Outage Probability of the Gaussian Free Space Optical Channel with Pulse-Position Modulation

Nick Letzepis
Institute for Telecommunications Research
University of South Australia
Mawson Lakes SA 5095
nick.letzepis@unisa.edu.au

Albert Guillén i Fàbregas
Engineering Department
University of Cambridge
Cambridge, CB2 1PZ, UK
guillen@ieee.org

***Abstract*— The main drawback in communicating via the free space optical channel is the detrimental effect the atmosphere has on a propagating laser beam. Atmospheric turbulence causes random fluctuations in the irradiance of the received optical laser beam, commonly referred to as *scintillation*. The scintillation fading process is slow compared to the large data rates typical of optical transmission. As such, we adopt a quasi-static block fading model and study the outage probability of the channel under the assumption of orthogonal pulse-position modulation. Non-ideal photodetection is also assumed such that the combined shot noise and thermal noise are considered as signal-independent additive Gaussian white noise. Two channel state information (CSI) scenarios are considered: CSI at the receiver only, and CSI at both transmitter and receiver. We compute the signal-to-noise ratio exponents with receiver CSI for the cases when the scintillation is lognormal and exponential distributed, corresponding to weak and strong turbulence regimes respectively. When CSI is also known at the transmitter we show that large gains are possible by using power allocation techniques to minimise the outage probability.**

I. Introduction

The free space optical (FSO) channel offers an attractive alternative to the radio frequency (RF) channel for the purpose of transmitting data at very high rates. By utilising a high carrier frequency in the optical range, digital communication on the order of gigabits per second is possible. In addition, FSO links are difficult to intercept, immune to interference or jamming from external sources, and are not subject to frequency spectrum regulations. FSO communications has received recent attention in applications such as satellite communications, fiber-backup, RF-wireless back-haul and last-mile connectivity [1].

The main drawback in communicating via the FSO channel is the detrimental effect the atmosphere has on a propagating laser beam. The atmosphere is composed of gas molecules, water vapor, pollutants, dust, and other chemical particulates that are trapped by Earth's gravitational field. Since the wavelength of a typical optical carrier is comparable to these molecule and particle sizes, the carrier wave is subject to various propagation effects that are uncommon to RF systems. One such effect is *scintillation*, caused by atmospheric turbulence, refers to random fluctuations in the irradiance of the received optical laser beam (analogous to fading experienced in RF systems) [2–4].

Recent works on the mitigation of scintillation concentrate on the use of multiple-lasers and multiple-apertures to create a multiple-input-multiple-output (MIMO) channel [5–13]. Many of these works consider scintillation as an ergodic fading process, and analyse the channel in terms of its ergodic capacity. However, compared to typical data rates, scintillation is a slow time varying process (with a coherence time on the order of milliseconds), and it is therefore more appropriate to analyse the outage probability of the channel. To some extent, this has been done in the works of [6, 10, 12–14]. In [6, 13] the outage probability of the MIMO FSO channel is analysed under the assumption of ideal photodetection (PD) (i.e. PD is modeled as a Poisson counting process) with no bandwidth constraints. Wilson *et al.* [10] also assume perfect PD, but with the further constraint of pulse-position modulation (PPM). Lee and Chan [12], study the outage probability under the assumption of on-off keying (OOK) transmission and non-ideal PD, i.e. the combined shot noise and thermal noise process is modeled as zero mean signal independent additive white Gaussian noise (AWGN). Farid and Hranilovic [14] extend this analysis to include the effects of pointing errors.

In this paper we study the outage probability of the single-input-single-output (SISO) FSO channel under the assumption of PPM and non-ideal PD. In particular, we model the channel as a quasi-static block fading channel whereby communication takes place over a finite number of blocks and each block of transmitted symbols experiences an independent identically distributed (i.i.d.) fading realisation. Given the slow time-varying nature of scintillation, channel state information (CSI) can be estimated at the receiver and fed back to the transmitter via a dedicated feedback link. We consider two types of CSI knowledge. First we assume perfect CSI is available only at the receiver (CSIR case), and the transmitter knows only the channel statistics. For this case we derive signal-to-noise ratio (SNR) exponents when the fading is lognormal and exponential distributed, corresponding to weak and strong turbulence conditions respectively. Moreover, we show that these exponents are composed of a channel related parameter (dependent on the scintillation distribution) times the Singleton bound [15–17]. Then we consider the case when perfect CSI is known at both the transmitter and receiver (CSIT case). For this case, the transmitter finds the optimal power allocation to minimise the outage probability [18]. Using results from [19], we derive the optimal power allocation that minimises the outage probability, subject to short- and long-term power constraints. We show that under a long-term power constraint, delay-limited capacity [20] always exists for lognormal distributed scintillation, whereas, for the exponential case, one must code over several blocks for delay-limited capacity to exist. The number of required blocks depends on the rate of the binary code via the SNR exponent.

