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ASYMPTOTIC CAPACITY OF STATIC MULTIUSER CHANNELS WITH AN UNKNOWN NUMBER OF USERS

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ABSTRACT

We examine a multiple-access communication system in which multiuser detection is performed without knowledge of the number of active interferers. Using a statistical-physics approach, we compute the single-user channel capacity and spectral efficiency in the large-system limit.

I. INTRODUCTION

In mobile multiple-access communications, the number of active users, their location, and other channel-state parameters, are variable with time. Estimating these parameters is crucial in several applications, e.g., user localization in wireless networks, neighbor discovery in ad hoc networks, and power-control strategy optimization.

The classic approach to multiuser detection theory [1] is based on the assumption that the number of active users is constant, known at the receiver, and equal to the maximum number of users entitled to access the system. However, this model is unrealistic in an environment where there is a considerable number of users that remain inactive at any given time. Furthermore, the assumption of a number of users larger than it actually is leads to a considerable performance loss for several families of detectors. In [2], a new class of detectors was introduced accounting for the dearth of information on the number of active users. In particular, the problem of jointly estimating the number, the identities, and the data of active users was solved in [2], whereas in [3] the extension to the estimation of the continuous parameters of the active users was addressed.

The object of interest in this paper is the large-system behavior of code-division multiple-access (CDMA) systems within the framework of [2]. Our analysis uses tools recently developed from statistical physics (see, e.g., [4] and the references therein). Specifically, here we aim at analyzing the multiuser and spectral efficiency of a CDMA system when its natural dimensions (number of users K, and spreading gain N) grow to infinity while their ratio $\beta \triangleq K/N$ (the *system load*) remains constant. Of particular interest is the large-system equivalent single-user channel introduced in [5] for randomly spread CDMA.

In [6], by applying a large-system analysis to the static channel model described in [2], asymptotic multiuser efficiency and bit error probability of a joint detector of data and user identities were derived. The results in [6] show the degradation of multiuser efficiency and bit error probability caused by the uncertainty on the activity of the users. Our goal here is to derive single-user capacity and spectral efficiency of a multiuser channel using the detector of [2].

II. SYSTEM MODEL

In [2, 7], Random-Set Theory (RST) was proved to be a natural, flexible, and mathematically sound framework for the multiuser-detection problem under study. In fact, RST allows one to account for the randomness present not only in the data to be detected, but also in the set of users that are active at any time. This new perspective can be used to derive novel, more robust multiuser detectors.

A *random set* [8] is a map from a sample space Ω to a family of subsets of a given space $\mathbb{S}$. In the model we are concerned with, $\mathbb{S}$ contains the unknown data and parameters of the active interferers. The information carried by the interferers at time t can be modeled as a random set:

$$\mathbf{X}_t = \{\mathbf{x}_t^{(1)}, \ldots, \mathbf{x}_t^{(k)}\} \tag{1}$$

The elements of $\mathbf{X}_t$ are random vectors whose components are the unknown user parameters, while k, the number of active interferers, is itself a random integer taking values in the interval $[0, K-1]$, with K the maximum number of users that can access the channel. If everything about the interferers is known except their number and identities, then all $\mathbf{X}_t$ are subsets of the power set $2^{\mathbb{K}}$, $\mathbb{K} \triangleq \{1, \ldots, K\}$. Similarly, if we aim at detecting the (binary) data that the interferers transmit, then $\mathbf{X}_t$ takes values in a set with 3^K elements, denoted as $3^{\mathbb{K}}$. Our goal here is the detection of the number and identities of active interferers as well as the data they carry, under the assumption that their remaining parameters remain constant. For this reason we shall not deal with the evolution of $\mathbf{X}_t$ with time (the "static-channel" assumption).

Specifically, we examine here a direct-sequence CDMA (DS-CDMA) system, whose received signal at time t is

$$\mathbf{y}_t = \mathbf{R}\mathbf{A}\mathbf{b}_\mathbf{t}(\mathbf{X}_\mathbf{t}) + \mathbf{z}_t, \quad t = 1, \ldots, T \tag{2}$$

where

- $\mathbf{X}_t$ is the random set of active users.
- $\mathbf{R}$ is the correlation matrix of the signature sequences.
- $\mathbf{A}$ is the diagonal matrix of the users' signal amplitudes.
- The vector $\mathbf{b}_t(\mathbf{X}_t)$ contains the user data, and has zero entries in the locations corresponding to inactive user.
- $\mathbf{z}_t$ is a Gaussian noise vector $\sim \mathcal{N}(0, (N_0/2))$ or $\sim \mathcal{N}(0, \sigma^2)$, with $N_0/2 = \sigma^2$ the power spectral density of the received noise.

