R)$ defined in (8). The following Proposition gives an upper bound on $d_1(R)$ [10, 14, 15].

*Proposition 1:* Assume $|\mathcal{X}| = 2^M$, transmit power $P_1 = P$, and consider transmission rate $R$ over the block-fading channel in (1) with $L = 1$. Asymptotic to $P$, the outage probability behaves like

$$p(1) \doteq P^{-d_1(R)}, \tag{9}$$

where the outage diversity $d_1(R)$ is upper bounded by

$$d_1(R) \le d(R) \triangleq N_r \left(1 + \left\lfloor B\left(N_t - \frac{R}{M}\right)\right\rfloor\right). \tag{10}$$

*Proof:* The upper bound is obtained by assuming a genie aided receiver that is able to completely remove the interference between the transmit antennas. This results in a system with $N_t$ parallel channels, each with one transmit antenna and $N_r$ receive antennas. Following the methods in [12, 16], we obtain the outage diversity $d(R)$ in (10). ■

Proposition 1 gives $d(R)$ as an upper bound on the outage diversity of MIMO block-fading channels. We now show that $d(R)$ is actually the optimal outage diversity.

*Proposition 2:* Consider communication with one transmission round over the MIMO block-fading channel given in (1) using the scheme described in Section II-B, where the coded symbols in $\mathcal{C}$ are uniformly drawn from $\mathcal{X}$. In the limit $T \to \infty$, the word error probability is asymptotically given by

$$P_e(1) \doteq P^{-d_1^{(r)}(R)}, \tag{11}$$

where the SNR-exponent $d_1^{(r)}(R)$ is lower bounded by

$$d_1^{(r)}(R) \ge N_r \left\lceil B\left(N_t - \frac{R}{M}\right)\right\rceil \tag{12}$$

*Proof:* The proof follows the argument in [12], where an upper bound on the word error probability is obtained via Chernoff and union bounding. The outage diversity achieved by random coding is obtained by analyzing the asymptotic word error probability (averaged over the ensemble of random codes) when $P \to \infty$. ■

Noting that $d_1^{(r)}(R) = d(R)$ at all rates $R$ such that $d(R)$ is continuous, the optimal outage diversity of the MIMO block-fading channel is given by $d_1(R) = d(R)$ in (10), except for the discontinuous points. Furthermore, the optimal outage diversity is achievable by random coding. This tradeoff plays a crucial role in the analysis of the rate-delay-diversity tradeoff of ARQ channels, as will be illustrated in the next section.

### B. ARQ rate-diversity-delay tradeoff

In this section, we study the optimal rate-diversity tradeoff of the ARQ system as a function of the maximum allowable number of transmission rounds, when power control is allowed. An upper bound on the outage diversity at transmission round $\ell$ is given as follows.

*Proposition 3:* Consider transmission with the ARQ scheme described in II-B over the block-fading channel given in (1). Assume that the optimal power adaptation rule with power constraint given in (3) is employed. The probability of having an outage at round $\ell$ is asymptotically given by

$$p(\ell) \doteq P^{-d_\ell(R)}, \tag{13}$$

where $d_\ell(R)$ is the optimal rate-diversity-delay tradeoff given by $d_1(R) = d(R)$ and

$$d_\ell(R) = BN_tN_r\left(\ell - 1 + \sum_{j=1}^{\ell-2} d_j(R)\right) + (1 + d_{\ell-1}(R))d(R) \tag{14}$$

for $\ell \ge 2$. The optimal tradeoff is achieved by the following power adaptation rule

$$P_\ell = \frac{P}{Lp(\ell-1)}, \ell = 1, \ldots, L, \tag{15}$$

*Proof:* See Appendix I. ■

Proposition 3 gives $d_\ell(R)$ as the optimal outage diversity at round $\ell$ for INR-ARQ transmission over the MIMO block-fading channel with a finite discrete input constellation. We now prove that $d_\ell(R)$ is achievable by random coding.