Throughout the paper, we will devote special attention to the single block transmission case i.e., the channel does not vary within a codeword. This scenario is relevant for FSO, since, due to the large data-rates, one is able to transmit millions of bits virtually over the same channel realisation. We will see that most results admit very simple forms, and some times, even closed form. This analysis allows for a system characterisation where the expressions highlight the roles of the key design parameters. The paper is organised as follows. In Section II, we define the channel model and assumptions. In Section III we present material on outage probability, mutual information and minimum mean squared error (MMSE) for PPM. Then in Sections IV and V we analyse outage probability for the CSIR and CSIT cases respectively. Concluding remarks are then given in Section VI.

II. System Model

The communication system of interest consists of a single laser and a single aperture receiver. Information data is first encoded by a binary code of rate R_c and then modulated according to a Q-ary PPM scheme, resulting in rate $R = R_c \log_2 Q$ (bits/channel use). The PPM signal is transmitted optically, via a laser beam, through an atmospheric turbulent channel and collected by the receiver aperture. The received optical signal is converted to an electrical signal via PD. Non-ideal PD is assumed such that the combined shot noise and thermal noise processes can be modeled as zero mean, signal independent AWGN (an assumption commonly used in the literature, see e.g. [3–5, 12, 14, 21–29]).

In FSO communications, channel variations are typically much slower than the signaling period. As such, we model the channel as a non-ergodic block-fading channel, for which a given codeword of length BL *sees* only a finite number B of scintillation realisations [30, 31]. Hence under these assumptions the received signal can be written as

$$\boldsymbol{y}_b[\ell] = \sqrt{p_b} h_b \boldsymbol{x}_b[\ell] + \boldsymbol{z}_b[\ell], \tag{1}$$

for $b = 1, \dots, B, \ell = 1, \dots, L$ where $\boldsymbol{y}_b[\ell], \boldsymbol{x}_b[\ell], \boldsymbol{z}_b[\ell] \in \mathbb{R}^Q$ are the received, transmitted and noise signals at block b and time instant ℓ, and h_b denotes the fading due to scintillation. Each transmitted symbol is drawn from a PPM alphabet, $\boldsymbol{x}_b[\ell] \in \mathcal{X}^{\text{ppm}} \triangleq \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_Q\}$, where $\boldsymbol{e}_q$ is the canonical basis vector, i.e., it has all zeros except for a one in position q, the time slot where the pulse is transmitted. The noise samples of $\boldsymbol{z}_b[\ell]$ are independent realisations of a random variable $Z \sim \mathcal{N}(0, 1)$, and p_b denotes the electrical power of block b. The fading coefficients h_b are independent realisations of a random variable H with probability density function (pdf) $f_H(h)$. Furthermore, we assume $\mathbb{E}[H^2] = 1$ so that the average received electrical SNR is $\mathsf{snr} \triangleq \mathbb{E}[p_b h_b^2] = \mathbb{E}[p_b]$.[2]

The scintillation pdf, $f_H(h)$, is parameterised by the *scintillation index* (SI),

$$\sigma_I^2 \triangleq \frac{\text{Var}(H)}{(\mathbb{E}[H])^2}. \tag{2}$$

Under weak atmospheric turbulence conditions (defined as those regimes for which $\sigma_I^2 < 1$), the SI is proportional to the so called *Rytov variance* which represents the SI of an unbounded plane wave in weak turbulence conditions, and is also considered as a measure of the strength of the optical turbulence under strong-fluctuation regimes [4]. The distribution of the irradiance fluctuations is dependent on the strength of the optical turbulence. For the weak turbulence regime, the fluctuations are generally considered to be lognormal distributed[3], and for strong turbulence, exponential distributed [2, 33].

For lognormal distributed scintillation,

$$f_H(h) = \frac{1}{h\sigma\sqrt{2\pi}} \exp\left(-(\log h - \mu)^2/(2\sigma^2)\right), \tag{3}$$

where μ and σ are related to the SI via $\mu = -\log(1 + \sigma_I^2)$ and $\sigma^2 = \log(1 + \sigma_I^2)$.

For exponential distributed scintillation,

$$f_H(h) = \lambda \exp(-\lambda h). \tag{4}$$

Note that this corresponds to the super-saturated turbulence regime, for which $\sigma_I^2 = 1$, which is easily verified since $\mathbb{E}[H] = 1/\lambda$ and $\text{var}[H] = 1/\lambda^2$. We assume $\lambda = \sqrt{2}$, so that $\mathbb{E}[H^2] = 1$, as in the lognormal case.

III. Outage Probability, Mutual Information and MMSE

The channel described by (1) under the quasi-static assumption is not information stable [34] and therefore, the channel capacity in the strict Shannon sense is zero. It can be shown that the codeword error probability of any coding scheme can be lower bounded by the information outage probability [30, 31],

$$P_{\text{out}}(\mathsf{snr}, R) = \Pr(I(\boldsymbol{p}, \boldsymbol{h}) < R), \tag{5}$$

where R is the transmission rate and $I(\boldsymbol{p}, \boldsymbol{h})$ is the instantaneous input-output mutual information for a given power allocation $\boldsymbol{p} \triangleq (p_1, \dots, p_B)$, and vector channel realisation $\boldsymbol{h} \triangleq (h_1, \dots, h_B)$. The instantaneous mutual information can be expressed as [35]

$$I(\boldsymbol{p}, \boldsymbol{h}) = \frac{1}{B} \sum_{b=1}^{B} I^{\text{awgn}}(p_b h_b^2), \tag{6}$$

[2] For the ideal PD model, the normalization $\mathbb{E}[H] = 1$ is used to keep optical power constant. Since we are assuming a non-ideal PD model and are working entirely in the electrical domain, we have chosen the normalization $\mathbb{E}[H^2] = 1$, commonly used in RF systems.