Under our static-channel assumption, the a posteriori probability (APP) of $\mathbf{X}_t$, given the whole received sequence, is

$$f(\mathbf{X}_t|\mathbf{y}_{1:T}) \triangleq f(\mathbf{X}_t|\mathbf{y}_t) \tag{3}$$

Hence, the sequential maximum a posteriori probability (MAP) estimator of users' identities has the form

$$\hat{\mathbf{X}} = \arg\max_{\mathbf{X}_t \in 2^{\mathbb{K}}} f(\mathbf{X}_t|\mathbf{y}_t) \tag{4}$$

Here we examine user-identity and user-and-data MAP detectors. For both detectors, we assume that the probability of interferer i to be active at time t, i.e. the *activity rate*, is equal to α, irrespective of i and t. Hence, the probability distribution of interferer set $\mathbf{X}_t$ depends only on its cardinality $|\mathbf{X}_t|$. By assuming that each user k transmits n antipodal binary information symbols, independent from time to time and across users, the a priori distributions of the random sets $\mathbf{B}$ containing either the user identities, or the user-and-data vectors, are

$$f_{\mathbf{X}}(\mathbf{B}) = \alpha^{|\mathbf{B}|}(1-\alpha)^{K-|\mathbf{B}|} \tag{5}$$

$$f_{\mathbf{X}}(\mathbf{B}) = 2^{-n|\mathbf{B}|}\alpha^{|\mathbf{B}|}(1-\alpha)^{K-|\mathbf{B}|} \tag{6}$$

respectively, where the time subscripts are omitted for simplicity. In the balance of this paper, we assume $n = 1$.

III. Large-system analysis

Sharp tools for large system-analyses were recently developed in [9] (see also [4] and references therein). In [9], statistical-physics concepts and methodologies were applied to multiuser detection, to derive large-system uncoded minimum BER and spectral efficiency with equal-power binary inputs. Other authors expanded the scope of [9]: in [10], channel capacity was derived, while [5] presented a unified treatment of Gaussian CDMA channels and multiuser detection in the large-system limit under arbitrary input distribution and flat-fading.

Central to the large-system analysis of [9] is the concept of *free energy*. In statistical physics, the free-energy $\mathcal{F}(\mathbf{X})$ (where $\mathbf{X}$ is the state variable) relates the energy $E(\mathbf{X})$ and the entropy $H(\mathbf{X})$ of a physical system in the following way:

$$\mathcal{F}(\mathbf{X}) = E(\mathbf{X}) - TH(\mathbf{X}) \tag{7}$$

where T is the temperature of the system. At thermal equilibrium, the free energy (7) is minimized as time evolves, since the entropy tends to a maximum according to the second law of thermodynamics. A key point here is that the free energy, normalized to the dimensionality of the system, is a self-averaging quantity. This is the reason why the free energy turns out to be the starting point for calculating macroscopic properties of a thermodynamic system.

In a communications system, the energy function becomes the metric of a detector in a multiuser channel. From this perspective, any detector, when parameterized with a certain metric, can be analyzed with the tools of statistical mechanics to derive results in the large-system limit. Hence, the calculation of the free energy can be associated to parameters of a multiuser detection performance such as spectral efficiency, channel capacity, and bit error probability.

At thermodynamic equilibrium, the free energy can also be expressed as $\mathcal{F} = -T \log Z$, where $Z = \sum_x \exp\left(-\frac{1}{T}\|\mathbf{x}\|\right)$ is called the *partition function*, $\|\cdot\|$ is the *energy operator*, and the temperature $T \geq 0$ reflects the energy constraints.