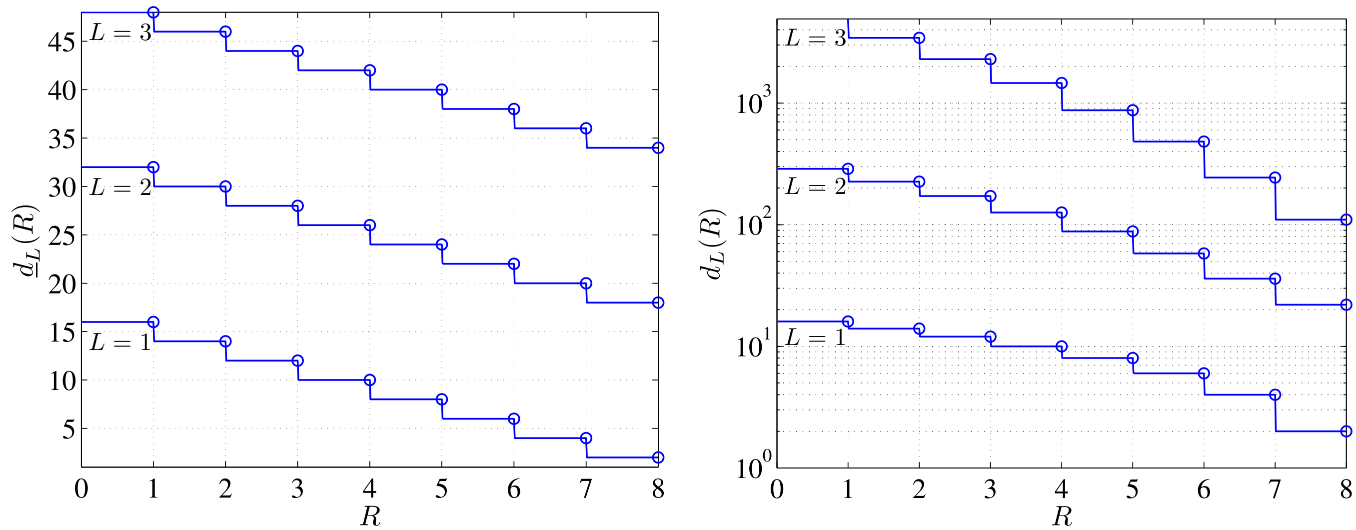
*Proposition 4:* Consider transmission over the MIMO block-fading channel in (1) using the scheme described in Section II-B, where the code symbols in $\mathcal{C}$ are uniformly drawn from $\mathcal{X}$. In the limit $T \to \infty$, the error probability at round $\ell$ is asymptotically given by

$$P_e(\ell) \doteq P^{-d_\ell^{(r)}(R)}, \tag{16}$$

where $d_\ell^{(r)}(R)$ is the achievable SNR-exponent at round $\ell$ satisfying

$$\begin{aligned} d_1^{(r)} &= N_r \left\lceil B\left(N_t - \frac{R}{M}\right)\right\rceil \\ d_\ell^{(r)} &= BN_tN_r\left(\ell - 1 + \sum_{j=1}^{\ell-2} d_j^{(r)}(R)\right) + \\ &\qquad (1 + d_{\ell-1}^{(r)}(R))d^{(r)}(R), \ell \ge 2. \end{aligned} \tag{17}$$

*Proof:* A sketch of the proof is given as follows. Firstly, consider transmission rates $R$ such that $\frac{BR}{M}$ is not an integer.




(a) Constant Power Tradeoff

(b) Optimal Tradeoff

Fig. 1. The rate-diversity-delay tradeoff for ARQ transmission over MIMO block-fading channels with $N_t = N_r = 2, B = 4$, $L = 1, 2, 3$ using 16-QAM constellations. The left figure illustrates the tradeoff achievable with constant power, whilst the right figure shows the optimal tradeoff when power adaptation is employed. The circles represent rates where the SNR-exponent of random codes does not achieve the outage diversity.

Consider a random coding scheme, where the coded symbols in $\mathcal{C}$ are uniformly drawn from $\mathcal{X}$, and the following decoder. After transmission round $\ell$, the decoder declares an outage if $I_{\overline{1,\ell}} < R$; otherwise, maximum likelihood decoding is performed. It can be shown that in the limit $T \to \infty$ and $P \to \infty$, the codeword is correctly decoded with probability 1 if $I_{\overline{1,\ell}} \geq R$. Therefore, $P_e(\ell) = p(\ell)$ and thus $d_\ell^{(r)}(R) = d_\ell(R)$ is achievable following Proposition 3. When $\frac{BR}{M}$ is an integer, $d_\ell^{(r)}(R)$ is achievable by following the same argument, where the decoder declares an error if $I_{\overline{1,\ell}} \leq R$. ∎

From Propositions 3 and 4, we have that $d_\ell^{(r)}(R) = d_\ell(R)$, and thus $d_\ell(R)$ is the achievable outage diversity at round $\ell$, for all transmission rates such that $\frac{BR}{M}$ is not an integer. This implies that the INR-ARQ system features the following rate-diversity-delay tradeoff.

