Turbo code design for block fading channels

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Abstract

Shannon capacity approaching codes have been extensively studied during the last decade for ergodic channels. We propose and analyze new channel multiplexing/interleaving techniques for parallel turbo codes over non-ergodic block fading channels. We reveal word error probability performance at less than 1dB from outage capacity. The achieved word error probability is almost insensitive to the code length. The race to the outage capacity limit on block fading channels is declared open.

1 Notations and channel coding model

We briefly recall the mathematical model of a wireless block fading channel [2]. This section also introduces our notation and the mathematical context in which convolutional codes and parallel turbo codes are studied. Let $C = (n_c N, K)_2$ be a linear binary block code of length $n_c N$, dimension K, and rate $R = K/(n_c N)$. The binary alphabet $GF(2) = \{0, 1\}$ defining the symbols of C is converted into a binary phase shift keying (BPSK) alphabet $\mathcal{B} = \{-1, +1\}$ before data transmission on the channel. For any code symbol $c_i \in GF(2)$, $i = 1 \ldots n_c N$, the observation made by the decoder at the channel output is

$$r_i = h_i \times (2c_i - 1) + z_i \tag{1}$$

where h_i is the real fading coefficient at time i, and z_i is an additive white Gaussian noise with zero mean and variance σ^2. In a block fading wireless channel, the fading varies slowly in time: h_i takes n_c different values during the transmission of one codeword. Thus, for $i = 1 \ldots N$ we have $h_i = \alpha_1$. For $i = N+1 \ldots 2N$ we have $h_i = \alpha_2$, and so on. The block fading channel is said to be an *n_c-state channel* because each codeword undergoes n_c independent fadings $\{\alpha_1, \alpha_2, \ldots, \alpha_{n_c}\}$. The fading is called *block fading* because it is constant in time during N bit periods. The random variables α_j are assumed to be iid (within a codeword, and also from one codeword to another) and Rayleigh distributed, $p(\alpha_j) = 2\alpha_j e^{-\alpha_j^2}$, $0 \leq \alpha_j < +\infty$. The left part of Fig.1 illustrates the model described above.

In order to improve the error correcting capability of C, we change the order of code symbols before channel transmission. This interleaving applied at the encoder output will be called *multiplexing* in the purpose of avoiding a possible confusion with the interleaver found inside the turbo code structure. The right part of Fig.1 illustrates the block fading model with channel multiplexing. Our objective is to minimize the word error probability (WEP) by a suitable selection of a channel multiplexing scheme.

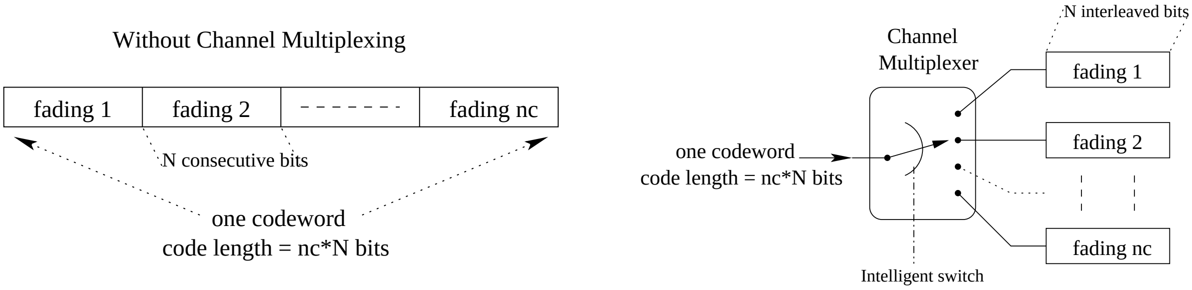

Figure 1: Block fading channel model without multiplexing (left), with multiplexing (right).

1.1 Pairwise error probability

Maximum likelihood (ML) decoding is assumed. We also suppose that the all-zero codeword is transmitted. The conditional pairwise error probability associated to a non-zero codeword c is

$$\begin{aligned} P(0 \rightarrow c|\alpha_1, \ldots, \alpha_{n_c}) &= P(dec.\ metric\ for\ c\ <\ dec.\ metric\ for\ 0 \mid 0\ transmitted) \\ &= P(\textstyle\sum_{i=1}^{n_c N} r_i\, h_i\, c_i > 0) \end{aligned} \tag{2}$$

On a Gaussian channel without fading ($h_i = 1$, $\forall i$), the well known expression for $P(0 \rightarrow c)$ is

$$P(0 \rightarrow c) = Q\left(\sqrt{\frac{2RE_b}{N_0} w_H(c)}\right) \tag{3}$$

where $w_H(c)$ is the codeword Hamming weight and the signal-to-noise ratio E_b/N_0 is defined from the variance σ^2 of the additive noise z_i, i.e., $\sigma^2 = \frac{1}{2E_b/N_0}$. Now, on an n_c-state Rayleigh fading channel, the conditional pairwise error probability becomes

$$P(0 \rightarrow c|\alpha_1, \ldots, \alpha_{n_c}) = Q\left(\sqrt{\frac{2RE_b}{N_0} \sum_{j=1}^{n_c} w_j(c)\alpha_j^2}\right) \tag{4}$$

