A Closed-Form Approximation for the Error Probability of Coded BPSK Fading Channels

Alfonso Martinez
Electrical Engineering Dept.
Technische Universiteit Eindhoven
Postbus 513, 5600 MB
Eindhoven, The Netherlands
e-mail: alfonso.martinez@ieee.org

Albert Guillén i Fàbregas
Institute for Telecommunications Research
Univ. of South Australia
SPRI Building - Mawson Lakes Blvd
Mawson Lakes SA 5095, Australia
e-mail: albert.guillen@unisa.edu.au

Giuseppe Caire
Electrical Engineering Dept.
University of Southern California
3740 McClintock Ave.
Los Angeles, CA 9008, USA
e-mail: caire@usc.edu

***Abstract*— This paper presents a simple closed-form expression to evaluate the error probability of binary fully-interleaved fading channels. The proposed expression does not require a numerical Laplace transform inversion, numerical integration or similar techniques, and captures the role of the relevant system parameters in the overall error performance. The expression has the same asymptotic behavior as the Bhattacharyya (Chernoff)-union bound but closes the gap with the simulation results. Its precision is numerically validated for coded and uncoded transmission over generic Nakagami fading channels.**

I. Introduction and Main Result

The computation of error probabilities in fading channels suffers from the absence of a simple formula akin to the $Q(\cdot)$ function in pure additive white Gaussian noise (AWGN) channels. The Chernoff bound [1] can be expressed in closed form but it is loose. An alternative method [2] uses Craig's expression of the $Q(\cdot)$ function, which usually requires numerical integration.

In a recent paper [3] we proposed the use of the saddle-point approximation to evaluate the error probability of bit-interleaved coded modulation (BICM) [4]. In this paper, we particularize the analysis for the case of binary transmission (BPSK) and show that the approximation admits a simple closed-form expression.

In particular, we shall show that for Nakagami fading with parameter m [1], average signal-to-noise ratio SNR, and a diversity scheme with D identical branches, the error probability between two codewords at Hamming distance d, or pairwise error probability (PEP), can be closely approximated by

$$\mathrm{PEP}(d, \mathrm{SNR}) \simeq \frac{1}{2\sqrt{\pi d\, \mathrm{SNR}}}\left(1 + \frac{\mathrm{SNR}}{mD}\right)^{-mdD+\frac{1}{2}}. \quad (1)$$

It is worth recalling that Nakagami fading subsumes as special cases the Rayleigh, Rice, and unfaded AWGN channels. The approximation is therefore valid for these cases as well. The precision of the proposed approximation is validated for uncoded and coded transmission using either convolutional or turbo-like codes, over generic Nakagami fading channels.

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II. Error Probability Analysis

A. Channel Model

We study coded modulation over binary-input (BPSK) Gaussian noise channels. The discrete-time received signal can be expressed as

$$y_k = \sqrt{\mathrm{SNR}}\, h_k\, x_k + z_k, \quad k = 1, \dots, N \quad (2)$$

where $y_k \in \mathbf{R}$ is the k-th received sample, $h_k \in \mathbf{R}$ is the k-th fading attenuation, $x_k \in \{-1, +1\}$ is the transmitted signal at time k, and $z_k \in \mathbf{R}$ is the k-th noise sample, assumed to be i.i.d. $\sim \mathcal{N}(0, \frac{1}{2})$. The codewords $\mathbf{x} = \big(x_1, \dots, x_N\big)$ are obtained by mapping the codewords $\mathbf{c} = \big(c_1, \dots, c_N\big)$ of the code $\mathcal{C}$, each of dimension K information bits and length N, with the labelling rule $0 \to -1, 1 \to +1$. The corresponding trasnsmission rate is $R = \frac{K}{N}$ bits per channel use. The average received signal-to-noise ratio is SNR. Perfect channel state information (CSI) at the receiver is assumed. We shall consider a general Nakagami-m fading [2] with $m \in (0, +\infty)$. Therefore the coefficients h_k follow the distribution

$$\mathrm{Pr}_{h_k}(h_k) = \frac{2m^m h_k^{2m-1}}{\Gamma(m)} e^{-mh_k^2}. \quad (3)$$

and the squared fading coefficient $\chi_k = |h_k|^2$ has the distribution

$$\mathrm{Pr}_{\chi_k}(\chi_k) = \frac{m^m \chi_k^{m-1}}{\Gamma(m)} e^{-m\chi_k}. \quad (4)$$

We recover the unfaded AWGN with $m \to +\infty$, the Rayleigh fading by letting $m = 1$ and the Rician fading with parameter K by setting $m = (K+1)^2/(2K+1)$.