*Theorem 1:* Consider INR-ARQ transmission over the MIMO block-fading channel given in (1) with $|\mathcal{X}| = 2^M$. Assume that the power constraint given in (3) and a delay constraint of $L$ transmission rounds is enforced. The outage diversity is asymptotically given by

$$p(L) \doteq P^{-d_L(R)}, \tag{18}$$

where $d_L(R)$ is given by Proposition 3. Furthermore, in the limit of infinite block length $T \to \infty$, at rates $R$ such that $d(R)$ is continuous, the outage diversity is achievable using random codes with the decoding scheme described in Section II-B.

### C. Numerical Results

The rate-diversity-delay tradeoff of a MIMO ARQ system with $N_t = N_r = 2, B = 4$ and $M = 4$ is illustrated in Figure 1. For comparison, we also consider INR-ARQ systems with constant transmit power. Let $\underline{d}_L(R)$ be the outage diversity achieved by INR-ARQ transmission with constant transmit power and delay a constraint of $L$ transmission rounds. Then $\underline{d}_L(R)$ is given by the outage diversity of a block-fading channel with $BL$ fading blocks and transmission rate $\frac{R}{L}$, thus [12, 16]

$$\underline{d}_L(R) = N_r \left(1 + \left\lfloor BL\left(N_t - \frac{R}{LM}\right)\right\rfloor\right). \tag{19}$$

The left and right subfigures, respectively, illustrate the constant-power tradeoff $\underline{d}_L(R)$ and the optimal tradeoff $d_L(R)$. Figure 1 shows that $d_1(R) = \underline{d}_1(R)$ as expected. For $L \geq 2$, we observe significant gains in the outage diversity when power adaptation is employed. Moreover, the outage diversity is achievable using random codes at most transmission rates. The cost to achieve the diversity gain is the potentially large transmit power in later ARQ rounds, which may be prohibitive due to physical limitations of the system.

We now study the gain in outage performance provided by power adaptation using the rule suggested in (15). Figure 2 illustrates the outage probability at transmission round $1, 2, 3$ of an ARQ system with $N_t = N_r = 1, B = 1, L = 3, R = 3$ using 16-QAM constellation. The solid and dashed lines correspondingly represent the outage diversity of the system with and without power adaptation. The system outage probability is illustrated by the curves corresponding to $\ell = 3$. The outage diversity at rounds 2 and 3 is $\underline{d}_2(R) = 2$ and $\underline{d}_3 = 3$ for system with constant transmit power as predicted in (19). For system with power adaptive rule given in (15), we observe that $d_2(R) = 3$ and that $d_3(R)$ is approaching 7 as expected from (14). This results in a significant gain in outage performance at high SNR. On the contrary, at low SNR, we observe that power adaptation results in a higher error probability at rounds $\ell < L$. The performance degradation arrives from the

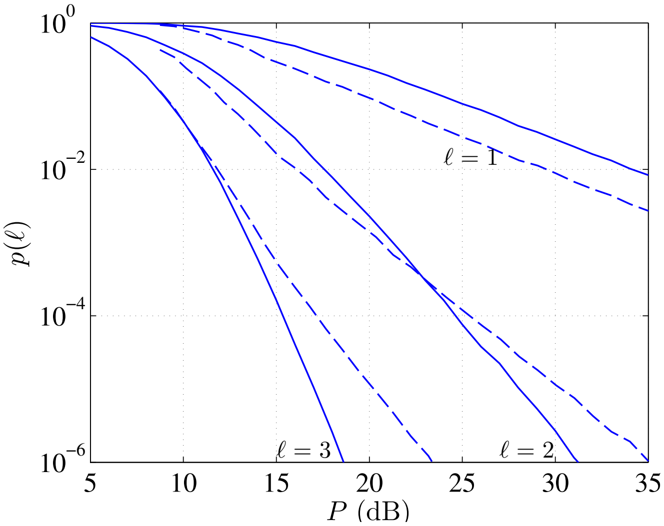



Fig. 2. Error probability $p(\ell)$ at round $\ell = 1, 2, 3$ in the ARQ system with $N_t = N_r = 1, B = 1, L = 3, R = 3$ using 16-QAM constellation. The dashed lines and solid lines correspondingly represent the error probability of system with constant transmit power and system with power adaptation.

suboptimal power allocation rule, where a significant portion of the power has been reserved for increasing the transmit power at later transmission rounds. Therefore, the suboptimal power adaptation scheme in (15) may negatively affect the throughput of the system, especially at low SNR, where the gain in outage performance obtained by power adaptation does not dominate.