The partial weight $w_j(c)$ is the Hamming weight of code symbols in c undergoing the fading coefficient α_j, $0 \leq w_j(c) \leq w_H(c)$ and $\sum_j w_j(c) = w_H(c)$. After a mathematical expectation over the n_c Rayleigh fadings, the pairwise error probability is upper bounded as follows [6]

$$P(0 \rightarrow c) \leq \frac{1}{2} \prod_{j=1}^{n_c} \frac{1}{\left(1 + w_j(c)\frac{RE_b}{N_0}\right)} \tag{5}$$

It is clear that channel multiplexing changes the values of $w_j(c)$ and hence makes a direct influence on the word error probability in block fading channels. The diversity order $d(c)$ achieved by the codeword c is the number of non-zero partial weights $w_j(c)$. Thus, the diversity order cannot exceed n_c (*full diversity*). The diversity order achieved by the linear binary code C is defined as $d(C) = Min_{c \in C-0} d(c)$. By applying the Singleton bound to the non-binary version of C, it has been shown that [4][5]

$$d(C) \leq 1 + \lfloor n_c(1-R) \rfloor \tag{6}$$

The maximal information rate R_{max} that achieves maximal diversity order $d = n_c$ is upper bounded by $R_{max} \leq 1/n_c$. Consequently, this paper considers only rate $1/n_c$ codes transmitted over block fading channels. One of our objectives is to achieve full-diversity $d = n_c$ with high probability, while transmitting a rate $R = R_{max} = 1/n_c$. Another objective is to maximize the product $\prod_{j=1}^{n_c} w_j(c)$ called the *coding gain*. Indeed, when $E_b/N_0 >> 1$, the pairwise error probability behaves like $P(0 \rightarrow c) \propto 1/\prod_{j=1}^{n_c} w_j(c) \times 1/(E_b/N_0)^{d(c)}$.

1.2 Information theoretical limits

Block fading channels with finite number of states n_c, i.e., non-ergodic channels, admit a null Shannon capacity [2]. The information theoretical limit for such channels is established by defining an outage probability $P(C < R)$, where C is a random variable representing the instantaneous capacity for a given fading instance, and R is the information rate that the encoder is willing to transmit. For an infinitely large block length, the outage probability is the lowest error probability that can be achieved by a channel encoder and decoder pair. The outage capacity limit drawn in this paper is derived from the probability distribution of the mutual information [1] on a block fading channel with n_c states and BPSK input.

2 Single convolutional codes on block fading channels

Let C_∞ be a non-recursive non-systematic (NRNSC) binary convolutional code with memory ν and coding rate $R_\infty = 1/n$, where $\nu \in \mathbb{N}$, $\nu \geq 1$, and $n \in \mathbb{N}$, $n \geq 2$. The convolutional encoder outputs $\{s_i\}_{i=1\ldots n}$ are related to its single input e via n generator polynomials $g_i(x)$

$$s_i(x) = \sum_{t=0}^{+\infty} s_{i,t}\, x^t = g_i(x)e(x) \qquad e(x) = \sum_{t=0}^{+\infty} e_t\, x^t \tag{7}$$

where $e_t \in GF(2)$ is the input at time instant $t \in \mathbb{Z}^+$, and $s_{i,t} \in GF(2)$ is the i^{th} encoder output at time t. The constraint length of C_∞ is $\nu + 1$ where $\nu = \max_i(\deg(g_i(x)))$. Convolutional codes of rate greater than $1/2$ can be built from lower rate codes by puncturing, i.e., dropping a fraction of coded bits. The semi-infinite trellis graph of C_∞ has 2^ν states and 2 transitions per state. A transition from state u to state ϑ at time t is denoted $T_{u\to\vartheta}(t)$ $u, \vartheta \in [0\ldots 2^\nu - 1]$. The notation $T_{u\to\vartheta}(t)$ is replaced by $u \to \vartheta$ if time position is not required. The binary label of a transition is $\Lambda[T_{u\to\vartheta}(t)] = (s_{1,t}, s_{2,t}, \ldots, s_{n,t})$. We write $T_{u\to\vartheta}(t) = \emptyset$ when the transition $u \to \vartheta$ does not exist. Here, the symbol $\emptyset$ denotes the null element, not the empty set. If $u = 0$, then $T_{0\to\vartheta}(t) \neq \emptyset$ for $\vartheta = \Phi$. Similarly, If $\vartheta = 0$, then $T_{u\to 0}(t) \neq \emptyset$ for $u = \Psi$. Since C_∞ is linear and the all-zero codeword is taken as a reference, we say that $0 \to \Phi$ is an *outgoing transition* and $\Psi \to 0$ is an *incoming transition*. Any non-zero codeword with finite Hamming weight includes the first transition $T_{0\to\Phi}(t_1)$ and the last transition $T_{\Psi\to 0}(t_2)$, at time instants t_1 and t_2 respectively, $t_2 > t_1$.