[3] Note that for $\sigma_I^2 \geq 1$, experimental studies [5, 32] have shown that the scintillation can still appear to be lognormal distributed.

where $I^{\mathrm{awgn}}(\rho)$ is the input-output mutual information of an AWGN channel with SNR ρ. For PPM [21]

$$I^{\mathrm{awgn}}(\rho) = \log_2 Q - \mathbb{E}\left[\log_2\left(1+\exp(-\rho)\sum_{q=2}^{Q}\exp\left(\sqrt{\rho}(Z_q - Z_1)\right)\right)\right], \quad (7)$$

where $Z_q \sim \mathcal{N}(0,1)$ for $q = 1, \ldots, Q$.

For the CSIT case we will use the recently discovered relationship between mutual information and the MMSE [36]. This relationship states that[4]

$$\frac{d}{d\rho} I^{\mathrm{awgn}}(\rho) = \frac{\mathsf{mmse}(\rho)}{\log(2)} \quad (8)$$

where $\mathsf{mmse}(\rho)$ is the MMSE in estimating the input from the output of a Gaussian channel as a function of the SNR ρ. For the case of PPM, we can express the MMSE as follows.

Theorem 3.1: Suppose QPPM symbols are transmitted across an AWGN channel with SNR ρ. The MMSE is

$$\mathsf{mmse}(\rho) = 1 - \mathbb{E}\left[\frac{\exp(2\sqrt{\rho}(\sqrt{\rho}+Z_1)) + (Q-1)\exp(2\sqrt{\rho}Z_2)}{\left(\exp(\rho)\exp(\sqrt{\rho}Z_1) + \sum_{k=2}^{Q}\exp(\sqrt{\rho}Z_k)\right)^2}\right], \quad (9)$$

where $Z_i \sim \mathcal{N}(0,1)$ for $i = 1, \ldots, Q$.

Proof: See the Appendix. ■

Both (7) and (9) can be evaluated using standard Monte-Carlo methods.

IV. Outage Probability Analysis with CSIR

In this section we analyse the outage probability for the case when perfect CSI is known at the receiver only. We pay special attention to the large SNR case. Hence, for this case the optimal power allocation is to distribute power uniformly over all blocks, i.e. $p_1 = \ldots = p_B = p = \mathsf{snr}$.

For codewords transmitted over B blocks, obtaining a closed form analytic expression for the outage probability is intractable. Instead we analyse the asymptotic behaviour of the outage probability. Toward this end, following the footsteps of [17, 37], we derive the *SNR exponent*.

Theorem 4.1: The outage SNR exponent for a SISO FSO communications system modeled by (1) is given as follows:

(a) lognormal distributed scintillation

$$d_{(\log \mathsf{snr})^2} = \frac{1}{8\log(1+\sigma_I^2)}\left(1 + \lfloor B(1-R_c)\rfloor\right), \quad (10)$$

(b) exponential distributed scintillation

$$d_{(\log \mathsf{snr})} = \frac{1}{2}\left(1 + \lfloor B(1-R_c)\rfloor\right), \quad (11)$$

where the SNR exponents $d_{(\log \mathsf{snr})^k}$ for $k = 1, 2$ are defined as

$$d_{(\log \mathsf{snr})^k} \triangleq -\lim_{\mathsf{snr}\to\infty} \frac{\log P_{\mathrm{out}}(\mathsf{snr}, R)}{(\log \mathsf{snr})^k} \quad (12)$$

and $R_c = R/\log_2(Q)$ is the rate of the binary code. ■

Proof: See the Appendix.

In (10) and (11) we see that the SNR exponent is a channel-related parameter times the Singleton bound, which is the optimal rate-diversity tradeoff for Rayleigh-faded block fading channels [15–17]. For the lognormal case, the channel-related parameter is $8\log(1+\sigma_I^2)$ and hence is directly linked to the SI. Moreover, for small $\sigma_I^2 < 1$, $8\log(1+\sigma_I^2) \approx 8\sigma_I^2$ and the SNR exponent is inversely proportional to the SI. For the exponential case, the channel-related parameter in (11) is a constant $1/2$ as expected, since the SI is constant. In comparing (10) and (11) we observe a striking difference. For the lognormal case (10) implies the outage probability is dominated by a $(\log(\mathsf{snr}))^2$ term, whereas for the exponential case it is dominated by a $\log(\mathsf{snr})$ term. Thus the outage probability decays much more rapidly with SNR for the lognormal case than it does for the exponential case. Furthermore, for the lognormal case, the slope of the outage probability curve, when plotted on a log-log scale, will not converge to a constant value. In fact, a constant slope curve will only be observed when plotting the outage probability on a log-$(\log)^2$ scale.