In a communications system such as (2), the detector goal is to infer the information-bearing symbols given the received signal $\mathbf{y}$ and knowledge about the channel state (which we denote $\mathbf{S}$) that yields $Z(\mathbf{y}, \mathbf{S}) = p_{\mathbf{Y}|\mathbf{S}}(\mathbf{Y}|\mathbf{S})$. The corresponding free-energy normalized by the number of users is expressed as:

$$\mathcal{F}_K = -\frac{1}{K}\log p_{\mathbf{Y}|\mathbf{S}}(\mathbf{Y}|\mathbf{S}) \tag{8}$$

To calculate this expression, we use the self-averaging assumption, which states that the randomness of (8) vanishes as $K \to \infty$. This is tantamount to saying that the free energy per user converges in probability to its expected value over the distribution of the random variables $\mathbf{y}$ and $\mathbf{S}$:

$$\mathcal{F} = \lim_{K\to\infty} \mathbb{E}\left\{-\frac{1}{K}\log p_{\mathbf{Y}|\mathbf{S}}(\mathbf{Y}|\mathbf{S})\right\} \tag{9}$$

A. Free energy and capacity of a CDMA

Statistical-physics tools can also be used to achieve insight on the information-theoretic capacity of CDMA channels. As shown in [9], the large-system single-user capacity of a CDMA channel with binary modulation can be written as

$$C = \lim_{K\to\infty} \frac{1}{K} I(\mathbf{b}^1(\mathbf{X}), \ldots, \mathbf{b}^K(\mathbf{X}); \mathbf{y}) = \mathcal{F} - \frac{1}{2\beta}\log(2\pi e) \tag{10}$$

where $I(\cdot\,;\cdot)$ denotes mutual information, $\mathbf{b}^1(\mathbf{X}), \ldots, \mathbf{b}^K(\mathbf{X})$ are the users' transmitted symbols corresponding to the random set $\mathbf{X}$, $\mathbf{y}$ is the received signal, and $\mathcal{F}$ is the free energy defined in (9).

In our analysis, we invoke the decoupling principle [4], which allows us to use an equivalent single-user channel model consisting of a scalar Gaussian channel having the original input distribution and SNR per active user (γ), and an inverse noise variance equal to the multiuser efficiency η. Its input–output relationship is

$$y = \sqrt{\gamma}\,\mathbf{b}^k(\mathbf{X}) + \frac{1}{\sqrt{\eta}} \tag{11}$$

where k denotes an arbitrary user, and η is the solution of the fixed-point equation

$$\eta^{-1} = 1 + \beta\,\mathbb{E}_\gamma[\gamma\,\mathrm{MMSE}(\gamma, \eta)]$$

where MMSE is the minimum mean square error of estimating $\mathbf{b}^k(\mathbf{X})$ under white noise with SNR equal to γ:

$$\mathrm{MMSE}(\gamma, \eta) = \mathbb{E}_{\mathbf{b}^k(\mathbf{X}),y}[(\mathbf{b}^k(\mathbf{X}) - \mathbb{E}\{\mathbf{b}^k(\mathbf{X})|\gamma, \eta, y\})^2] \tag{12}$$

which in turn can be computed as:

$$\alpha - \int \frac{\mathbb{E}^2_{\mathbf{b}^k(\mathbf{X})}\{\mathbf{b}^k(\mathbf{X})P(y|\gamma, \eta, \mathbf{b}^k(\mathbf{X}))\}}{\mathbb{E}_{\mathbf{b}^k(\mathbf{X})}\{P(y|\gamma, \eta, \mathbf{b}^k(\mathbf{X}))\}}\mathrm{d}y$$

where $P(y|\gamma,\eta,\mathbf{b}^k(\mathbf{X}))$ is the Gaussian density describing the single-user equivalent channel (11).

Under the decoupling principle, the mutual information between the input symbol and the output of any MAP estimator turns out to be equal to the input–output mutual information of the Gaussian equivalent channel, so that

$$I(\mathbf{b}^k(\mathbf{X}),y) = \mathbb{E}_{\mathbf{b}^k(\mathbf{X}),y}\left\{\log\left(\frac{P(y|\gamma,\eta,\mathbf{b}^k(\mathbf{X}))}{P(y)}\right)\right\} \quad (13)$$

where $P(\mathbf{b}^k(\mathbf{X}))$ is the input distribution and $P(y) = \sum_{\mathbf{b}^k(\mathbf{X})} P(\mathbf{b}^k(X)P(y|\gamma,\eta,\mathbf{b}^k(\mathbf{X}))$ is the output density. Finally, the spectral efficiency under joint decoding is obtained by adding the single-user mutual information (normalized to the dimensionality of the multiuser channel—the spreading length) to a divergence factor depending on the multiuser efficiency η [5]:

$$\mathcal{C} = \beta\,\mathbb{E}_\gamma\left\{I(\mathbf{b}^k(X),y)\right\} + \frac{1}{2}\left[(\eta-1)\log e - \log\eta\right] \quad (14)$$