B. Error Probability Under ML Decoding

For maximum likelihood decoding (ML) the error probability of linear binary codes is accurately given by the union bound in a region above the cut-off rate [5]. The codeword error probability Pr_e is very closely upper bounded by

$$\mathrm{Pr}_e \le \sum_d A_d\ \mathrm{PEP}(d, \mathrm{SNR}), \quad (5)$$

[2]Even though the case $m \geq 0.5$ is usually considered in the literature [1], [2], the distribution is well-defined for $0 < m < 0.5$. In general reliable transmission is possible for $m > 0$.

where A_d denotes the number of codewords in $\mathcal{C}$ with Hamming weight d, $\mathrm{PEP}(d,\mathrm{SNR})$ is the pairwise error probability (PEP) for two codewords differing in d bits. Similarly, the bit-error probability P_b is given by the right-hand side of Eq. (5) with A_d replaced by $\tilde{A}_d = \sum_i \frac{i}{K} A_{i,d}$, $A_{i,d}$ being the number of codewords in $\mathcal{C}$ with output Hamming weight d and input weight i. Besides, for a memoryless channel and a binary linear code, the pairwise error probability is given by the tail probability of a sum of random variables

$$\mathrm{PEP}(d,\mathrm{SNR}) = \Pr\left(\sum_{j=1}^{d} \Lambda_j > 0\right), \tag{6}$$

where the variables Λ, to which we shall refer in the following as a posteriori log-likelihood ratios, are independent and identically distributed, with value

$$\Lambda_j = \log \frac{\Pr(\hat{c_j} = \bar{c} | \mathcal{V}(c))}{\Pr(\hat{c_j} = c | \mathcal{V}(c))}, \tag{7}$$

that is, the ratio of the a posteriori likelihoods of the bit j-th taking the values $\bar{c}$ and c, having transmitted bit c. The ratio Λ depends on all the random elements in the channel, that is the noise and fading realizations z and h respectively. In order to avoid cumbersome notation, we have grouped them in a vector $\mathcal{V}(c) \stackrel{\Delta}{=} (z,h)$. Conditioned on a realization of the fading h_k, it is straightforward to show that $\Lambda(h_k)$ is normally distributed $\mathcal{N}(-4\chi_k \mathrm{SNR}, 8\chi_k \mathrm{SNR})$, where $\chi_k = |h_k|^2$. Note that it does not depend on the transmitted bit c.

In estimates of tail probabilities, the cumulant transform $\kappa(s)$ (or cumulant generating function) of Λ is a more convenient representation than the density. The transform is given by $\kappa(s) \stackrel{\Delta}{=} \log \mathrm{E}_{\mathcal{V}(c)}\left[e^{s\Lambda}\right]$, with $s \in \mathbf{C}$ [6], [7]. Using the definition of Λ, we rewrite $\kappa(s)$ as

$$\kappa(s) = \log \mathrm{E}_{h,z}\left[e^{s\Lambda(h)}\right] \tag{8}$$

$$= \log \mathrm{E}_{\chi}\left[e^{-4s\chi\,\mathrm{SNR} + 4s^2\chi\,\mathrm{SNR}}\right] \tag{9}$$

$$= \log \int_0^{+\infty} \frac{m^m \chi^{m-1}}{\Gamma(m)} e^{-m\chi}\, e^{-4s\chi\,\mathrm{SNR} + 4s^2\chi\,\mathrm{SNR}}\, d\chi. \tag{10}$$

Using [8] we can explicitly express Eq. (10) as

$$\kappa(s) = -m \log\left(1 + \frac{4s\,\mathrm{SNR}}{m} - \frac{4s^2\,\mathrm{SNR}}{m}\right). \tag{11}$$

The saddlepoint $\hat{s}$ is the value for which $\kappa'(\hat{s}) = 0$. It can be shown that this point exists and is unique [6]. Symmetry dictates that the saddlepoint is placed at $\hat{s} = 1/2$ [7]. At the saddlepoint the first derivative $\kappa'(\hat{s}) = 0$ and

$$\kappa(\hat{s}) = -m \log\left(1 + \frac{\mathrm{SNR}}{m}\right) \tag{12}$$

$$\kappa''(\hat{s}) = \frac{8\,\mathrm{SNR}}{1 + \frac{\mathrm{SNR}}{m}}. \tag{13}$$