## V. Conclusion

We have studied the optimal outage diversity of the INR-ARQ MIMO block-fading channel with power adaptation, and we showed that significant gains can be obtained. Furthermore, the optimal outage diversity can be achieved by employing random codes across the multiple transmit antennas. We also presented a simple adaptation scheme that gives optimal outage diversity, and significant outage probability gains at high SNR.

## Appendix I
## Proof of Proposition 3

For readability, we first provide an overview of the proof. At the first step, we prove that the optimal power adaptation rule is asymptotically given by $P_\ell \doteq P^{1+d_{\ell-1}(R)}$. The asymptotic outage probability $d_{\ell+1}(R)$ in round $\ell+1$ can be obtained by applying Proposition 1 if $d_\ell(R)$ is known. The tradeoff in (14) can therefore be obtained via induction. The details of the proof are given as follows.

From the power constraint in (3), we have that $P_\ell \le \frac{P}{P_e(\ell-1)} = \frac{P}{p(\ell-1)}$. Therefore, we obtain a lower bound on the outage probability by considering the following power adaptation rule

$$\overline{P}_\ell = \frac{P}{p(\ell-1)} \doteq P^{1+d_{\ell-1}(R)}. \tag{20}$$

Furthermore, the following suboptimal power adaptation rule

$$\underline{P}_\ell = \frac{P}{Lp(\ell-1)}, \tag{21}$$

which satisfies the power constraint in (3), also gives

$$\underline{P}_\ell \doteq P^{1+d_{\ell-1}(R)}. \tag{22}$$

Therefore, the optimal power adaptation rule asymptotically satisfies

$$P_\ell \doteq P^{1+d_{\ell-1}(R)}. \tag{23}$$

We derive the optimal outage diversity by analyzing the asymptotic outage probability achieved by the power adaptation rule (23).

Since $P_1 = P$, we first have from Propositions 1 and 2 that

$$p(1) \doteq P^{-d_1(R)}, \tag{24}$$

where $d_1(R) = d(R)$.

For $k = 0, \ldots, BN_t$, let $\hat{I}_k = \frac{Mk}{B}$. Let $\tau$ be an integer satisfying $\hat{I} < R \le \hat{I}_{\tau+1}$. Applying Proposition 1 for $I \in \left(\hat{I}_k, \hat{I}_{k+1}\right]$, we have that

$$\Pr\{I_1 < I\} \doteq P^{-d_1^\dagger(k)}, \tag{25}$$

where $d_1^\dagger(k) \triangleq d_1(\hat{I}_{k+1})$. Furthermore,

$$\begin{aligned}\Pr\left\{I_1 \in \left[\hat{I}_k, \hat{I}_{k+1}\right)\right\} &= \Pr\left\{I_1 < \hat{I}_{k+1}\right\} - \Pr\left\{I_1 < \hat{I}_k\right\} \\ &\doteq P^{-d_1^\dagger(k)}. \end{aligned} \tag{26}$$

For $k = 0, \ldots BN_t - 1$ and a given $I \in \left(\hat{I}_k, \hat{I}_{k+1}\right]$, we now prove by induction that for $\ell = 1, \ldots, L$,

$$\Pr\left\{I_{\overline{1,\ell}} < I\right\} \doteq \Pr\left\{I_{\overline{1,\ell}} \in \left[\hat{I}_k, \hat{I}_{k+1}\right)\right\} \doteq P^{-d_\ell^\dagger(k)}, \tag{27}$$

where $d_0^\dagger(k) = 0$ and

$$\begin{aligned} d_\ell^\dagger(k) = d_\ell(\hat{I}_{k+1}) = N_t N_r B \left(\ell - 1 + \sum_{j=1}^{\ell-2} d_j^\dagger(\tau)\right) + \\ (1 + d_{\ell-1}^\dagger(\tau)) d_1^\dagger(k+1) \end{aligned} \tag{28}$$

Equations (25) and (26) prove that (27) is correct at round 1. Assume that (27) is correct at round $\ell$. Since $R \in \left(\hat{I}_\tau, \hat{I}_{\tau+1}\right]$, we have that $d_\ell(R) = d_\ell^\dagger(\tau)$. Therefore, the power adaptation rule (23) gives $P_{\ell+1} \doteq P^{1+d_\ell^\dagger(\tau)}$.