For the special case of single block transmission, $B = 1$, it is straightforward to express the outage probability in terms of the cumulative distribution function (cdf) of the scintillation random variable, i.e.

$$P_{\mathrm{out}}(\mathsf{snr}, R) = F_H\left(\sqrt{\frac{\mathsf{snr}_R^{\mathrm{awgn}}}{\mathsf{snr}}}\right) \quad (13)$$

where $F_H(h)$ denotes the cdf of H, and

$$\mathsf{snr}_R^{\mathrm{awgn}} \triangleq I^{\mathrm{awgn},-1}(R) \quad (14)$$

denotes the SNR value at which the mutual information is equal to R, i.e., the solution of the equation $I^{\mathrm{awgn}}(\rho) = R$. Table I reports these values for $Q = 2, 4, 8, 16$ and $R = R_c \log_2 Q$, with $R_c = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. Therefore, for $B = 1$, using (13) we can compute the outage probability analytically for most cases of interest.

TABLE I

Minimum signal-to-noise ratio $\mathsf{snr}_R^{\mathrm{awgn}}$ (in decibels) for reliable communication for target rate $R = R_c \log_2 Q$.

Q	$R_c = \frac{1}{4}$	$R_c = \frac{1}{2}$	$R_c = \frac{3}{4}$
2	−0.7992	3.1821	6.4109
4	0.2169	4.0598	7.0773
8	1.1579	4.8382	7.7222
16	1.9881	5.5401	8.3107

Outage probability curves for the $B = 1$ case are shown in Fig. 1. For the lognormal case, as expected, we see that the curves do not have constant slope for large SNR. Whereas, for the exponential case, a constant slope is clearly visible.

[4] The $\log(2)$ term arises because we have defined $I^{\mathrm{awgn}}(\rho)$ in bits/channel usage.

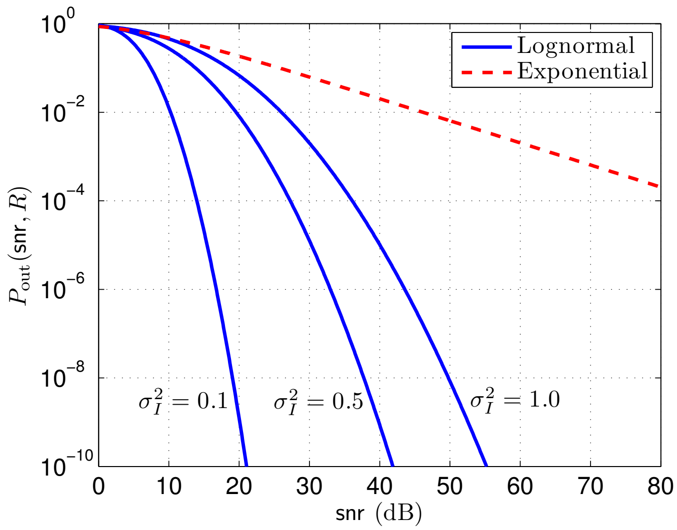

Fig. 1. Outage probability for lognormal (solid) and exponential distributed scintillation (dashed) with $B = 1$, $Q = 2$, $\mathsf{snr}_{1/2}^{\text{awgn}} = 3.18$ dB.

V. Outage Probability Analysis with CSIT

In this section we consider the case where the transmitter and receiver both have perfect CSI knowledge. In this case, the transmitter determines the optimal power allocation that minimises the outage probability subject to a power constraint [18]. In particular, we will consider both short- and long-term power constraint scenarios. Each constraint leads to a different power allocation algorithm, and we therefore study them separately. Using results from [19] we uncover new insight as to how key design parameters influence the performance of the system.

A. Short-Term Power Allocation

Under a short-term power constraint, the sum of the powers of each block for any given codeword is constrained by P, i.e. $\frac{1}{B}\sum_{b=1}^{B} p_b \leq P$. As such, the optimal power allocation scheme is the solution to the following optimisation problem.[5]

$$\begin{cases} \text{Minimise} & P_{\text{out}}(P, R) \\ \text{Subject to} & \frac{1}{B}\sum_{b=1}^{B} p_b \leq P \\ & p_b \geq 0,\ b = 1, \ldots, B \end{cases} \tag{15}$$

which is equivalent to the maximising the mutual information at every channel realisation [18]

$$\begin{cases} \text{Maximise} & I(\boldsymbol{p}, \boldsymbol{h}) \\ \text{Subject to} & \frac{1}{B}\sum_{b=1}^{B} p_b \leq P \\ & p_b \geq 0,\ b = 1, \ldots, B \end{cases} \tag{16}$$

The solution to the above problem is given by mercury-waterfilling at each channel realisation [19, 38]

$$p_b = \frac{1}{h_b^2}\mathsf{mmse}^{-1}\left(\min\left\{1, \frac{\eta}{h_b^2}\right\}\right), \tag{17}$$

for $b = 1, \ldots, B$ where η is chosen to satisfy the power constraint.

The following result follows immediately as a consequence of the analysis in [19].

Corollary 5.1: The short-term SNR exponents are given by (10) and (11), for lognormal and exponential scintillation, respectively.