In the sequel, the convolutional code C_∞ is converted into a linear binary block code C of dimension N and length $n(N+\nu)$. The encoder starts in state 0 at $t = 0$, i.e., $T_{u\to\vartheta}(0) = \emptyset$ for all states $u \neq 0$. The encoder terminates in state 0 at $t = N+\nu$, i.e., $T_{u\to\vartheta}(N+\nu-1) = \emptyset$ for all states $\vartheta \neq 0$. It is also assumed that $N >> \nu \geq 1$, the coding rate of C is $R \approx R_\infty = 1/n$.

A non-zero codeword of C that includes a unique outgoing transition and a unique incoming transition is called a *simple error event*. Let $\nabla \in C$ be non-zero codeword. If ∇ is a simple error event, then we write $\nabla(\omega, t_1, t_2, L)$ to indicate that the corresponding trellis path includes $T_{0\to\Phi}(t_1)$ and $T_{\Psi\to 0}(t_2)$, its input Hamming weight is ω, and its *length* is $L = t_2 - t_1 + 1$, $\nu + 1 \leq L \leq N + \nu$. Similarly, a *double error event* $\nabla(\omega, t_1, t_2, t_3, t_4, L) = \nabla(\omega_1, t_1, t_2, L_1) + \nabla(\omega_2, t_3, t_4, L_2)$ is a codeword obtained by the sum of two simple error events, its length is $L = L_1 + L_2$ and its input weight is $\omega = \omega_1 + \omega_2$. In general, a *compound error event* is the sum of two or many simple error events.

Definition 1 *Consider a rate $1/n$ binary convolutional code C with constraint length $\nu+1$ defined by the generator polynomials $g_1(x), g_2(x), \ldots, g_n(x)$. The code C is said to be a full-span convolutional code if the generators satisfy* $\deg(g_i(x)) = \nu$ *and* $g_i(0) = 1$, $\forall i$.

All known rate $1/n$ non-recursive non-systematic binary convolutional codes with best minimum Hamming distance are full-span codes [6], e.g., the 4-state NRNSC$(7,5)_8$, the 8-state NRNSC$(13,15)_8$, and the 8-state NRNSC$(17,15)_8$.

Fig. 2 shows four different ways of multiplexing the 3 outputs s_1, s_2, s_3 of a rate $1/3$ convolutional code. These four multiplexing ways are similar for any rate $1/n$ convolutional code, $n \geq 2$, without puncturing of code symbols.