IV. Main results

To the purpose of our analysis, we assume that $\mathbf{S} = \mathbf{RA}$, where $\mathbf{S}$ is an $N \times K$ matrix whose columns are the spreading sequences of the users and N is the spreading gain, or sequence length, and that the noise is white: $\mathbf{z} \sim \mathcal{N}(0,\sigma^2\mathbf{I})$ where $\sigma = 1$. Thus, our DS-CDMA system is modeled as follows:

$$\mathbf{y} = \mathbf{Sb}(\mathbf{X}) + \mathbf{z} \quad (15)$$

and the energy operator $\|\cdot\|$, as derived from the free-energy definition, represents the logarithm of the joint distribution $f(\mathbf{y}|\mathbf{S},\mathbf{X})f_{\mathbf{X}}(\mathbf{B})$:

$$\|\mathbf{b}(\mathbf{X})\| = \frac{1}{2\sigma^2}\left(\mathbf{y}-\mathbf{Sb}(\mathbf{X})\right)^T\left(\mathbf{y}-\mathbf{Sb}(\mathbf{X})\right) - \log(f_{\mathbf{X}}(\mathbf{B})) \quad (16)$$

where $f_{\mathbf{X}}(\mathbf{B})$ is the input random-set distribution (5) or (6).

Evaluation of the free energy $\mathcal{F}$ is made possible by the *replica method*, which consists of computing $\mathcal{F}$ as follows:

$$\mathcal{F} = -\lim_{n\to 0}\frac{\partial}{\partial n}\left(\lim_{K\to\infty}\frac{1}{K}\log\mathbb{E}\left\{p_{\mathbf{Y}|\mathbf{S}}(\mathbf{Y}|\mathbf{S})\right\}\right) \quad (17)$$

Our main results regarding to user identification and jointly user identification and data detection are as follows.

Theorem 1 *In randomly spread DS-CDMA communication with constant equal power per user, the large-system single-user channel capacity of a multiuser system with MAP estimation of user identities is*

$$\begin{aligned} C^{\text{users}}(\alpha) = & \quad (18)\\ -\alpha\frac{1}{\sqrt{2\pi}}&\int e^{-\eta y^2/2}\log\left(\alpha + (1-\alpha)e^{(-\eta\gamma/2-\eta\sqrt{\gamma}y)}\right)dy\\ -(1-\alpha)\frac{1}{\sqrt{2\pi}}&\int e^{-\eta y^2/2}\log\left(\alpha e^{(-\eta\gamma/2+\eta\sqrt{\gamma}y)} + (1-\alpha)\right)dy \end{aligned}$$

where η, the multiuser efficiency*, is the solution of the following fixed-point equation:*

$$\eta = \frac{1}{1+\beta\left(\gamma\left[\alpha - \int\sqrt{\frac{\eta}{2\pi}}\frac{\alpha^2 e^{-\eta/2(y-\sqrt{\gamma}))^2}}{\alpha+(1-\alpha)e^{-\eta(\sqrt{\gamma}y-\gamma/2)}}dy\right]\right)} \quad (19)$$

that minimizes the free energy

$$\begin{aligned}\mathcal{F} = &-\mathbb{E}\left[\int(\phi\log\phi)dy\right] - \frac{1}{2}\log\frac{2\pi e}{\eta}\\ &+\frac{1}{2\beta}\left((\eta-1)\log e + \log\frac{2\pi e}{\eta}\right) \end{aligned}\quad (20)$$

Here α is the activity rate of users, β the system load, γ the signal to noise ratio (SNR) per user, $\mathbb{E}$ denotes expectation over all random variables, and $\phi = p(y|\gamma,\eta,\mathbf{b}^k(\mathbf{X}))$ describes the large-system equivalent single-user Gaussian channel.

Corollary 2 *With model* (15)*, the large-system optimal spectral efficiency of a detector performing user identification is*

$$\mathcal{C}^{\text{users}}(\alpha) = C^{\text{users}}(\alpha) + \frac{1}{2}\left[(\eta-1)\log e - \log\eta\right] \quad (21)$$