B. Long-Term Power Allocation

Under a long-term power constraint, the average power of a codeword is constrained, i.e. $\mathbb{E}\left[\frac{1}{B}\sum_{b=1}^{B} p_b\right] \leq P$. In this case, the optimal power allocation is the solution to the following problem.

$$\begin{cases} \text{Minimise} & P_{\text{out}}(P, R) \\ \text{Subject to} & \mathbb{E}\left[\frac{1}{B}\sum_{b=1}^{B} p_b\right] \leq P \\ & p_b \geq 0,\ b = 1, \ldots, B. \end{cases} \tag{18}$$

From [19], the solution is given by

$$\boldsymbol{p} = \begin{cases} \boldsymbol{\wp}, & \sum_{b=1}^{B} \wp_b \leq s \\ \boldsymbol{0}, & \text{otherwise}, \end{cases} \tag{19}$$

where $\boldsymbol{\wp}$ is the solution to the following problem

$$\begin{cases} \text{Minimise} & \frac{1}{B}\sum_{b=1}^{B} \wp_b \\ \text{Subject to} & I(\boldsymbol{\wp}, \boldsymbol{h}) \geq R \\ & \wp_b \geq 0,\ b = 1, \ldots, B \end{cases}. \tag{20}$$

In (19), s is a threshold such that $s = \infty$ if $\lim_{s\to\infty} \mathbb{E}_{\mathcal{R}(s)}\left[\frac{1}{B}\sum_{b=1}^{B} \wp_b\right] \leq P$, where

$$\mathcal{R}(s) \triangleq \left\{\boldsymbol{h} \in \mathbb{R}_+^B : \frac{1}{B}\sum_{b=1}^{B} \wp_b \leq s\right\}, \tag{21}$$

otherwise, s is chosen such that $P = \mathbb{E}_{\mathcal{R}(s)}\left[\frac{1}{B}\sum_{b=1}^{B} \wp_b\right]$.

The solution to problem (20) is given by [19]

$$\wp_b = \frac{1}{h_b^2}\mathsf{mmse}^{-1}\left(\min\left\{1, \frac{1}{\eta h_b^2}\right\}\right),\ b = 1, \ldots, B \tag{22}$$

where η is now chosen to satisfy the rate constraint

$$\frac{1}{B}\sum_{b=1}^{B} I^{\text{awgn}}\left(\mathsf{mmse}^{-1}\left(\min\left\{1, \frac{1}{\eta h_b^2}\right\}\right)\right) = R \tag{23}$$

Corollary 5.2: The long-term SNR exponents are given by

(a) lognormal distributed scintillation

$$d_{(\log \mathsf{snr})}^{\text{lt}} = \infty, \tag{24}$$

(b) exponential distributed scintillation

$$d_{(\log \mathsf{snr})}^{\text{lt}} = \frac{d_{(\log \mathsf{snr})}^{\text{st}}}{1 - d_{(\log \mathsf{snr})}^{\text{st}}} \tag{25}$$

where $d_{(\log \mathsf{snr})}^{\text{st}} = \frac{1}{2}\left(1 + \lfloor B(1 - R_c)\rfloor\right)$ is the corresponding short-term SNR exponent.

Proof: As shown in [19], whenever the short-term SNR exponent is $d_{(\log \mathsf{snr})}^{\text{st}} > 1$, then the exponent $d_{(\log \mathsf{snr})}^{\text{lt}}$ of the

[5] We parameterise P_{out} by the power constraint P (rather than snr as in (5)) since this constraint dictates the optimal allocation scheme and hence the average SNR.

long-term scheme is infinity, i.e, the curve is vertical, and the delay-limited capacity exists [20]. On the other hand, when $d^{\rm st}_{(\log \mathsf{snr})} < 1$, then $d^{\rm lt}_{(\log \mathsf{snr})} = \frac{d^{\rm st}_{(\log \mathsf{snr})}}{1-d^{\rm st}_{(\log \mathsf{snr})}}$. For lognormal scintillation, (10) implies that the short-term exponent $d^{\rm st}_{(\log \mathsf{snr})} \to \infty$ and $d^{\rm st}_{(\log \mathsf{snr})^2} < \infty$, then the long-term exponent $d^{\rm lt}_{(\log \mathsf{snr})} = \infty$ and therefore the corresponding outage curves will be vertical even when $B = 1$. ■

Interestingly, for exponential scintillation, from (11), we see that $d^{\rm st}_{(\log \mathsf{snr})} > 1$ only if $\lfloor B(1-R_c)\rfloor > 1$, which means that to support higher code rates one must code over more blocks for delay-limited capacity to exist, e.g. for: $R_c = 0.25$, $B \geq 3$; $R_c = 0.5$, $B \geq 4$; and $R_c = 0.75$, $B \geq 8$. Moreover, for $B = 1$, the delay-limited capacity does not exist at all. Given that the typical coherence time of scintillation is on the order of tens of milliseconds, coding over multiple blocks (although not unfeasible) may be undesirable in applications with strict latency requirements. This problem can be overcome by using multiple-lasers and multiple apertures. In particular, in [39] it was shown that the number of lasers times the number of apertures must be greater than two for delay-limited capacity to exist with $B = 1$ and exponential distributed scintillation.