Horizontal Multiplexing

$\mathbf{s_1}$	1	1	1	1	1	1
$\mathbf{s_2}$	2	2	2	2	2	2
$\mathbf{s_3}$	3	3	3	3	3	3

Vertical Multiplexing

$\mathbf{s_1}$	1	2	3	1	2	3
$\mathbf{s_2}$	1	2	3	1	2	3
$\mathbf{s_3}$	1	2	3	1	2	3

Diagonal Multiplexing

$\mathbf{s_1}$	1	2	3	1	2	3
$\mathbf{s_2}$	2	3	1	2	3	1
$\mathbf{s_3}$	3	1	2	3	1	2

Random Multiplexing

$\mathbf{s_1}$	2	3	2	3	2	2
$\mathbf{s_2}$	1	2	3	2	1	3
$\mathbf{s_3}$	1	2	1	1	2	3

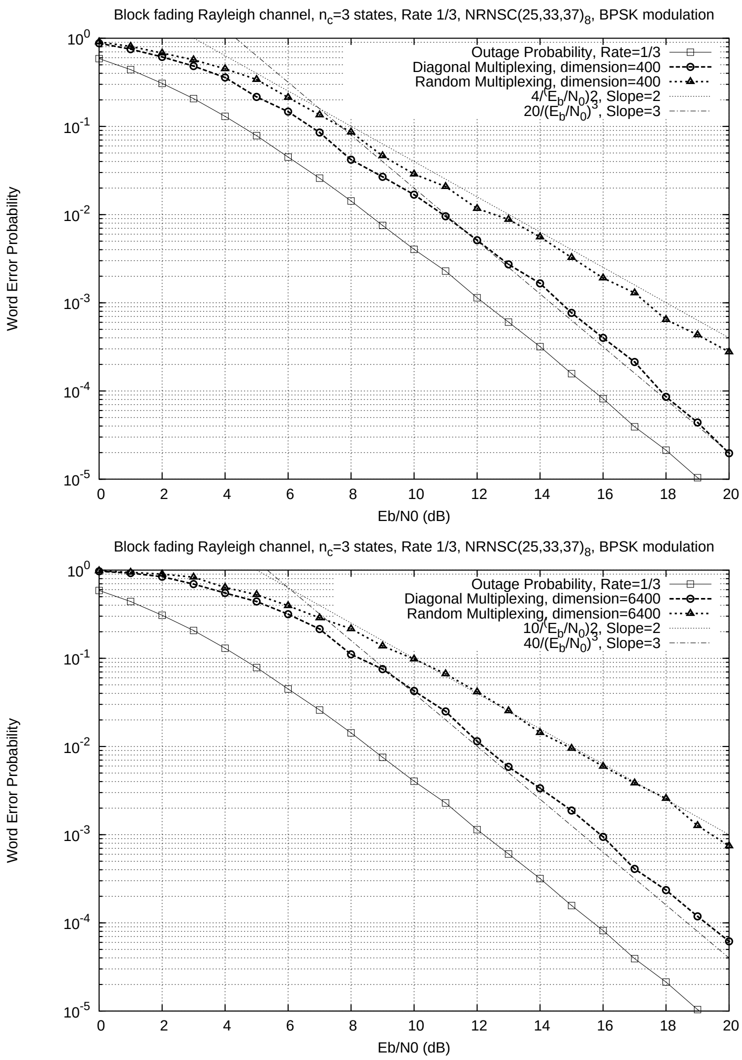

Figure 2: Left: Four different ways of channel multiplexing are described for rate 1/3 NRNSC codes on a 3-state block fading channel. Right: The word error probability performance of a rate 1/3 16-state binary NRNSC$(25,33,37)_8$ is compared to the outage capacity limit for code dimensions 400 bits and 6400 bits respectively.

Proposition 1 *Diagonal channel multiplexing of a full-span rate $1/n_c$ convolutional code achieves full diversity on an n_c-state block fading channel.*

Proof: Consider an outgoing transition $T_{0\to\Phi}(t_1)$ of a codeword $\nabla \in C$. Since C is full-span, then $\Lambda(0 \to \Phi) = 1^{n_c}$, where 1^{n_c} is the all-1 binary label. Any column in the diagonal multiplexing table includes n_c distinct fadings. Hence, ∇ achieves diversity order n_c. □

The above proposition is also true for horizontal multiplexing, but it is false for both vertical and random multiplexing. Numerical evaluation of the coding gain, based on the weight distribution of C, shows that diagonal multiplexing outperforms all other proposed methods. Indeed, for a given Hamming weight $w_H(\nabla)$, diagonal channel interleaving produces quasi-equal partial weights $w_j(\nabla)$, $j = 1 \dots n_c$. Computer simulations confirm the supremacy of diagonal multiplexing. Fig. 2 illustrates the WEP performance versus the signal-to-noise ratio for a 16-state rate 1/3 convolutional code. Diagonal multiplexing yields a WEP at 2 to 3dB distance from the outage capacity limit. Notice also that WEP of a convolutional on a block fading channel does depend on the code length.

3 Parallel concatenation of two RSC codes

A rate $1/n$ recursive systematic convolutional (RSC) code is defined from its NRNSC equivalent by dividing all generators with $g_1(x)$. The RSC encoder outputs $\{s_i\}_{i=1\dots n}$ are related to its single input e via the following expressions

$$s_1(x) = e(x) \quad \text{and} \quad s_i(x) = \frac{g_i(x)}{g_1(x)} e(x) \quad for \;\; i = 2\dots n \tag{8}$$

We limit our study in this section to rate 1/3 parallel turbo codes [3] built from two identical rate 1/2 RSC constituents separated by a pseudo-random interleaver π of size N bits. A rate 1/2 turbo code can be obtained by puncturing one parity bit out of two parity bits generated by the RSC constituents. Figures 3 and 4 illustrate four ways of channel multiplexing for rate 1/3 and rate 1/2 turbo codes respectively. The systematic output is s_1, the parity bit produced by the first RSC is s_2 and the parity bit produced by the second RSC is s_3. The symbol X denotes a punctured symbol. Many other channel multiplexing methods can be constructed, but we limit the description in this paper to those given in Figures 2, 3 and 4.

π-diagonal Multiplexing							
s1	1	2	3	···	1	2	3
s2	2	3	1	···	2	3	1
s3	$\pi($ 3	1	2	···	3	1	2 $)$

h-diagonal Multiplexing						
s1	1	2	1	2	1	2
s2	2	1	2	1	2	1
s3	3	3	3	3	3	3

Figure 3: Rate 1/3 turbo code, $n_c = 3$ channel states, π-diagonal and h-diagonal multiplexing.

h-diagonal Multiplexing						
s1	1	2	1	2	1	2
s2	2	X	2	X	2	X
s3	X	1	X	1	X	1

h-π-diagonal Multiplexing						
s1	1	2	1	2	1	2
s2	2	X	2	X	2	X
s3	$\pi($ X	1	X	1	X	1 $)$

Figure 4: Rate 1/2 turbo code, $n_c = 2$ channel states, h-diagonal and h-π-diagonal multiplexing.