C. Effect of Diversity

Conditioned on a realization of the fading coefficients in the D identical receiver branches, $\underline{h} = (h_1 \dots, h_D)$, and assuming that the total average received signal-to-noise ratio is SNR, $\Lambda(\underline{h})$ are normally distributed $\mathcal{N}(-4D^{-1}\sum_d \chi_d \mathrm{SNR}, 8D^{-1}\sum_d \chi_d \mathrm{SNR})$, where $\chi_d = |h_d|^2$. Let us define the vector $\underline{\chi} = (\chi_1, \dots, \chi_D)$. The cumulant transform is now given by

$$\kappa(s) = \log \mathrm{E}_{\underline{h}}\, \mathrm{E}_z\left(e^{s\Lambda(\underline{h})}\right) \tag{14}$$

$$= \log \mathrm{E}_{\underline{\chi}}\left(e^{-4sD^{-1}\sum_d \chi_d \mathrm{SNR} + 4s^2 D^{-1}\sum_d \chi_d \mathrm{SNR}}\right) \tag{15}$$

$$= \sum_{d=1}^{D} \log \mathrm{E}_{\chi_d}\left(e^{-4sD^{-1}\chi_d \mathrm{SNR} + 4s^2 D^{-1}\chi_d \mathrm{SNR}}\right). \tag{16}$$

Using the formula for the distribution of the Nakagami-m fading, the last equation can be evaluated and gives

$$\kappa(s) = -mD \log\left(1 + \frac{4s\,\mathrm{SNR}}{mD} - \frac{4s^2\,\mathrm{SNR}}{mD}\right). \tag{17}$$

This is equivalent to a Nakagami fading with parameter $\tilde{m} = mD$. In the limit $D \to \infty$ it is equal to that of unfaded AWGN.

D. Saddlepoint Approximation

In [3] we present a derivation of the saddlepoint approximation and an estimate of the approximation error to the PEP. Keeping only the first order term in the asymptotic series, the PEP can be approximated by

$$\mathrm{PEP}(d,\mathrm{SNR}) \simeq \frac{1}{\sqrt{2\pi d \kappa''(\hat{s})}\hat{s}}\, e^{d\kappa(\hat{s})} \tag{18}$$

$$= \frac{1}{2\sqrt{\pi d\,\mathrm{SNR}}}\left(1 + \frac{\mathrm{SNR}}{m}\right)^{-md+\frac{1}{2}}. \tag{19}$$

The effect of the correction is found to be negligible in practical calculations, which implies that we need not sum over any more terms in the asymptotic series.

For Rayleigh fading ($m = 1$) (19) improves on Chernoff's bound (Bhattacharyya) [1],

$$\mathrm{PEP}(d,\mathrm{SNR}) \le e^{d\kappa(\hat{s})} = (1 + \mathrm{SNR})^{-d}. \tag{20}$$

Similarly, under diversity with D identical branches, the PEP can be approximated by

$$\mathrm{PEP}(d,\mathrm{SNR}) \simeq \frac{1}{\sqrt{2\pi d \kappa''(\hat{s})}\hat{s}}\, e^{d\kappa(\hat{s})} \tag{21}$$

$$= \frac{1}{2\sqrt{\pi d\,\mathrm{SNR}}}\left(1 + \frac{\mathrm{SNR}}{mD}\right)^{-mdD+\frac{1}{2}}. \tag{22}$$

For $m \to \infty$ or $D \to \infty$, i. e., AWGN, it gives the expansion of the $Q(\cdot)$ function into an exponential, that is

$$\Pr\left(\sum_{l=1}^{d} \Lambda_i > 0\right) \simeq \frac{1}{2\sqrt{\pi d\,\mathrm{SNR}}} e^{-d\,\mathrm{SNR}} \simeq Q\left(\sqrt{2d\,\mathrm{SNR}}\right). \tag{23}$$

III. Numerical Results and Discussion

In this section we show some numerical results that illustrate the accuracy of the proposed methods as well as its asymptotic behavior. In particular, we show the following: the Bhattacharyya union bound (B-UB), the saddlepoint approximation