For $I \in \left(\hat{I}_k, \hat{I}_{k+1}\right]$, we have that

$$\begin{aligned} &\Pr\left\{I_{\overline{1,\ell+1}} < I\right\} = \\ &\sum_{j=0}^{k} \Pr\left\{I_{\overline{1,\ell}} \in \left[\hat{I}_j, \hat{I}_j + I - \hat{I}_k\right)\right\} \times \\ &\qquad \Pr\left\{I_{\ell+1} + I_{\overline{1,\ell}} < I \,\middle|\, I_{\overline{1,\ell}} \in \left[\hat{I}_j, \hat{I}_j + I - \hat{I}_k\right)\right\} \\ &+ \sum_{j=0}^{k} \Pr\left\{I_{\overline{1,\ell}} \in \left[\hat{I}_j + I - \hat{I}_k, \hat{I}_{j+1}\right)\right\} \times \\ &\qquad \Pr\left\{I_{\ell+1} + I_{\overline{1,\ell}} < I \,\middle|\, I_{\overline{1,\ell}} \in \left[\hat{I}_j + I - \hat{I}_k, \hat{I}_{j+1}\right)\right\} \end{aligned}$$

Since $\hat{I}_j + I - \hat{I}_k \in \left(\hat{I}_j, \hat{I}_{j+1}\right]$, it follows from (27) that

$$\Pr\left\{ I_{\overline{1,\ell}} \in \left[\hat{I}_j, \hat{I}_j + I - \hat{I}_k\right) \right\} \doteq \Pr\left\{ I_{\overline{1,\ell}} \in \left[\hat{I}_j + I - \hat{I}_k, \hat{I}_{j+1}\right) \right\} \doteq P^{-d_\ell^\dagger(j)}. \quad (29)$$

Therefore,

$$\Pr\left\{ I_{\overline{1,\ell+1}} < I \right\} \doteq \sum_{j=0}^{k} P^{-d_\ell^\dagger(j)} \Pr\left\{ I_{\ell+1} < I - I_{\overline{1,\ell}} \,\middle|\, I_{\overline{1,\ell}} \in \left[\hat{I}_j, \hat{I}_j + I - \hat{I}_k\right) \right\}.$$

Since $I_{\overline{1,\ell}} \in \left(\hat{I}_j, \hat{I}_j + I - \hat{I}_k\right]$ and $I \in \left(\hat{I}_k, \hat{I}_{k+1}\right]$, we have that $I - I_{\overline{1,\ell}} \in \left(\hat{I}_{k-j}, \hat{I}_{k-j+1}\right]$. Therefore, applying Proposition 1, noting that the transmit power is $P_{\ell+1} \doteq P^{1+d_\ell^\dagger(\tau)}$, we have that

$$\Pr\left\{ I_{\overline{1,\ell+1}} < I \right\} \doteq \sum_{j=0}^{t} P^{-d_\ell^\dagger(j)} \left(P^{1+d_\ell^\dagger(\tau)}\right)^{-d(\hat{I}_{k-j+1})}. \quad (30)$$

Since the term with $j = 0$ dominates in (30), applying the assumption in (28) for $d_\ell^\dagger(j)$, we have that

$$\Pr\left\{ I_{\overline{1,\ell+1}} < I \right\} \doteq P^{-d_{\ell+1}^\dagger(k)}, \quad (31)$$

where $d_{\ell+1}^\dagger(k)$ is as given in (28). Therefore, the assumption in (28) is valid for round $\ell + 1$. By induction, (28) is valid for $\ell = 1, \ldots, L$. Noting that $R \in \left(\hat{I}_\tau, \hat{I}_{\tau+1}\right]$, we have that $d_\ell(R) = d_\ell^\dagger(\tau)$, which concludes the proof of the Proposition.

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