Proposition 2 *Let C be a rate 1/3 turbo code built from $RSC(7,5)$ and transmitted on a 3-state block fading channel. Under* <u>random</u> *channel multiplexing, the expected number η of codewords in C with incomplete diversity and input weight $w = 2$ is*

$$\eta(w=2, d<n_c) \approx 3\left(\frac{9}{5}\right)^2\left(\frac{2}{3}\right)^{10} \approx 0.168$$

Proof: The Hamming weight distribution $H(W,D,L) = \sum_{j=1}^{N} A^j$ of RSC$(7,5)$ can be fully determined from the following transition matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & WD^2L & 0 \\ WD^2L & 0 & L & 0 \\ 0 & WDL & 0 & DL \\ 0 & DL & 0 & WDL \end{bmatrix}$$

where the indeterminate W represents the input weight, D the output weight, and L the trellis path length. Any codeword $\nabla(2,t_1,t_2,L)$ has length $L = L_{min} + 3i$, and weight $w_H(\nabla) = 6 + 2i$, $i \geq 0$. The minimum length is $L_{min} = 4$. For a fixed time instant t_1 and a fixed length L, the multiplicity of $\nabla(2,t_1,t_2,L)$ is 1. Now, a turbo codeword is the direct sum of two RSC codewords: $\nabla_1(2,t_1,t_2,L_1)$ in RSC1 and $\nabla_2(2,\tau_1,\tau_2,L_2)$ in RSC2, where $\tau_1 = \pi(t_1)$ and $\tau_2 = \pi(t_2)$. Here, π denotes the interleaver pseudo-random permutation of the turbo code. For finite dimension N, the expected number of turbo codewords attaining diversity 2 is (the number of codewords with diversity 1 is negligible)

$$\eta = 3\sum_{L_1,L_2}\frac{(N-L_1)(N-L_2)}{N^2}\left(\frac{2}{3}\right)^{w_H(\nabla_1)+w_H(\nabla_2)-2} = 3\sum_{i=0}^{\frac{(N-4)}{3}}\sum_{j=0}^{\frac{(N-4)}{3}}\frac{(N-4-3i)(N-4-3j)}{N^2}\left(\frac{2}{3}\right)^{10+2i+2j}$$

If $N \to +\infty$, then $\eta \to 3(9/5)^2(2/3)^{10}$ which completes the proof. □

In a similar manner, it can be shown that under random channel multiplexing

$$\eta(w=2, d<n_c) = 3\left(\frac{81}{65}\right)^2\left(\frac{2}{3}\right)^{14} \approx 0.0159 \quad for\ \ RSC(13,15)$$

and

$$\eta(w=2, d<n_c) = 3\left(\frac{9}{5}\right)^2\left(\frac{2}{3}\right)^{10} \approx 0.168 \quad for\ \ RSC(17,15)$$

The number $\eta(w=2, d<n_c)$ is relatively small for RSC$(13,15)$ because $g_1(x) = (13)_8$ is primitive. Nevertheless, η does not decrease with N. The use of random multiplexing does not guarantee full-diversity on block fading channels. The two following propositions give some insight when $w = 3$ and $w = 4$ (results are similar to the Gaussian channel case).

Proposition 3 *Let C be a rate 1/3 turbo code built from $RSC(7,5)$ and transmitted on a 3-state block fading channel. Under* <u>random</u> *channel multiplexing, the expected number η of codewords in C with incomplete diversity and input weight $w = 3$ and $w = 4$ is*

$$\eta(w=3, d<n_c) \approx 1.70/N \quad and \quad \eta(w=4, d<n_c) \geq 0.0094$$

Proof: The weight distribution of RSC$(7,5)$ yields two distinct weight profiles when $w = 3$. Any codeword $\nabla(3,t_1,t_2,L)$ satisfies the first profile: $L = 3+3i$, $w_H(\nabla) = 5+2i$ with multiplicity $i+1$, $i \geq 0$, or the second profile: $L = 5+3i$, $w_H(\nabla) = 7+2i$ with multiplicity $i+1$, $i \geq 0$. Now, we can write

$$\eta = 3\sum_{i=0}^{\frac{(N-3)}{3}}\sum_{j=0}^{\frac{(N-3)}{3}}\frac{(N-3-3i)(N-3-3j)}{N^3}(i+1)(j+1)\left(\frac{2}{3}\right)^{7+2i+2j}$$

$$+3\sum_{i=0}^{\frac{(N-5)}{3}}\sum_{j=0}^{\frac{(N-5)}{3}}\frac{(N-5-3i)(N-5-3j)}{N^3}(i+1)(j+1)\left(\frac{2}{3}\right)^{11+2i+2j}$$

$$+6\sum_{i=0}^{\frac{(N-3)}{3}}\sum_{j=0}^{\frac{(N-5)}{3}}\frac{(N-3-3i)(N-5-3j)}{N^3}(i+1)(j+1)\left(\frac{2}{3}\right)^{9+2i+2j}$$

When $N \to +\infty$, we obtain $\eta(w=3, d<n_c) \approx 1.70/N$.
For input weight $w=4$ codewords, three cases are distinguished: 1- ∇_1 and ∇_2 are simple error events, 2- ∇_1 is simple but ∇_2 is a double error event, and 3- ∇_1 and ∇_2 are double error events. In the last case, $\eta \approx 0.0094$ (algebraic details are omitted). □.
Note that in the case RSC$(13,15)$ with random multiplexing, we also have $\eta(w=3, d<n_c) \propto 1/N$. Due to its special weight distribution, all RSC$(17,15)$ codewords have even weight because $g_1(x) = (x+1)^3$, then RSC$(17,15)$ satisfies $\eta(w=3, d<n_c) = 0$.