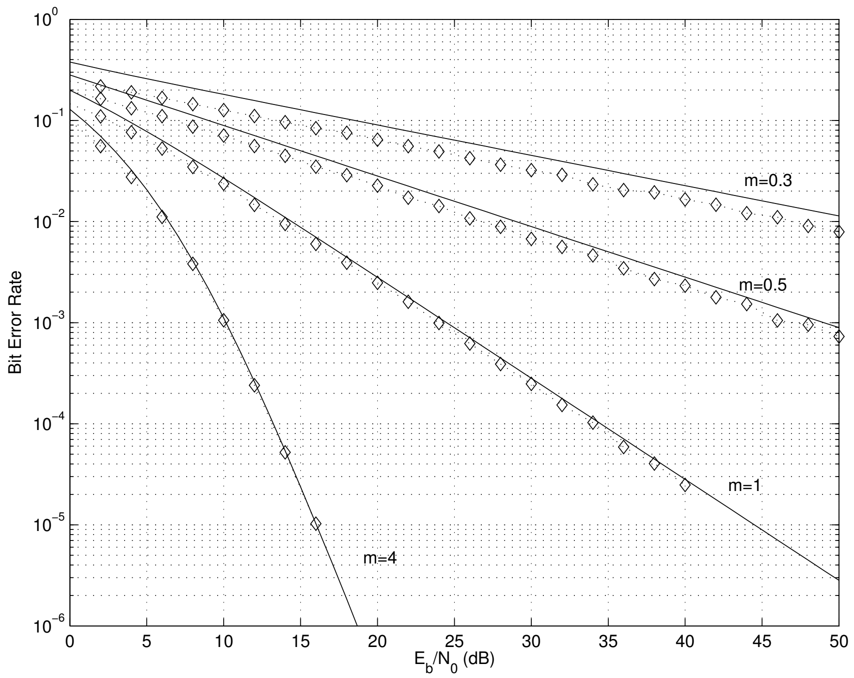

Fig. 1. Comparison of simulation and saddlepoint approximation for uncoded BPSK in Nakagami fading of parameter $m = 0.3, 0.5, 1, 4$.

(19) union bound (SP-UB), and the simulation of the bit-error rate (BER sim) for both convolutional and turbo-like code ensembles.

Figure 1 shows the bit-error probability simulation and saddlepoint approximation for uncoded BPSK in Nakagami fading with parameter $m = 0.3, 0.5, 1$ and 4. We observe that both curves are very close.

Figure 2 shows the bit-error probability simulation and bounds for the 64-state rate 1/2 convolutional code in Nakagami fading with parameter $m = 0.3, 0.5, 1$ and 4. As we see, the closed-form saddlepoint union bound yields an accurate estimation of the error probability. Figure 3 shows the corresponding curves for a repeat-accumulate code of rate 1/4 with $K = 512$. For the sake of presentation clarity, Figure 3 does not include the curves of the Bhattacharyya union bound which gives a looser bound. For every block of information bits a different interleaver is randomly generated. The simulation points correspond to an iterative decoder with 20 iterations. The saddlepoint union bound gives an accurate estimation of the error floor region. Furthermore, it does also provide an accurate approximation to the "knee" of the error curve, i. e., the transition between the waterfall and the error floor regions.

In all the cases the saddlepoint approximation gives an extremely accurate result at a fraction of the complexity required by alternative computation methods [2], such as the (exact) formula for the uncoded case (a Gauss hypergeometric function), or numerical integration of Craig's form of the $Q(\cdot)$ function. Furthermore, as opposed to the numerical integration method, the saddlepoint approximation is useful in an engineering sense, as it highlights the role of all relevant system parameters in the overall error probability.

IV. Conclusions

In this paper we have presented a simple method to compute a tight closed-form approximation to the error probability of binary transmission over fully-interleaved fading channels. This probability is found to correspond in a natural way to the tail probability of a sum of independent random variables, which is calculated using the saddlepoint approximation. In contrast to numerical integration methods, the proposed saddlepoint approximation yields a simple expression that highlights the design tradeoffs among the different system parameters. We have verified the validity of the approximation for uncoded and coded (convolutional and turbo-like code ensembles) transmission with various fading parameters. The general underlying method allows for straightforward extensions to other fading models, for instance, with correlation among successive fading realizations.