For the special case of $B = 1$, the solution (22) can be simplified since $\eta = (h^2 \mathsf{mmse}(I^{\rm awgn,-1}(R)))^{-1} = (h^2 \mathsf{mmse}(\mathsf{snr}^{\rm awgn}_R))^{-1}$. Hence, for $B = 1$

$$\wp^{\rm opt} = \frac{\mathsf{snr}^{\rm awgn}_R}{h^2}. \qquad (26)$$

Intuitively, (26) implies that for single block transmission, whenever $\mathsf{snr}^{\rm awgn}_R/h^2 \leq s$, one simply transmits at the minimum power necessary so that the received instantaneous SNR is equal to the SNR threshold ($\mathsf{snr}^{\rm awgn}_R$) of the code.

As in Section IV, the outage probability for single block transmission can be determined analytically in terms of the scintillation cdf. This is given in the following theorem.

Theorem 5.1: The outage probability of the channel (1), for the case when $B = 1$ and CSIT subject to a long-term power constraint is

$$P_{\rm out}(\mathsf{snr}, R) = F_H\left(\sqrt{\frac{\mathsf{snr}^{\rm awgn}_R}{\gamma^{-1}(\mathsf{snr})}}\right), \qquad (27)$$

where $\gamma^{-1}(\mathsf{snr})$ is the unique solution to the equation $\gamma(s) = \mathsf{snr}$,

$$\gamma(s) = \mathsf{snr}^{\rm awgn}_R \int_\nu^\infty \frac{f_H(h)}{h^2}\, dh, \qquad (28)$$

where $\nu = \sqrt{\frac{\mathsf{snr}^{\rm awgn}_R}{s}}$.

Proof: See the Appendix. ■

In (27), $\gamma(s)$ is the average SNR for a threshold s. For lognormal and exponential scintillation $\gamma(s)$ can be determined explicitly,

$$\gamma^{\ln}(s) = \frac{1}{2}\mathsf{snr}^{\rm awgn}_R (1+\sigma_I^2)^4 \,\mathrm{erfc}\left(\frac{3\log(1+\sigma_I^2) + \frac{1}{2}\log \mathsf{snr}^{\rm awgn}_R - \frac{1}{2}\log s}{\sqrt{2\log(1+\sigma_I^2)}}\right) \qquad (29)$$

$$\gamma^{\exp}(s) = \sqrt{2\mathsf{snr}^{\rm awgn}_R s}\exp\left(-\sqrt{\frac{2\mathsf{snr}^{\rm awgn}_R}{s}}\right) + 2\mathsf{snr}^{\rm awgn}_R \mathrm{Ei}\left(-\sqrt{\frac{2\mathsf{snr}^{\rm awgn}_R}{s}}\right), \qquad (30)$$

where $\mathrm{Ei}(z) = \int_{-z}^\infty \exp(-t)/t\, dt$ is the exponential integral.

Fig. 2 compares the outage probability for the CSIR and CSIT (with long-term power constraints). For lognormal scintillation it can be seen that CSIT vastly improves the performance, and as expected, there exists a threshold SNR at which $P_{\rm out} \to 0$, i.e. the delay-limited capacity. For $B = 1$ this threshold can be computed as

$$\mathsf{snr}^{\ln}_R(\sigma_I^2) = \lim_{s\to\infty} \gamma^{\ln}(s) = \mathsf{snr}^{\rm awgn}_R (1+\sigma_I^2)^4. \qquad (31)$$

For example, $\mathsf{snr}^{\ln}_{1/2}(1) = 15.2$ dB, as clearly shown in Fig. 2. For exponential scintillation, whilst CSIT significantly improves performance, we see that delay-limited capacity doesn't exist in this case, and hence neither does the limit $\lim_{s\to\infty}\gamma^{\exp}(s)$. In this case, one must code over more blocks for delay-limited capacity to exist.

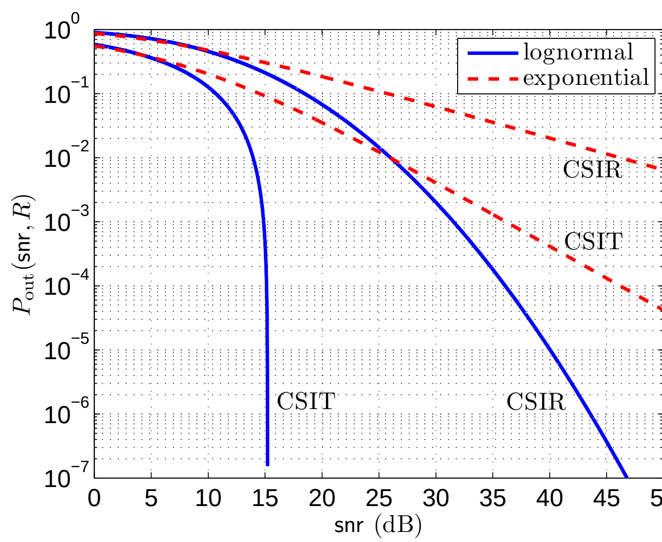

Fig. 2. Comparison of CSIR and CSIT outage probabilities for lognormal (solid) and exponential (dashed) distributed scintillation with $\sigma_I^2 = 1$, $B = 1$, $Q = 2$, $\mathsf{snr}^{\rm awgn}_{1/2} = 3.18$ dB.