Proposition 4 *Let C be a rate 1/2 turbo code built from punctured RSC$(7,5)$ and transmitted on a 2-state block fading channel. Under random channel multiplexing, the expected number η of codewords in C with incomplete diversity and input weight $w=2$ is*

$$\eta(w=2, d<n_c) \approx 0.440$$

Proof: The weight distribution is determined by the two following transition matrices. Matrices A_1 and A_2 correspond respectively to the unpunctured and punctured parity bit transitions.

$$\mathbf{A_1} = \begin{bmatrix} 0 & 0 & WD^2L & 0 \\ WD^2L & 0 & L & 0 \\ 0 & WDL & 0 & DL \\ 0 & DL & 0 & WDL \end{bmatrix} \qquad \mathbf{A_2} = \begin{bmatrix} 0 & 0 & WDL & 0 \\ WDL & 0 & L & 0 \\ 0 & WDL & 0 & L \\ 0 & L & 0 & WDL \end{bmatrix}$$

The omitted details to get $\eta \approx 0.440$ are similar to the proof of proposition 2. □

Proposition 5 *Let C be a rate 1/3 turbo code built from a full-span RSC$(g_1(x), g_2(x))$ and transmitted on a 3-state block fading channel. Under π-diagonal channel multiplexing, the expected number η of codewords in C with incomplete diversity and input weight $w=2$ and $w=3$ is*

$$\eta(w=2, d<n_c) = 0 \quad \textit{and} \quad \eta(w=3, d<n_c) = 0$$

Proof : Input weight $w=2$: Let $\nabla_1(2,t_1,t_2,L_1)$ and $\nabla_2(2,\tau_1,\tau_2,L_2)$ be two simple error events defining a turbo codeword. The outgoing transition $T_{0\to\Phi}(t_1)$ in ∇_1 is connected via the pseudo-random turbo interleaver to $T_{0\to\Phi}(\tau_1)$ or to $T_{\Psi\to 0}(\tau_2)$ in ∇_2. Since all incoming and outgoing transitions are full-span, then the label $(s_{1,t_1}, s_{2,t_1}, s_{3,\tau_1})$ is equal to $(1,1,1)$. The π-diagonal multiplexer guarantees that $(\alpha_{(i)}, \alpha_{(i+1)})$ is associated to $\alpha_{(i+2)}$. Therefore, $(s_{1,t_1}, s_{2,t_1}, s_{3,\tau_1}) = (1,1,1)$ undergoes the fading vector $(\alpha_{(i)}, \alpha_{(i+1)}, \alpha_{(i+2)})$ which yields a diversity order 3.
Input weight $w=3$: The outgoing transition of $\nabla_1(3,t_1,t_2,L_1)$ is connected to three possible positions via the turbo code pseudo-random interleaver. If $\pi(t_1)=\tau_1$ or $\pi(t_1)=\tau_2$, then diversity order is 3 (see the proof in the case $w=2$). On the other hand, if $\tau_1 < \pi(t_1) < \tau_2$, then diversity 3 is collected by the incoming transition of the codeword ∇_1. □

Even if $\eta(w=2, d<n_c) = \eta(w=3, d<n_c) = 0$ for any RSC with π-diagonal multiplexing, this nice property does not help improving the WEP in practice. Indeed, let A_i denotes the transition matrix of RSC1 (upper turbo code constituent) with diagonal and π-diagonal multiplexing. Matrix A_i includes both information and parity bits undergoing fadings α_i and $\alpha_{(i+1)}$ respectively,

$$\mathbf{A}_i = \begin{bmatrix} L & 0 & WD_iD_{(i+1)}L & 0 \\ WD_iD_{(i+1)}L & 0 & L & 0 \\ 0 & WD_iL & 0 & D_{(i+1)}L \\ 0 & D_{(i+1)}L & 0 & WD_iL \end{bmatrix}$$

By evaluating the generalized weight enumerator $H(W,D_1,D_2,D_3,L)=\sum_{i=1}^{3}\sum_{j=1}^{N}(A_iA_{(i+1)}A_{(i+2)})^j$, we can state the following for both diagonal and π-diagonal multiplexing of rate 1/3 turbo codes:

- The 4-state RSC$(7,5)$: $H(W,D_1=1,D_2=1,D_3=0,L)=0\cdot W^2+0\cdot W^3+1\cdot W^4L^7+0\cdot W^5+1\cdot W^6(L^{10}+L^{10}+L^{10}+\ldots)+\ldots$. Therefore, diagonal multiplexing of RSC1 already insures $\eta=0$ for $w=2$ and $w=3$. Multiplexing via π does not help in this case. Fortunately, π-multiplexing will be of great effi ciency in the rate 1/2 case.
- The 8-state RSC$(17,15)$: $H(W,D_1=1,D_2=1,D_3=0,L)=0\cdot W^2+0\cdot W^3+0\cdot W^4L^7+0\cdot W^5+1\cdot W^6(L^{10}+L^{13}+L^{16}+L^{19}+L^{22})+0\cdot W^7+1\cdot W^8(L^{16}+\ldots)+\ldots$.
- The 8-state RSC$(13,15)$: $H(W,D_1=1,D_2=1,D_3=0,L)=0\cdot W^2+1\cdot W^3L^4+1\cdot W^4L^7+1\cdot W^5L^{10}+1\cdot W^6L^{13}+\ldots$.

Proposition 6 *The h-diagonal channel multiplexing of a full-span rate $1/3$ parallel turbo code achieves full diversity on a 3-state block fading channel, i.e., $\eta=0$ for any input weight w.*

Proof: Similarly to the proofs of above propositions, the outgoing transition $0\rightarrow\Phi$ in ∇_1 guarantees a diversity order equal to 2. Any non-zero binary element in ∇_2 will collect the 3rd diversity order. □

Proposition 7 *Let C be a rate 1/2 turbo code built from punctured RSC$(7,5)$ and transmitted on a 2-state block fading channel. Under h-diagonal channel multiplexing, the expected number η of codewords in C with incomplete diversity and input weight $w=2$ and $w=3$ is*

$$\eta(w=2,d<n_c)=0 \quad \textit{and} \quad \eta(w=3,d<n_c)\approx N/144$$

Proof: The proof is omitted due to the lack of space. □

Proposition 8 *Let C be a rate 1/2 turbo code built from a full-span punctured RSC$(g_1(x),g_2(x))$ and transmitted on a 2-state block fading channel. Under h-π-diagonal channel multiplexing, the expected number η of codewords in C with incomplete diversity and input weight $w=2$ and $w=3$ is*

$$\eta(w=2,d<n_c)=0 \quad \textit{and} \quad \eta(w=3,d<n_c)=0$$

Proof: The arguments are identical to the proof of proposition 5. □

4 Experimental results

We illustrated our experimental results in Figures 5, 6 and 4. The WEP of a rate 1/3 turbo code with dimension 400 and length 1212 bits is depicted in Fig. 5. The h-diagonal channel multiplexing guarantees a WEP at distance less than 1dB from the outage capacity limit. Random channel multiplexing also performs well for WEP greater than 10^{-5} since the expected number η of bad codewords is relatively low. Fig. 6 shows the same situation with dimension 6400 and length 19212 bits. It can be noticed that WEP is insensitive to code length and still less than 1dB from outage capacity limit. Finally, Fig. 4 illustrates the WEP of a rate 1/2 turbo code with h-π-diagonal versus random multiplexing. The h-π-diagonal also exhibits a WEP at distance less than 1dB from outage capacity limit and shows no sensitivity to code length. Our results indicate that parallel turbo codes broadly outperform the serial concatenation (repeat-accumulate codes) designed for block fading channels [7]. In all our computer simulations, the number of turbo decoding iterations is greater than 10 (Forward-Backward algorithm on both RSC constituents), and 100 erroneous blocks have been counted to estimate the word error probability.

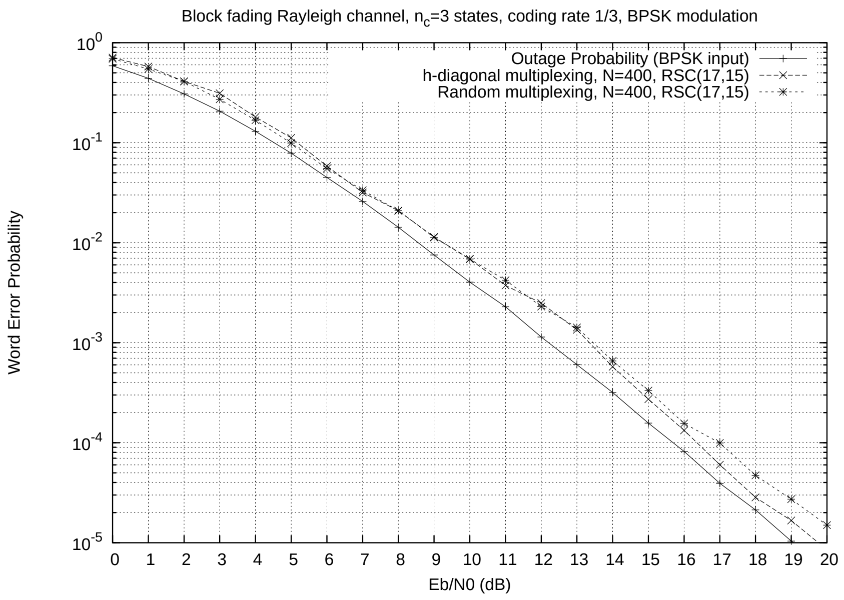

Figure 5: Rate 1/3 parallel turbo code with 8-state RSC(17,15) constituent, dimension=400, length=1212 bits. Random and h-diagonal multiplexing versus outage probability limit. At least 10 double Forward-backward decoding iterations for each turbo coding block.