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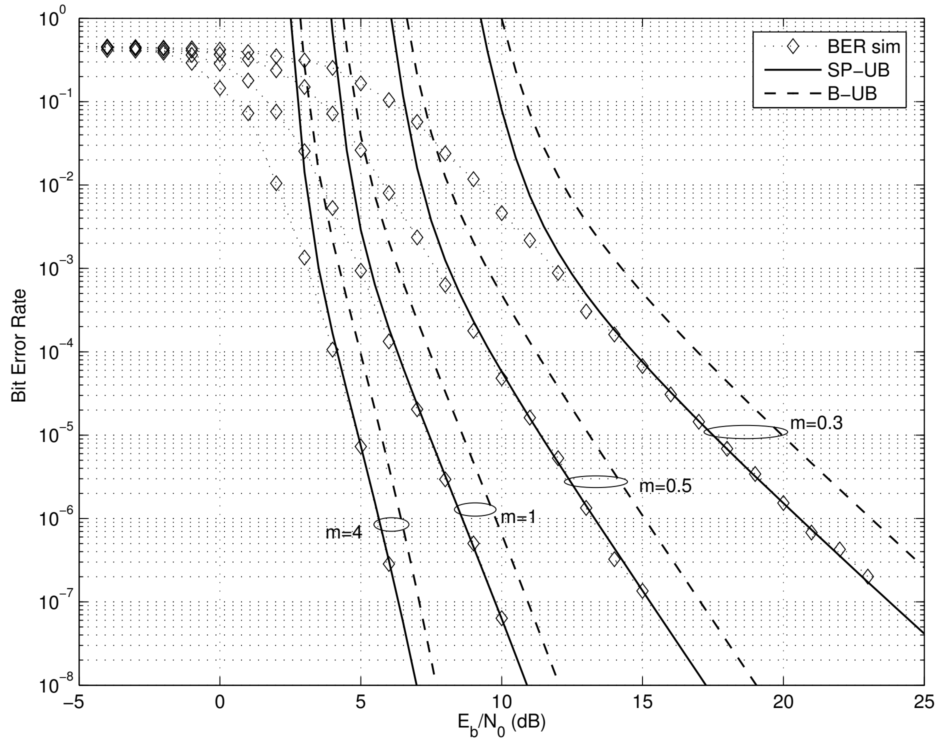

Fig. 2. Comparison of simulation, Bhattacharyya union bound and saddlepoint approximation for the 64-state rate 1/2 convolutional code in Nakagami fading of parameter $m = 0.3, 0.5, 1, 4$.

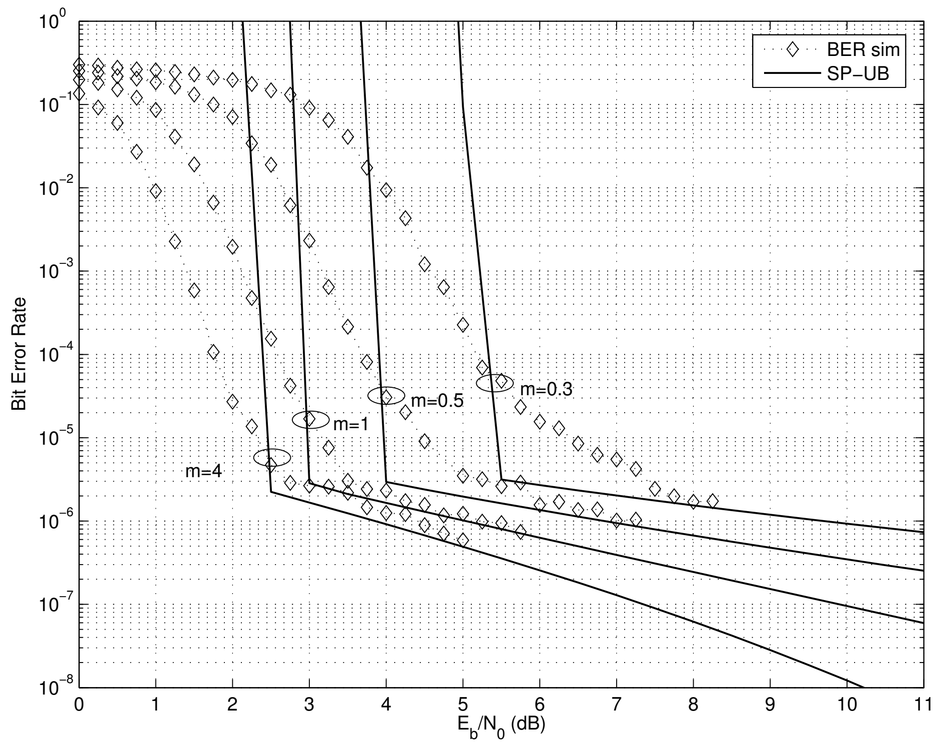

Fig. 3. Comparison of simulation and saddlepoint approximation for a rate 1/4 repeat-accumulate code in Nakagami fading of parameter $m = 0.3, 0.5, 1, 4$.