VI. Conclusions

In this paper we have analysed the outage probability of the Gaussian FSO channel under the assumption of PPM and non-ideal PD. When CSI is known only at the receiver, we have shown that the SNR exponent is composed of a channel related parameter (dependent on the scintillation distribution) times the Singleton bound. When the scintillation is lognormal distributed, we have shown that the outage probability is dominated by a $(\log(\mathsf{snr}))^2$ term, whereas for the exponential case it is dominated by a $\log(\mathsf{snr})$ term. When CSI is also known at the transmitter, we derived the optimal power allocation that minimises outage probability subject to short- and long-term power constraints with PPM. In the later case, for lognormal scintillation, we showed that delayed-limited capacity exists, even when coding over a single block (channel realisation). Whereas for exponential scintillation, one must code over multiple blocks. Moreover, the number of blocks required depends on the binary code rate through the SNR exponent.

APPENDIX

PROOF OF THEOREM 3.1

Suppose PPM symbols are transmitted over an AWGN channel, the non-fading equivalent of (1). The received noisy symbols are given by $\boldsymbol{y} = \sqrt{\rho}\boldsymbol{x} + \boldsymbol{z}$, where $\boldsymbol{x} \in \mathcal{X}^{\text{ppm}}$ (we have dropped the time index ℓ for brevity of notation).

Using Bayes' rule [40], the MMSE estimate is

$$\hat{\boldsymbol{x}} = \mathbb{E}\left[\boldsymbol{x}|\boldsymbol{y}\right] = \sum_{q=1}^{Q} \frac{\boldsymbol{e}_q \exp(\sqrt{\rho}y_q)}{\sum_{k=1}^{Q}\exp(\sqrt{\rho}y_k)}. \quad (32)$$

From (32) the ith element of $\hat{\boldsymbol{x}}$ is

$$\hat{x}_i = \frac{\exp(\sqrt{\rho}y_i)}{\sum_{k=1}^{Q}\exp(\sqrt{\rho}y_k)}. \quad (33)$$

Using the orthogonality principle [41] $\mathsf{mmse}(\rho) = \mathbb{E}\left[\|\boldsymbol{x}-\hat{\boldsymbol{x}}\|^2\right] = \mathbb{E}[\|\boldsymbol{x}\|^2] - \mathrm{E}[\|\hat{\boldsymbol{x}}\|^2]$. Since $\|\boldsymbol{e}_q\|^2 = 1$ for all $q = 1,\ldots,Q$, then $\mathbb{E}[\|\boldsymbol{x}\|^2] = 1$. Due to the symmetry of QPPM we need only consider the case when $\boldsymbol{x} = \boldsymbol{e}_1$ was transmitted. Hence,

$$\mathsf{mmse}(\rho) = 1 - \left(\mathbb{E}[\hat{x}_1^2] + (Q-1)\mathbb{E}[\hat{x}_2^2]\right). \quad (34)$$

Now $y_1 = \sqrt{\rho} + z_1$ and $y_i = z_i$ for $i = 2,\ldots,Q$, where z_q is a realisation of a random variable $Z_q \sim \mathcal{N}(0,1)$ for $q = 1,\ldots,Q$. Hence, substituting these values in (33) and taking the expectation (34) yields the result given the theorem.

PROOF OF THEOREM 4.1

We begin by defining a normalised (with respect to SNR) fading coefficient, $\zeta_b \triangleq -\frac{2\log h_b}{\log \mathsf{snr}}$, which has a pdf given by

$$f_{\zeta_b}(\zeta) = \frac{\log\mathsf{snr}}{2}\exp\left(-\frac{1}{2}\zeta\log\mathsf{snr}\right) \cdot f_H\left(\exp\left(-\frac{1}{2}\zeta\log\mathsf{snr}\right)\right). \quad (35)$$

Hence the instantaneous SNR for block b is given by $\rho_b = \mathsf{snr}h_b^2 = \mathsf{snr}^{1-\zeta_b}$ for $b = 1,\ldots,B$. Therefore,

$$\lim_{\mathsf{snr}\to\infty} I^{\text{awgn}}(\rho_b) = \lim_{\mathsf{snr}\to\infty} I^{\text{awgn}}\left(\mathsf{snr}^{1-\zeta_b}\right) = \log_2 Q\left(1 - 1\!\!1\{\zeta_b > 1\}\right) \quad (36)$$

From the definition of outage probability (5), we have that

$$P_{\text{out}}(\mathsf{snr}, R) = \Pr(I_{\boldsymbol{h}}(\mathsf{snr}) < R) = \int_{\mathcal{A}} f(\boldsymbol{\zeta})d\boldsymbol{\zeta} \quad (37)$$

where $\boldsymbol{\zeta} \triangleq (\zeta_1,\ldots,\zeta_B)$ is a $1\times B$ vector of normalised fading coefficients, $f(\boldsymbol{\zeta})$ denotes their joint pdf, and

$$\mathcal{A} \triangleq \left\{\boldsymbol{\zeta}\in\mathbb{R}^B : \sum_{b=1}^{B} 1\!\!1\{\zeta_b > 1\} > B\left(1-R_c\right)\right\} \quad (38)$$

is the asymptotic outage set. We now compute the asymptotic behaviour of the outage probability, i.e.