Theorem 3 *In randomly spread DS-CDMA with constant equal power per user, the large-system single-user capacity of a multiuser channel with MAP user identification and data detection under BPSK transmission is*

$$\begin{aligned} &C^{\text{users}-\text{data}}(\alpha)\\ &= -\alpha\int\frac{e^{-y^2/2}}{\sqrt{2\pi}}\log\left(\alpha\cosh(\eta\gamma - y\sqrt{\eta\gamma}) + (1-\alpha)e^{\eta\gamma/2}\right)dy\\ &-(1-\alpha)\int\frac{e^{-(y-\sqrt{\eta\gamma})^2/2}}{\sqrt{2\pi}}\log\Bigg(\alpha\cosh(\eta\gamma - y\sqrt{\eta\gamma})\\ &+(1-\alpha)e^{\eta\gamma/2}\Bigg)dy + \eta\gamma\frac{(\alpha+1)}{2}\log(e) \end{aligned}\quad (22)$$

where η, the multiuser efficiency*, is the solution of the following fixed-point equation:*

$$\eta = \frac{1}{1+\beta\left(\gamma\left[\alpha - \int\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\frac{\alpha^2\sinh[\eta\gamma - y\sqrt{\eta\gamma}]}{\alpha\cosh[\eta\gamma-y\sqrt{\eta\gamma}]+(1-\alpha)e^{\eta\gamma/2}}dy\right]\right)} \quad (23)$$

that minimizes the free-energy (20)*.*

Corollary 4 *With model* (15)*, the optimal spectral efficiency of a detector performing user identification and data detection is*

$$\mathcal{C}^{\text{users}-\text{data}}(\alpha) = C^{\text{users}-\text{data}}(\alpha) + \frac{1}{2}\left[(\eta-1)\log e - \log\eta\right] \quad (24)$$

The fixed-point equations (20) and (23) are derived according to the prior distributions of random sets given in (5) and (6), which consider users' activity in a static channel. Under MAP estimation, detection requires the knowledge not only of the a priori probabilities of the data, but also of the activity rate α. Then, the fixed-point equations are determined by the minimum mean-square error. Finally, the single-user channel capacity and the joint spectral efficiency for each case are calculated from (13) and (14), by using the multiuser-efficiency fixed-point solution that yields the smallest free energy (or, equivalently, the smallest spectral efficiency).

V. NUMERICAL RESULTS

Some numerical results will now be shown, to illustrate the theory just developed. We exhibit the large-system performance of an individually optimum detector that knows the user's activity rate α. In this context, we examine this detector in two different types of random set estimation: *user identification*, where the a priori distribution of the random set is (5), and *user identification and data detection*, where it is (6).

Fig. 1 plots the large-system multiuser efficiency for MAP user identification based on the knowledge of the activity rate alone. The values of the multiuser efficiency are obtained numerically through the resolution of the fixed-point equation (19) for a system load $\beta = 1$. Fig. 2 plots the multiuser efficiency of MAP detection of users and data based on the knowledge of the activity rate alone and assuming binary antipodal modulation. In this case, the values of multiuser efficiency are obtained numerically through the resolution of the fixed-point equation (23) for $\beta = 1$. It is interesting to notice that the performance of the user detector is symmetrical with respect to the worst case $\alpha = 0.5$. In the case of user-and-data detection, the multiuser efficiency curves are degraded with respect to the standard case $\alpha = 1.0$, and although the behavior is not symmetrical, the worst case also tends to $\alpha = 0.5$ for large γ.

Figs. 3 and 4 compare capacity and spectral efficiency of the two MAP detectors described above for different values of α. With detection of user identities, the single-user capacity is symmetric with respect to $\alpha = 0.5$, the value at which capacity is maximum (1 bit/sec). The cases $\alpha = 0$ and $\alpha = 1$ correspond to full certainty of users' activities, and therefore multiuser efficiency and capacity (and also spectral efficiency) are trivially 1 and 0, respectively (see (18)). Unless otherwise specified, base-2 logarithms are assumed throughout. Regarding the users-and-data detector, the capacity curves show the cost incurred by the estimation of active-user identities with respect to the case where all users are known to be active (α=1.0). With the latter detector, symbols are assumed to take on three possible values $\{-1, 0, 1\}$ (0 corresponds to an inactive user) and the maximum capacity is achieved for $\alpha = 2/3$, which corresponds to equal prior distribution of data and activity (base-3 logarithms are used here, except for $\alpha = 1$). Perusal of both figures shows that, in the low-SNR regime, the multiuser efficiency tends to compensate the differences in capacity due to the effect of users' activity in their a priori distribution. However, for high SNR, multiuser efficiency approaches 1, and therefore the knowledge of the activity rate is the main factor that determines the difference in performance for each case.