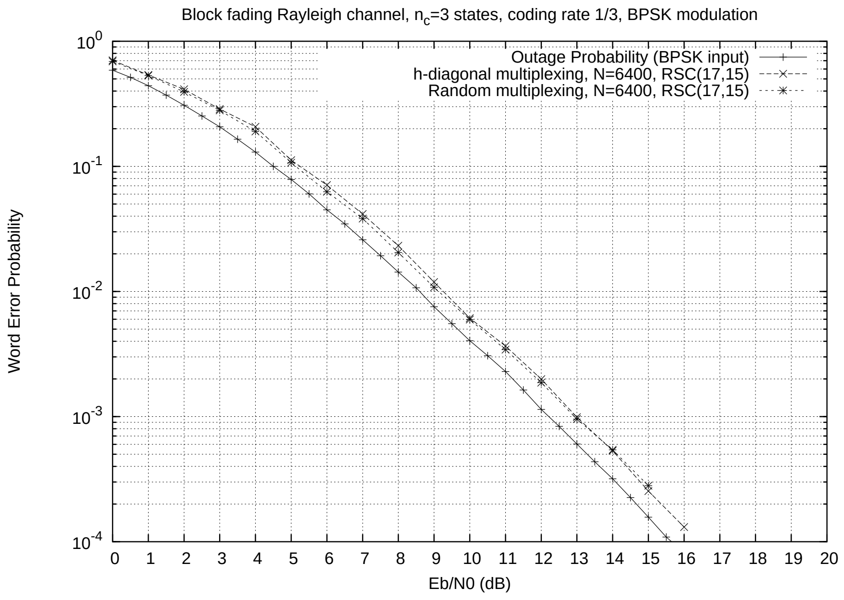

Figure 6: Rate 1/3 parallel turbo code with 8-state RSC(17,15) constituent, dimension=6400, length=19212 bits. Random and h-diagonal multiplexing versus outage probability limit. At least 10 double Forward-backward decoding iterations for each turbo coding block.

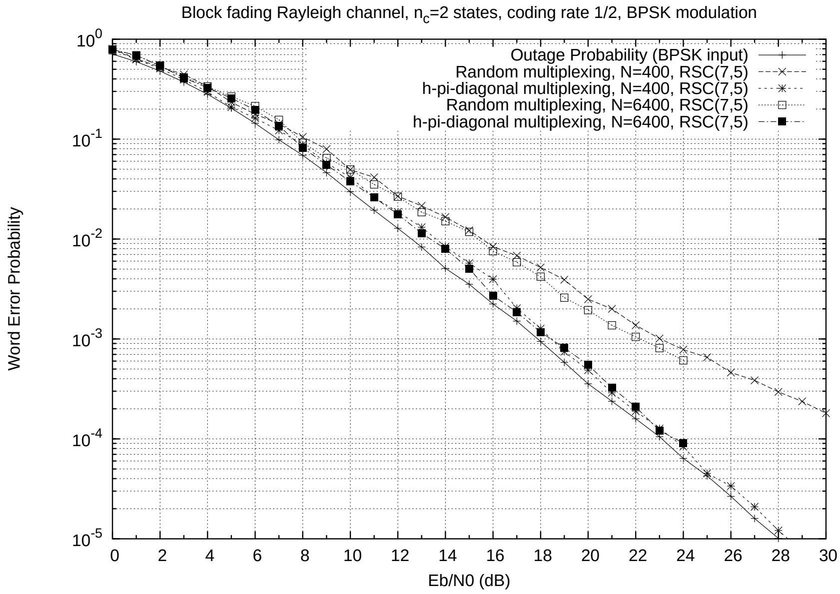

Figure 7: Rate 1/2 parallel turbo code with 4-state RSC(7,5) constituent, code dimension=400 and 6400, code length=808 and 12808 bits. Random and h-π-diagonal multiplexing versus outage probability limit. At least 10 double Forward-backward decoding iterations for each turbo coding block.

5 Conclusions and further work

Parallel turbo codes are very efficient on block fading channels, especially when intelligent channel multiplexing is cascaded with turbo encoding. The WEP performances presented in this paper versus the outage capacity limit are the best known results in the literature (up to now) for rate 1/2 and rate 1/3 binary codes on block fading channels. Our future work should include the analysis of higher input weight configurations, a universal description of properties valid for all recursive systematic convolutional codes, and a coding gain analysis for the schemes proposed in this paper. We declare *open* the race to the outage capacity limit.

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