$$-\lim_{\mathsf{snr}\to\infty}\log P_{\text{out}}(\mathsf{snr}, R) = -\lim_{\mathsf{snr}\to\infty}\log\int_{\mathcal{A}} f(\boldsymbol{\zeta})d\boldsymbol{\zeta}. \quad (39)$$

A. Lognormal case

Suppose h_b is lognormal distributed. Hence, from (35) and (3) the joint pdf of $\boldsymbol{\zeta}$ is

$$f(\boldsymbol{\zeta}) = \frac{(\log\mathsf{snr})^B}{(8\pi\sigma^2)^{\frac{B}{2}}}\exp\Bigg(-\frac{1}{8\sigma^2}\sum_{b=1}^{B}\big((\log\mathsf{snr})^2(\zeta_b)^2 + 4\mu\log\mathsf{snr}\zeta_b + 4\mu^2\big)\Bigg). \quad (40)$$

Ignoring terms of order less than $(\log\mathsf{snr})^2$ in the exponent and constant terms independent of $\boldsymbol{\zeta}$ in front of the exponential, then

$$f(\boldsymbol{\zeta}) \doteq \exp\left(-\frac{(\log\mathsf{snr})^2}{8\sigma^2}\sum_{b=1}^{B}\zeta_b^2\right). \quad (41)$$

Hence, from (39) we have

$$\begin{aligned} -\lim_{\mathsf{snr}\to\infty}&\log P_{\text{out}}(\mathsf{snr}, R) \\ &= -\lim_{\mathsf{snr}\to\infty}\log\int_{\mathcal{A}}\exp\left(-\frac{(\log\mathsf{snr})^2}{8\sigma^2}\sum_{b=1}^{B}\zeta_b^2\right)d\boldsymbol{\zeta} \\ &= \frac{(\log\mathsf{snr})^2}{8\sigma^2}\inf_{\mathcal{A}}\left\{\sum_{b=1}^{B}\zeta_b^2\right\}, \end{aligned} \quad (42)$$

where the second line follows from Varadhan's lemma [42].

It is straightforward to show that the above infimum is achieved by setting any κ of the ζ_b equal to one and the rest equal to zero, where κ is the unique integer satisfying $\kappa < B\left(1-R_c\right) \leq \kappa + 1$. Hence it follows that

$$-\lim_{\mathsf{snr}\to\infty}\log P_{\text{out}}(\mathsf{snr}, R) = \frac{(\log\mathsf{snr})^2}{8\sigma^2}\left(1 + \lfloor B\left(1-R_c\right)\rfloor\right), \quad (43)$$

Dividing both sides of (43) by $(\log\mathsf{snr})^2$ the SNR exponent (10) is obtained.

B. Exponential case

Suppose h_b is exponential distributed with parameter $\lambda = \sqrt{2}$. For this case, the joint pdf of $\boldsymbol{\zeta}$ is Ignoring the exponential terms in the exponent and constant terms independent of $\boldsymbol{\zeta}$ in front of the exponential,

$$f(\boldsymbol{\zeta}) \doteq \exp\left(-\frac{\log\mathsf{snr}}{2}\sum_{b=1}^{B}\zeta_b\right). \quad (44)$$

Following the same steps as the lognormal case, i.e. defining the same asymptotic outage set and application of Varadhan's lemma [42], then we find that

$$-\lim_{\mathsf{snr}\to\infty}\log P_{\text{out}}(\mathsf{snr}, R) = \frac{\log\mathsf{snr}}{2}\left(1 + \lfloor B\left(1-R_c\right)\rfloor\right), \quad (45)$$

Dividing both sides of (45) by $\log\mathsf{snr}$ the SNR exponent (11) is obtained.

PROOF OF THEOREM 5.1

An outage occurs only if the optimal power allocation for a transmitted block is zero, which occurs whenever $\wp^{\rm opt} > s$. Hence,

$$P_{\rm out}(\mathsf{snr}, R) = \Pr\left(\frac{\mathsf{snr}_R^{\rm awgn}}{h^2} > s\right) = F_H\left(\sqrt{\frac{\mathsf{snr}_R^{\rm awgn}}{s}}\right). \quad (46)$$

Now we relate the parameter s to the average SNR as follows.

$$\gamma(s) = \mathsf{snr}_R^{\rm awgn}\mathbb{E}_{H\in\mathcal{R}(s)}\left[H^{-2}\right] = \mathsf{snr}_R^{\rm awgn}\int_{\mathcal{R}(s)}\frac{f_H(h)}{h^2}\,dh. \quad (47)$$

where $\mathcal{R}(s)$ is defined as in (21). Given the optimal power allocation (26) for $B = 1$, $\mathcal{R}(s) = [\nu, \infty]$, where $\nu = \sqrt{\frac{\mathsf{snr}_R^{\rm awgn}}{s}}$. Hence the theorem follows.

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