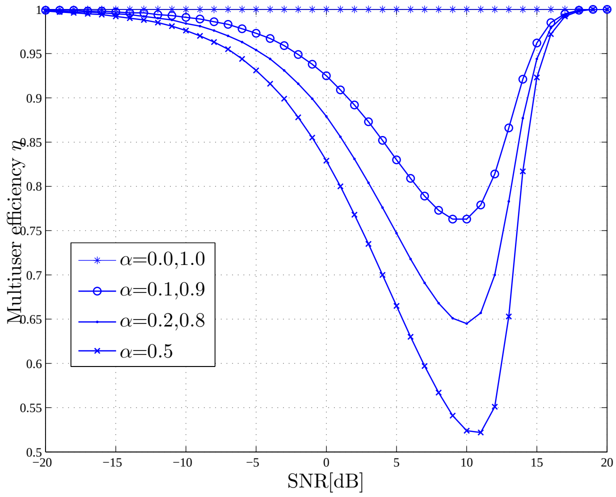

Figure 1: Large-system multiuser efficiency of MAP user identification with a priori knowledge of α and β=1.

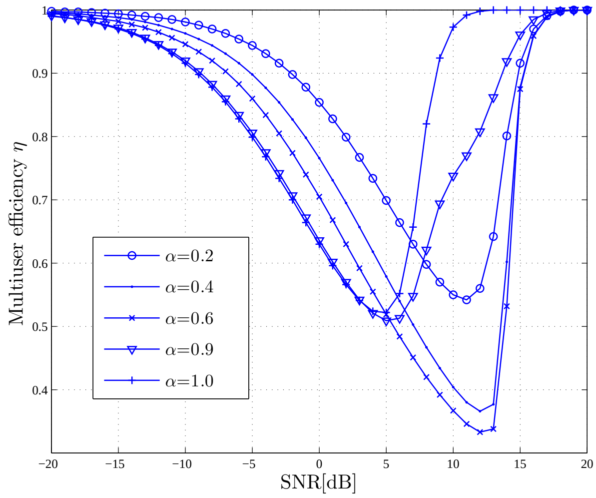

Figure 2: Large-system multiuser efficiency of user identification and data detection under MAP detection with prior knowledge of α and β=1.

VI. CONCLUSIONS

We have derived single-user channel capacity and the joint spectral efficiency of a static CDMA multiuser system with an unknown, large number of users. Our large-system analysis uses statistical-physics tools, and focuses on the performance of detectors under the assumption that the system parameters do not change. Multiuser efficiency and channel ca-

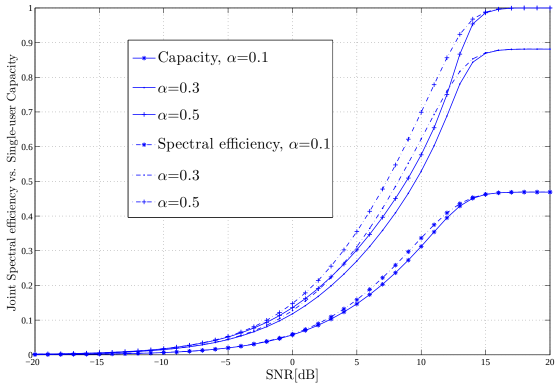

Figure 3: Large-system joint spectral efficiency and channel capacity with MAP detection of user identities with prior knowledge of α and β=1.

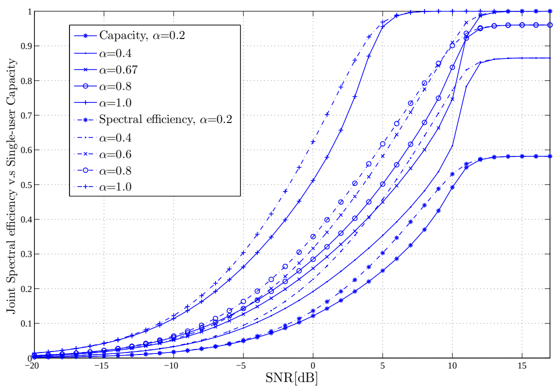

Figure 4: Large-system joint spectral efficiency and channel capacity with MAP detection of user identities and their data with prior knowledge of α and β=1.

pacity are computed through fixed point equations derived under the single-user equivalent channel assumption. Numerical results show the sensitivity of CDMA performance to the activity rate, and the loss due to the uncertainty on the number of users accessing the system. The limiting case $\alpha = 1$, corresponding to all users being active, matches the results previously known from the multiuser-detection literature.

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